Chapter 4 Linear Second Order Equations

Definition: A linear 2^{nd} order DE has the form

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

where $a(t)$, $b(t)$, $c(t)$, $f(t)$ are continuous functions on an interval $I = (\alpha, \beta)$.

Theorem Let a, b, c be constants & suppose $a \neq 0$.
Suppose r is a real number that satisfies

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = 0$$ (Characteristic or Auxiliary Equation).

Then $y = \_\_\_\_$ is a solution of the homogeneous equation

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_.$$

Theorem If y_1, y_2 are solutions of

(*) $ay'' + by' + cy = 0$,

then

$$y = \_\_\_\_\_\_\_\_\_\_,$$

is also a solution where ____________________.

Existence Uniqueness Theorem for 2^{nd} order linear IVPs with constant coefficients Let $t_0, a, b, c, Y_0, Y_1 \in \mathbb{R}$, $a \neq 0$.
The IVP

$$ay'' + by' + cy = 0; \quad y(t_0) = \_\_\_\_, \; y'(t_0) = \_\_\_\_,$$

has a _________ solution valid on the interval $I = \_\_\_\_\_\_\_\_\_$.

Definition Two functions $y_1(t), y_2(t)$ (defined on an interval I) are *linearly dependent on I* if __________

____

____

for _____. Otherwise, they are __________.

Lemma Two functions $y_1(t), y_2(t)$ (on I) are *linearly dependent on I* if and only if __________

__________.

Theorem Let a, b, c, d, e, f be real constants.

(i) If $ad - bc$ ____ then the system of equations

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

has a ______ solution (x, y)

namely $x = \boxed{}$, $y = \boxed{}$.

(ii) If $ad - bc =$ ____ then the system of equations

$$\begin{cases} ax + by = 0 \\ cx + dy = 0 \end{cases}$$

has __________ solutions.

Definition: Let $y_1(t), y_2(t)$ be two differentiable functions defined on an open interval I. The *Wronskian* of y_1, y_2 is defined by

$$W[y_1, y_2](t) := \boxed{} = \text{__________}.$$

<u>Theorem</u>: Suppose $y_1(t)$, $y_2(t)$ are differentiable and linearly dependent functions on an open interval I. Then

$$W[y_1, y_2](t) = \_\_\_,$$

for ________.

<u>NOTE</u>: The converse of this Theorem ___ ___ true.

<u>Corollary</u> Suppose $y_1(t)$, $y_2(t)$ are differentiable functions on an open interval I and

$$W[y_1, y_2](\_\_\_) \_\_\_\_,$$

for ________. Then $y_1(t)$, $y_2(t)$ are linearly ________ on I.

<u>General Existence & Uniqueness Theorem for 2nd order Linear DEs</u>

Let $a(t)$, $b(t)$, $c(t)$, $f(t)$ be ______ functions on an open interval I and suppose $a(t)$ ___ for ___ t in I. Suppose t_0 ___ and Y_0, Y_1 are _____. Then the IVP

____________, ____, ____,

has a ______ solution valid on ________.

<u>Theorem</u> Let $a(t), b(t), c(t)$ be __ __ __ __ functions on an open interval I and $a(t)$ __ __ for __ __ t in I. Suppose t_0 __ __ and Y_0, Y_1 are __ __ __ __ __. Suppose $y_1(t), y_2(t)$ are solutions of the DE

$$(*) \qquad a(t)y'' + b(t)y' + c(t)y = \_\_\_.$$

(i) Suppose

$$W[y_1, y_2](t_0) = \_\_\_.$$

Then $y_1(t), y_2(t)$ are linearly __ __ __ __ on I and hence

$$W[y_1, y_2](t) = \_\_\_$$

for __ __ t in I.

(ii) Suppose $y_1(t), y_2(t)$ are linearly independent. Then

$$W[y_1, y_2](t) \ \_\_\_$$

for __ __ t in I.

