"MODULAR FUNCTIONS & DIRICHLET SERIES IN NUMBER THEORY", Tom M. Apostol.

Introduction & Chapter 1 Elliptic Functions

~~Let p(n) denote~~

A partition of n is a sequence of non-increasing positive integers whose sum is n.

Let $p(n)$ denote the number of partitions of n.

$n=1$ $p(1)=1$

$n=2$ The partitions of 2 are 2, $1+1$. $p(2)=2$.

$n=3$ The partitions of 3 are 3, $2+1$, $1+1+1$. $p(3)=3$.

$n=4$ The partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$, $1+1+1+1$. $p(4)=5$.

$n=5$ The partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$.

$p(5)=7$

$p(6)=11$

$p(7)=15$

Theorem (Euler)

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \qquad |q|<1.$$

Idea of proof

$$\frac{1}{1-q^1} = 1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots$$

$$\frac{1}{1-q^2} = 1 + q^2 + q^{2+2} + \cdots$$

$$\frac{1}{1-q^3} = 1 + q^3 + q^{3+3} + \cdots$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{1-q^1} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdots$$

$$= (1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots)(1 + q^2 + q^{2+2} + \cdots)(1 + q^3 + q^{3+3} + \cdots)\cdots$$

$$= 1 + q^1 + (q^{1+1} + q^2) + (q^{1+1+1} + q^{1+2} + q^3) + \cdots$$
$$= 1 + p(1)q^1 + p(2)q^2 + p(3)q^3 + \cdots \qquad \square$$

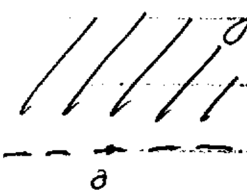
<u>Asymptotics</u> $\quad p(n) \sim \dfrac{\exp\left(\pi\sqrt{\frac{2n}{3}}\right)}{4n\sqrt{3}}$ (Hardy & Ramanujan 1918)

$f(n) \sim g(n)$ as $n \to \infty$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

<u>The Dedekind eta-function</u>

$$\eta(\tau) := \exp(\pi i \tau/12) \prod_{n=1}^{\infty} (1 - \exp(2\pi i \tau n))$$

for $\mathrm{Im}\,\tau > 0$.



Let $q = \exp(2\pi i \tau) = \exp(2\pi i (x+iy))$

$|q| = \exp(-2\pi y) < 1$, for $\mathrm{Im}\,\tau > 0$, $y = \mathrm{Im}\,\tau$.

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

<u>Transformation formula</u>

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\ \eta(\tau), \quad \mathrm{Im}\,\tau > 0.$$

$\eta(\tau)$ is a modular form of weight $\frac{1}{2}$.

<u>Arithmetic properties of $p(n)$</u> (Ramanujan)

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

Kolberg (196?) $\quad p(n) \equiv 0 \pmod 2$ for infinitely many n.

<u>Open Problem</u> $\quad p(n) \equiv 0 \pmod 3$ for infinitely many n.

Chapter 1 Elliptic Functions

Doubly periodic functions

A function f of a complex variable is called **periodic** with period ω if

$$f(z+\omega) = f(z) \qquad (\omega \neq 0)$$

whenever z, $z+\omega$ are in the domain of f.

If ω is a period then so is $n\omega$ for every integer n. If ω_1, ω_2 are periods then so is

$$m\omega_1 + n\omega_2$$

for any integers m, n.

Defn:

f is called **doubly-periodic** if it has two periods ω_1, ω_2 where ω_2/ω_1 is not real.

Note:

(1) If ω_2/ω_1 is real & rational then there is a complex number ω such that $\omega_1 = b\omega$, $\omega_2 = a\omega$ for some integers a, b.

Proof: Let $\frac{\omega_2}{\omega_1} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $(a,b)=1$.

$\exists\, m, n \in \mathbb{Z}$: $am + bn = 1$.

