# Chapter 4 Linear Second Order Equations



Definition: A linear $2^{nd}$ order DE has the form

$$a(t)\,y''(t) + b(t)\,y'(t) + c(t)\,y(t) = f(t)$$

where $a(t)$, $b(t)$, $c(t)$, $f(t)$ are continuous functions on an interval $I = (\alpha, \beta)$.

Theorem Let $a, b, c$ be constants & suppose $a \neq 0$.
Suppose $r$ is a real number that satisfies

$$ar^2 + br + c = 0 \quad \text{(Characteristic or Auxiliary Equation).}$$

Then $y = e^{rt}$ is a solution of the homogeneous equation

$$ay'' + by' + cy = 0.$$

Theorem If $y_1, y_2$ are solutions of

$$(*) \quad ay'' + by' + cy = 0,$$

then

$$y = c_1 y_1 + c_2 y_2,$$

is also a solution where $c_1, c_2$ are any constants.

Existence Uniqueness Theorem for $2^{nd}$ order linear IVPs with constant coefficients Let $t_0, a, b, c, Y_0, Y_1 \in \mathbb{R}$, $a \neq 0$.
The IVP

$$ay'' + by' + cy = 0; \qquad y(t_0) = Y_0, \; y'(t_0) = Y_1,$$

has a unique solution valid on the interval $I = (-\infty, \infty)$.

<u>Definition</u> Two functions $y_1(t), y_2(t)$ (defined on an interval $I$) are <u>linearly dependent on $I$</u> if there are constants $c_1, c_2$ not both zero such that
$$c_1 y_1(t) + c_2 y_2(t) = 0$$
for all $t \in I$. Otherwise, they are <u>linearly independent</u>.

<u>Lemma</u> Two functions $y_1(t), y_2(t)$ (on $I$) are linearly dependent on $I$ if and only if one of the functions is a constant multiple of the other on $I$.

<u>Theorem</u> Let $a, b, c, d, e, f$ be real constants.

(i) If $ad - bc \neq 0$ then the system of equations
$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases} \qquad \begin{vmatrix} a & e \\ c & f \end{vmatrix} \quad \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$
has a ______ solution $(x, y)$

namely $x = \dfrac{ed - bf}{ad - bc}$, $y = \dfrac{af - ce}{ad - bc}$

(ii) If $ad - bc = 0$ then the system of equations
$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$
has infinitely many solutions.

<u>Definition</u>: Let $y_1(t), y_2(t)$ be two differentiable functions defined on an open interval $I$. The <u>Wronskian</u> of $y_1, y_2$ is defined by
$$W[y_1, y_2](t) := \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t) y_2'(t) - y_1'(t) y_2(t)$$

**Theorem**: Suppose $y_1(t), y_2(t)$ are differentiable and linearly dependent functions on an open interval $I$. Then

$$W[y_1, y_2](t) = 0,$$

for all $t$ in $I$.

**NOTE**: The converse of this Theorem is NOT true.

**Corollary** Suppose $y_1(t), y_2(t)$ are differentiable functions on an open interval $I$ and

$$W[y_1, y_2](t_0) \neq 0,$$

for some $t_0$ in $I$. Then $y_1(t), y_2(t)$ are linearly independent on $I$.

**General Existence & Uniqueness Theorem for 2nd order Linear DEs**

Let $a(t), b(t), c(t), f(t)$ be continuous functions on an open interval $I$ and suppose $a(t) \neq 0$ for all $t$ in $I$. Suppose $t_0 \in I$ and $Y_0, Y_1$ are constants. Then the IVP

$$a(t)y'' + b(t)y' + c(t)y = f(t), \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1,$$

has a unique solution valid on the interval $I$.

Theorem Let $a(t), b(t), c(t)$ be continuous functions on an open interval $I$ and $a(t) \neq 0$ for all $t$ in $I$. Suppose $t_0 \in I$ and $Y_0, Y_1$ are constants. Suppose $y_1(t), y_2(t)$ are solutions of the DE

(*) $a(t)y'' + b(t)y' + c(t)y = 0$.

(i) Suppose

$$W[y_1, y_2](t_0) = 0.$$

Then $y_1(t), y_2(t)$ are linearly dependent on $I$ and hence

$$W[y_1, y_2](t) = 0$$

for all $t$ in $I$.

(ii) Suppose $y_1(t), y_2(t)$ are linearly independent. Then

$$W[y_1, y_2](t) \neq 0$$

for all $t$ in $I$.

There exists unique constants $c_1, c_2$ such that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is the solution to the IVP

(**) $a(t)y'' + b(t)y' + c(t)y = 0$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$.

The general solution of (*) is given by

$$y = d_1 y_1(t) + d_2 y_2(t)$$

where $d_1, d_2$ are any constants.

Theorem (Characteristic Equation has real roots)

Let $a, b, c$ be constants, $a \neq 0$ and $\Delta = b^2 - 4ac$.