There exists __ __ __ constants c_1, c_2 such that

$$y = \_\_\_\_\_\_\_\_$$

is the solution to the IVP

$$(**) \qquad a(t)y'' + b(t)y' + c(t)y = \_\_\_, \ \_\_\_\_, \ \_\_\_\_.$$

The general solution of $(*)$ is given by

$$y = \_\_\_\_\_\_\_\_$$

where __ __ __ __ __ __ __ __ __ __ __.

Theorem (Characteristic Equation has real roots)

Let a, b, c be constants, $a \neq 0$ and $\Delta = $ ______.

(1) If $\Delta > 0$ then the auxiliary equation

$$ar^2 + br + c = 0$$

has ___ _____ real roots r_1, r_2

$$r_1 = \_\_\_\_\_\_, \quad r_2 = \_\_\_\_\_\_,$$

and $y_1 = $ ____, $y_2 = $ ____

are two linearly _________ solutions of

(*) $ay'' + by' + cy = 0$,

and the general solution of (*) is given by

$$y = \_\_\_\_\_\_\_\_\_.$$

where ______________.

(2) If $\Delta = 0$ then the auxiliary equation

$$ar^2 + br + c = a\left(r + \_\_\_\right)^2 = 0$$

has a ______ root $r = \_\_\_\_$.

Also $y_1 = $ ____, $y_2 = $ ____ are

linearly _______ solutions of (*) and the general solution of (*) is given by

$y = $ ________ where __________.

<u>Theorem</u> (Characteristic Equation has complex roots)
Let a, b, c be constants, $a \neq 0$ and $\Delta =$ _____.
Suppose Δ _____. Then the auxiliary equation

$$ar^2 + br + c = 0$$

has two ______ roots

$$r = \alpha \pm i\beta$$

where $\alpha =$ ____, $\beta =$ _____.

The differential equation

$$(*) \qquad ay'' + by' + cy = 0$$

has two linearly ________ solutions

$$y_1 = \_\_\_\_\_\_\_, \quad y_2 = \_\_\_\_\_\_\_,$$

and the general solution of $(*)$ is given by

$$y = \_\_\_\_\_\_\_\_\_\_\_\_,$$

where _____________.

Theorem [The Method of Undetermined Coefficients]
Let a, b, c be constants with $a \neq 0$ and let $f(t)$ be a given function. Consider the non-homogeneous DE

$$(*) \qquad ay'' + by' + cy = f(t),$$

with associated auxiliary equation

$$(**). \qquad ar^2 + br + c = 0.$$

TYPE (I) $f(t) = p_n(t)\, e^{\alpha t}$ where $p_n(t)$ is a polynomial of degree n.

(a) If $r = \alpha$ is _____ root of the auxilary eq$^{\underline{n}}$ $(**)$ then a particular solution of $(*)$ has the form

$y_p(t) =$ ________________,
where ____________________.

(b) If $r = \alpha$ is a _____ root of the auxiliary eq$^{\underline{n}}$ $(**)$ then a particular solution of $(*)$ has the form

$y_p(t) =$ ________________,
where ____________________.

(c) If $r = \alpha$ is a _______ root of the auxiliary eq$^{\underline{n}}$ $(**)$ then a particular solution of $(*)$ has the form

$y_p(t) =$ ________________,
where ____________________.

TYPE (II) $f(t) = p_n(t)\, e^{\alpha t} \cos\beta t + q_m(t)\, e^{\alpha t} \sin\beta t$
where β ___, $p_n(t)$ is a polynomial of degree ___, $q_m(t)$ is a polynomial of degree ___. Let $N =$ _____.

(a) If $r =$ ____ is _______ of the auxiliary eqn (**)
then a particular solution of (*) has the form

$y_p(t) =$ _______________,

where _________________.

(b) If $r =$ _____ is ________ of the auxiliary eqn (**)
then a particular solution of (*) has the form

$y_p(t) =$ ________________,

where ___________________.