Let $\omega = n\omega_1 + m\omega_2$. Then

$$\omega = \omega_1\left(n + m\frac{\omega_2}{\omega_1}\right) = \omega_1\left(n + \frac{ma}{b}\right)$$

$$= \frac{\omega_1}{b}(nb + ma) = \frac{\omega_1}{b}, \qquad \omega_1 = b\omega.$$

$$\omega_2 = \frac{a\omega_1}{b} = a\omega. \ \square$$

(2) If ω_2/ω_1 is real & irrational, then f has arbitarily small periods.

We need

<u>Dirichlet's Approx. Thm</u> Given any real number θ, & positive integer N, $\exists$ integers h, k, $0 < k \le N$ such that

$$|k\theta - h| < \frac{1}{N}.$$

<u>Proof:</u> Let $\{x\} = x - [x]$ denote the fractional part of x. Consider the $N+1$ numbers

$$0, \{\theta\}, \{2\theta\}, \dots, \{N\theta\} \in [0,1)$$

$[0,1) = [0, \frac{1}{N}) \cup [\frac{1}{N}, \frac{2}{N}) \cup \dots \cup [\frac{N-1}{N}, 1)$ (N subintervals)

One subinterval must contain at least two of $\{m\theta\}$.

So for some a, b, $0 \le a < b \le N$,

$$|\{b\theta\} - \{a\theta\}| < \frac{1}{N}.$$

But $\{b\theta\} - \{a\theta\} = b\theta - [b\theta] - a\theta + [a\theta]$

$$= \underbrace{(b-a)}_{k}\theta - \underbrace{([b\theta] - [a\theta])}_{h}. \qquad \square$$

<u>Theorem</u> Suppose ω_1, ω_2 are periods of f, ω_2/ω_1 real and irrational then f has arbitrarily small periods.

<u>Proof:</u> Let $\theta = \frac{\omega_2}{\omega_1}$. Let $\varepsilon > 0$. By Dirichlet's approx. Thm $\exists$ integers h, k such that

$$|k\theta - h| < \frac{\varepsilon}{|\omega_1|},$$

and $|k\omega_2 - h\omega_1| < \varepsilon$.

But $\omega = k\omega_2 - h\omega_1$ is a nonzero period since ω_2/ω_1 is irrational. $\square$

<u>Corollary</u>: If f is analytic and has periods ω_1, ω_2,
[redacted] ω_2/ω_1 real and irrational then f must be constant on every open connected subset of its domain.

<u>Proof:</u> $\exists$ sequence of periods $\{z_n\}$, $z_n \ne 0$, $z_n \to 0$.

Let $z \in$ domain of f. $f'(z) = \lim_{n\to\infty} \frac{f(z+z_n) - f(z)}{z_n} = 0$. $\square$

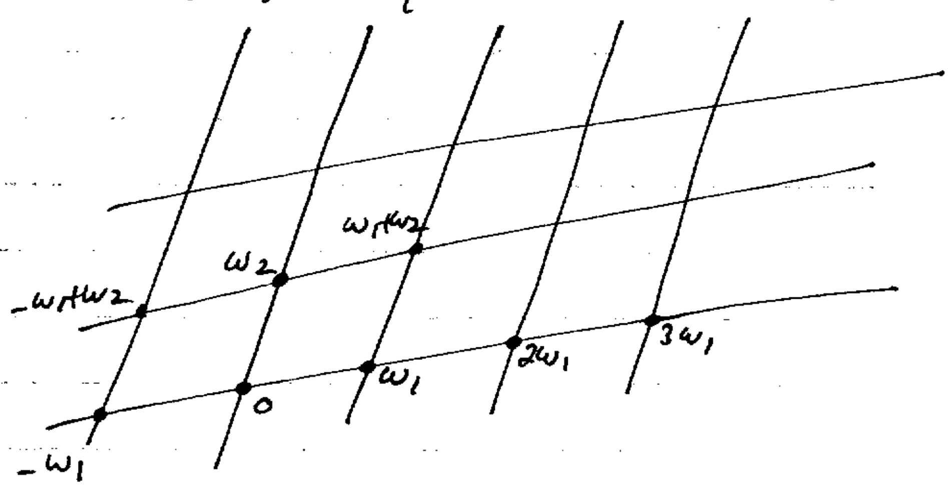
Fundamental pairs of periods

<u>Defn</u>: Let f have periods ω_1, ω_2, where $\omega_2/\omega_1 \notin \mathbb{R}$. Then ω_1, ω_2 is a <u>fundamental pair</u> of periods if for every period ω of f $\exists$ integers m, n :