(1) If $\Delta > 0$ then the auxiliary equation

$$ar^2 + br + c = 0$$

has two distinct real roots $r_1, r_2$

$$r_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad r_2 = \frac{-b - \sqrt{\Delta}}{2a},$$

and $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

are two linearly linearly independent solutions of

(*) $\quad ay'' + by' + cy = 0,$

and the general solution of (*) is given by

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

where $c_1, c_2$ are any constants.

(2) If $\Delta = 0$ then the auxiliary equation

$$ar^2 + br + c = a\left(r + \frac{b}{2a}\right)^2 = 0$$

has a double root $r = \frac{-b}{2a}$.

Also $y_1 = e^{-bt/2a}$, $y_2 = t e^{-bt/2a}$ are

linearly independent solutions of (*) and the general solution of (*) is given by

$y = (c_1 + c_2 t) e^{-bt/2a}$ where $c_1, c_2$ are any constants.

<u>Theorem</u> (Characteristic Equation has complex roots)

Let $a, b, c$ be constants, $a \neq 0$ and $\Delta = b^2 - 4ac$.

Suppose $\Delta < 0$. Then the auxiliary equation

$$ar^2 + br + c = 0$$

has two complex roots

$$r = \alpha \pm i\beta$$

where $\alpha = \frac{-b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$.

The differential equation

$$(*) \qquad ay'' + by' + cy = 0$$

has two linearly independent solutions

$$y_1 = e^{\alpha t} \cos\beta t, \quad y_2 = e^{\alpha t} \sin\beta t,$$

and the general solution of $(*)$ is given by

$$y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t),$$

where $c_1, c_2$ are any constants.

**Theorem** [The Method of Undetermined Coefficients]
Let $a, b, c$ be constants with $a \neq 0$ and let $f(t)$ be a given function. Consider the non-homogeneous DE

(*) $\quad ay'' + by' + cy = f(t)$,

with associated auxiliary equation

(**). $\quad ar^2 + br + c = 0$.

**TYPE (I)** $f(t) = p_n(t)e^{\alpha t}$ where $p_n(t)$ is a polynomial of degree $n$.

(a) If $r = \alpha$ is NOT a root of the auxiliary eqn (**) then a particular solution of (*) has the form
$$y_p(t) = e^{\alpha t}(A_0 + A_1 t + \cdots + A_n t^n),$$
where $A_0, A_1, \ldots, A_n$ are some constants.

(b) If $r = \alpha$ is a single root of the auxiliary eqn (**) then a particular solution of (*) has the form
$$y_p(t) = t\, e^{\alpha t}(A_0 + A_1 t + \cdots + A_n t^n),$$
where $A_0, A_1, \ldots, A_n$ are some constants.

(c) If $r = \alpha$ is a double root of the auxiliary eqn (**) then a particular solution of (*) has the form
$$y_p(t) = t^2 e^{\alpha t}(A_0 + A_1 t + \cdots + A_n t^n),$$
where $A_0, A_1, \ldots, A_n$ are some constants.

**TYPE (II)** $f(t) = p_n(t)e^{\alpha t}\cos\beta t + q_m(t)e^{\alpha t}\sin\beta t$
where $\beta > 0$, $p_n(t)$ is a polynomial of degree $n$, $q_m(t)$ is a polynomial of degree $m$. Let $N = \max(m, n)$.

(a) If $r = \alpha \pm i\beta$ is NOT a root of the auxiliary eqn (**) then a particular solution of (*) has the form
$$y_p(t) = P_N(t)\cos\beta t\, e^{\alpha t} + Q_N(t)\sin\beta t\, e^{\alpha t},$$
where $P_N(t), Q_N(t)$ are generic polynomials of degree $N$.

(b) If $r = \alpha \pm i\beta$ is a root of the auxiliary eqn (**) then a particular solution of (*) has the form
$$y_p(t) = t\left(P_N(t)\cos\beta t\, e^{\alpha t} + Q_N(t)\sin\beta t\, e^{\alpha t}\right),$$
where $P_N(t), Q_N(t)$ are generic polynomials of degree $N$.

<u>A $2^{nd}$ order linear differential operator has the form</u>
$$L[y] = y'' + p_1(t)\,y' + p_2(t)\,y,$$
where $p_1(t)$, $p_2(t)$ are given continuous functions on an interval $I$. If $y_1(t), y_2(t)$ are twice differentiable functions and $c_1, c_2$ are constants, then
$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2].$$

<u>Theorem on Nonhomogeneous $2^{nd}$ Order Linear DEs</u>

Let $a(t), b(t), c(t), f(t)$ be continuous functions on an open interval $I$ and suppose $a(t) \neq 0$ for $t \in I$. Suppose $y_1(t)$, $y_2(t)$ are two linearly independent solutions of
$$(*) \quad a(t)y'' + b(t)y' + c(t)y = 0,$$
and $y_p(t)$ is a particular solution of
$$(**) \quad a(t)y'' + b(t)y' + c(t)y = f(t).$$