<u>A 2^{nd} order linear differential operator has the form</u>

$L[y] =$ ____________,

where $p_1(t)$, $p_2(t)$ are given ______ functions on an interval I. If $y_1(t), y_2(t)$ are twice differentiable functions and c_1, c_2 are _____, then

$$L[c_1 y_1 + c_2 y_2] = \text{\_\_\_\_\_\_\_\_\_}.$$

<u>Theorem on Nonhomogeneous 2^{nd} Order Linear DEs</u>

Let $a(t), b(t), c(t), f(t)$ be continuous functions on an open interval I and suppose $a(t)$ ____ for _____.
Suppose $y_1(t)$, $y_2(t)$ are two ________________ of

(*) $a(t)y'' + b(t)y' + c(t)y =$ ___,

and $y_p(t)$ is a _______ solution of

(**) $a(t)y'' + b(t)y' + c(t)y =$ ____.

Then the general solution of (**) is given by

$$y = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,$$

where ____________________. Further, the IVP

(***) $a(t)y'' + b(t)y' + c(t)y = \_\_\_\_\_, \_\_\_\_\_\_, \_\_\_\_\_\_,$

has a __________ solution for given constants ______, ______, and t_0 in ______.

<u>The Superposition Principle</u>

Suppose $y_1(t)$ is a solution to

(1) $a(t)y'' + b(t)y' + c(t)y = g_1(t)$,

and $y_2(t)$ is a solution to

(2) $a(t)y'' + b(t)y' + c(t)y = g_2(t)$.

Then

$$y = \_\_\_\_\_\_\_\_\_\_\_\_\_\_, \text{ where } \_\_\_\_\_\_\_\_\_\_\_\_,$$

is a solution to

(3) $a(t)y'' + b(t)y' + c(t)y = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$

<u>Variation of Parameters</u> is a method for finding a ____________ solution $y_p(t)$ of a nonhomogeneous linear DE

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

provided we know ________________________ $y_1(t), y_2(t)$ of the corresponding homogeneous DE

$$a(t)y'' + b(t)y' + c(t)\,y = \_\_\_.$$

We assume

$$y_p(t) = \_\_\_\_\_\_\,y_1(t) + \_\_\_\_\_\_\,y_2(t).$$

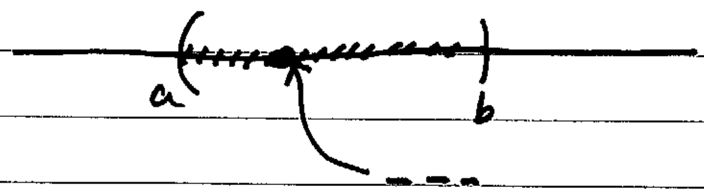
We solve the equations

$$\_\_\_\_ v_1' + \_\_\_\_ v_2' = \_\_\_,$$
$$\_\_\_\_ v_1' + \_\_\_\_ v_2' = \_\_\_\_$$

for the functions ___ and ____.
Then we find ___, ___, and ____.

The Existence-Uniqueness Theorem for 2nd order linear IVPs

Suppose $p(t)$, $q(t)$, $g(t)$ are c______ on an _____ interval (a,b) which _____ the point t_0



Let Y_0, Y_1 be _____. The IVP

$$y'' + p(t)y' + q(t)y = \_\_\_, \_\_\_, \_\_\_\_$$

has a _____ solution $y(t)$ valid on the interval ___ ie valid for ________.

Theorem If

$$\_\_\_\_\_\_\_\_\_\_\_\_ = 0$$

then $y =$ ____ is a solution to homogeneous

Cauchy-Euler Equation

$$\_\_\_\_\_\_\_\_\_\_\_\_\_ = 0.$$

TYPE of ROOTS		Linearly Independent Solutions
Single roots	$r = r_1, r_2$	$y_1 =$ ___, $y_2 =$ ___
Double root	$r = r_1$	$y_1 =$ ___, $y_2 =$ ___
Complex roots	$r = \alpha \pm i\beta$	$y_1 =$ _____, $y_2 =$ _____