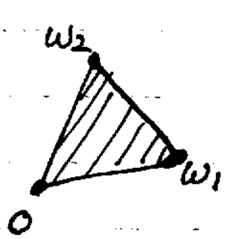
$$\omega = m\omega_1 + n\omega_2.$$

The lattice $\Omega(\omega_1, \omega_2)$ generated by ω_1, ω_2 is

$$\Omega(\omega_1, \omega_2) := \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

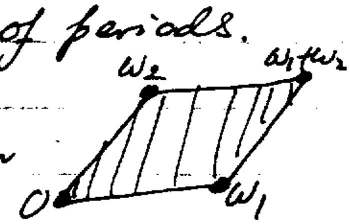


<u>Theorem</u> If ω_1, ω_2 is a fundamental pair of periods, then the triangle with vertices $0, \omega_1, \omega_2$ contains no further periods in its interior or on its boundary.

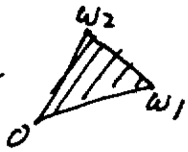


Conversely, any pair of periods with this property is fundamental.

<u>Proof</u>: Suppose ω_1, ω_2 is a fundamental pair of periods. Then ω_1, ω_2 are indept. over the reals.



If $\omega = \alpha\omega_1 + \beta\omega_2$ is in the parallelogram then $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. If ω is a period then $\alpha, \beta \in \mathbb{Z}$ so $\omega = 0, \omega_1, \omega_2$ or $\omega_1 + \omega_2$.
Hence the only periods in the triangle are $0, \omega_1, \omega_2$.

Conversely, suppose the triangle 

contains (6)

(*) [no other periods other than the vertices $0, \omega_1, \omega_2$.

Let ω be a period. $\exists\ t_1, t_2 \in \mathbb{R}$ s.th.

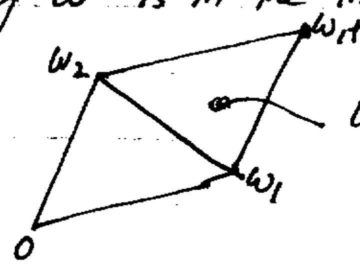
$$\omega = t_1 \omega_1 + t_2 \omega_2 .$$

Let $t_1 = [t_1] + r_1, \quad t_2 = [t_2] + r_2$

so that $0 \le r_1 < 1, \quad 0 \le r_2 < 1.$ Then

$$\omega' = r_1 \omega_1 + r_2 \omega_2 \quad \text{is a period.}$$

If ω' is in the triangle then $r_1 = r_2 = 0$. (by assumption (*))



Suppose ω' is not in the triangle.

Then $1 < r_1 + r_2 < 2$.

Let $\omega'' = \omega_1 + \omega_2 - \omega'$ (is a period)

$$= (1-r_1)\omega_1 + (1-r_2)\omega_2 .$$

$$(1-r_1) + (1-r_2) = 2 - (r_1 + r_2) < 1$$

and ω'' is in the triangle, & $1-r_1, 1-r_2 \in \mathbb{Z}$ (since ω'' is a period in triangle)

In both cases $r_1, r_2 \in \mathbb{Z}$ & $r_1 = r_2 = 0$, $t_1, t_2 \in \mathbb{Z}$.

Therefore ω_1, ω_2 are fundamental periods.

<u>Definition:</u> Two pairs of complex numbers (ω_1, ω_2), (ω_1', ω_2') each with non real ratio are <u>equivalent</u> if they generate the same period lattice; ie $\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2')$.

~~$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}$.~~

<u>Theorem</u>: Two pairs (ω_1, ω_2), (ω_1', ω_2') are equivalent iff $\exists\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries $\det A = \pm 1$ such that

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

<u>Theorem</u>: Two pairs (ω_1, ω_2), (ω_1', ω_2') are equivalent iff $\exists\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, $\det(A) = \pm 1$ such that
$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}.$$

<u>Proof</u>: ($\Rightarrow$) Suppose (ω_1, ω_2), (ω_1', ω_2') are equivalent.
Then $\exists A \in M_2(\mathbb{Z})$: $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$

& $B \in M_2(\mathbb{Z})$: $\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = B \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$

Hence $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = AB \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$.
Let $AB = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then
$$\omega_2' = \alpha \omega_2' + \beta \omega_1'$$
$\Rightarrow$ $\alpha = 1, \beta = 0$ since ω_1', ω_2' indept over $\mathbb{R}$.
Similarly $\gamma = 0, \delta = 1$, $AB = I$. $\det(A)\det(B) = 1$
But $\det(A), \det(B) \in \mathbb{Z}$ & $\det(A) = \pm 1$.

($\Leftarrow$) Suppose $\exists A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$, $\det(A) = \pm 1$, s.t. $\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$

Then $\omega_2', \omega_1' \in \Omega(\omega_1, \omega_2)$ &
$$\Omega(\omega_1', \omega_2') \subset \Omega(\omega_1, \omega_2).$$
$A^{-1} \in M_2(\mathbb{Z})$ so similarly
$$\Omega(\omega_1, \omega_2) \subset \Omega(\omega_1', \omega_2)$$
$$\& \ \Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2'). \quad \square$$

~~Thm Any nonce~~

<u>Definition</u> A function $f(z)$ is <u>elliptic</u> if

(a) f is doubly periodic

and (b) f is meromorphic on $\mathbb{C}$ ($f(z)$ is analytic on $\mathbb{C}$ except for possible poles).

<u>Thm</u> Any nonconstant elliptic function has a fundamental pair of periods.

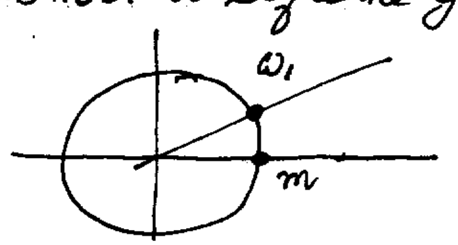
<u>Proof:</u> Suppose f is a nonconstant elliptic function. Let $P = \{ \omega : \omega \neq 0 \ \& \ \omega \text{ is a period of } f \}$. Since f is analytic & nonconstant f cannot have arbitrarily small periods.

Let $m = \inf\limits_{\omega \in P} |\omega|$. Then $m > 0$.

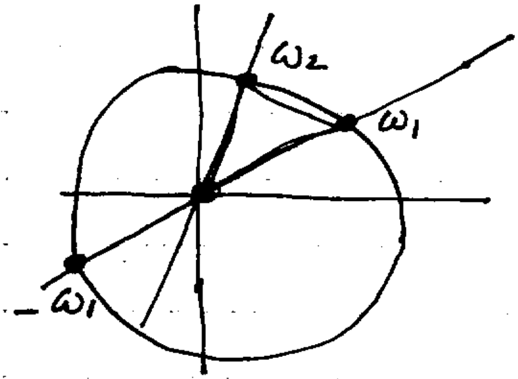
<u>Claim</u> $\exists\, \omega' \in P : \ |\omega'| = m$.

$\exists \{\omega_n\} \subset P$ s.t. $|\omega_n| \to m$ and hence there must be a convergent subsequence $\{\omega_{n_k}\}$, $\omega_{n_k} \to \omega'$. However $\omega_{n_k} - \omega_{n_{k+1}} \longrightarrow 0$ so $\exists K : \quad \omega_{n_k} = \omega_{n_{k+1}}$ for $k \geq K$ and $\omega' \in P$.

Of all $\omega \in P$ with $|\omega| = m$ there is one ω_1 with the smallest nonnegative argument otherwise we could construct a sequence of arbitrarily small nonzero periods.

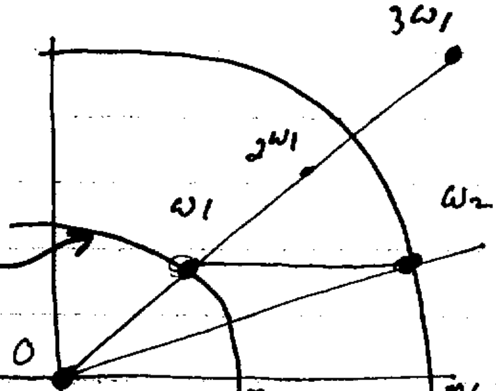


<u>Case 1</u> There are other periods ω with $|\omega| = m$ besides $\pm\omega_1$. Let ω_2 be the one with smallest arg greater than that of ω_1.



There are no periods in the triangle $0, \omega_1, \omega_2$

<u>Case 2</u>



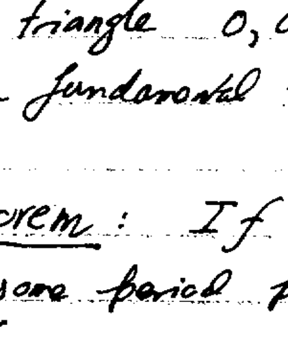
No other periods on $|z| = m$ except $\pm\omega_1$

There are no other periods with $|\omega| = m$ besides $\pm\omega_1$. Choose smallest $m_1 > m$ whose circle contains a period $\omega \neq n\omega_1$ $(n \in \mathbb{Z})$. Such a period exists because there can not be arbitrarily small periods and all periods can not be all multiples of ω_1. Let ω_2 be the period on this circle with the smallest nonneg. arg. There are no other periods in the triangle $0, \omega_1, \omega_2$ besides the vertices. So (ω_1, ω_2) is a fundamental pair. $\square$

<u>Theorem</u>: If f is an elliptic function with no poles in some period parallelogram then f is constant.

<u>Proof</u>:

$P = \{c_1\omega_1 + c_2\omega_2 : 0 \le c_1, c_2 < 1\}$



Let $z \in \mathbb{C}$.

$z = \alpha\omega_1 + \beta\omega_2 \qquad \alpha, \beta \in \mathbb{R}$

$f(z) = f((\alpha - [\alpha])\omega_1 + (\beta - [\beta])\omega_2)$

since $-[\alpha]\omega_1 - [\beta]\omega_2$ is a period.

$(\alpha - [\alpha])\omega_1 + (\beta - [\beta])\omega_2 \in P$ & $f(\mathbb{C}) = f(P)$.

f cts so f is bounded & f is constant by Liouville's Thm. $\square$

~~Theorem~~

Corollary: If f is an elliptic function and f has no zeros in a period parallelogram then f is constant.

Proof: If f is elliptic then $\frac{1}{f}$ is elliptic and the result follows from the Theorem. □

Theorem: Suppose $f(z)$ is analytic on a simple closed +ve contour C and analytic inside C except for poles.

Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \#\text{ of zeros} - \#\text{ of poles}$$

(counted according to multiplicity).

Proof: Suppose near $z=z_0$

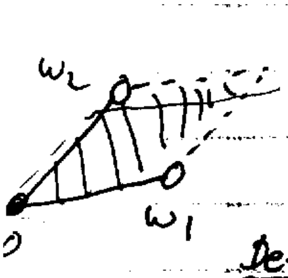
$$f(z) = a_m(z-z_0)^m + \cdots \qquad (m\neq 0,\ a_m\neq 0)$$

$$f'(z) = m a_m (z-z_0)^{m-1} + \cdots$$

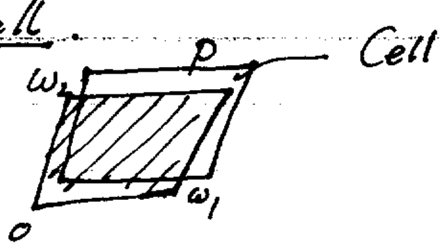
$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \cdots$$

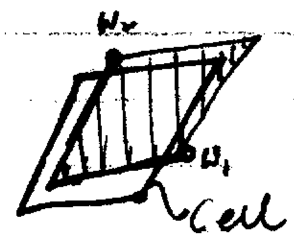
$$\operatorname*{Res}_{z=z_0} \frac{f'(z)}{f(z)} = m.$$

The result follows by the Residue Thm.

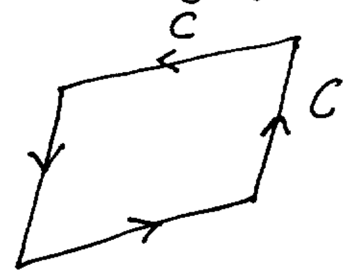


Definition: Suppose P is a period parallelogram of an elliptic function f. We translate P so that no zeros or poles occur on the boundary. The translated parallelogram is called a cell.





Proposition Let C be a contour which is the boundary of the cell of an elliptic function f. Then
$$\int_C f(z)\,dz = 0.$$



Theorem The sum of the residues of an elliptic function at the poles in any period parallelogram is zero.

Note: Suppose
$$f(z) = a_m (z - z_0)^m + \cdots \qquad \text{for } z \text{ near } z_0.$$
Let ω be a period
$$f(z) = f(z - \omega) = a_m\left(z - (z_0+\omega)\right)^m + \cdots \quad \text{for } z \text{ near } z_0+\omega$$

NOTE: Any nonconstant elliptic function can not have just one simple pole in a period parallelogram.

Theorem: The number of zeros of an elliptic function in any period parallelogram is equal to the number of poles each counted with multiplicity.

Proof If $f(z)$ is elliptic then so is $f'(z)$ and $\dfrac{f'(z)}{f(z)}$. Result follows from previous Thm.

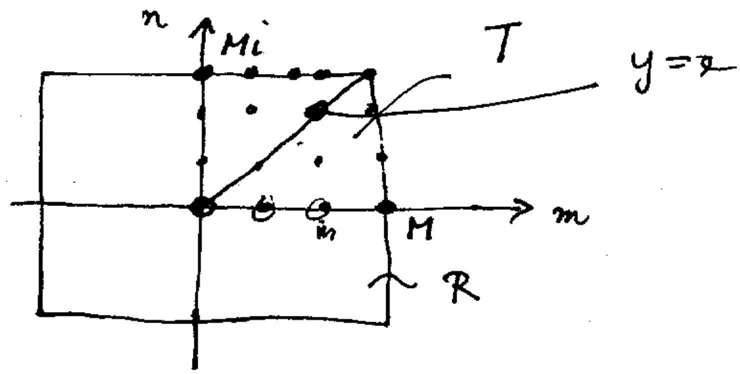
$$\frac{1}{2\pi i}\int_C \frac{f'(t)}{f(t)}\,dt = 0$$

<u>Construction of elliptic functions</u> Given $\omega_1/\omega_2 \notin \mathbb{R}$, Following Weierstrass we will construct an elliptic function $\wp(z) = \wp(z; \omega_1, \omega_2)$ with a pole of order 2 at $z=0$ and fundamental periods ω_1, ω_2.

<u>Lemma 1</u> Let ω_1, ω_2 be two nonzero complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$. Let $\Omega = \Omega(\omega_1, \omega_2)$, $\alpha \in \mathbb{R}$. Then $\displaystyle\sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \frac{1}{\omega^\alpha}$ converges absolutely iff $\alpha > 2$.

<u>Proof</u>

<u>CASE 1</u>: $\omega_1 = 1, \ \omega_2 = i$ $\qquad \omega = m + ni$



$$\sum_{\substack{\omega \neq 0 \\ \omega \in R}} \frac{1}{|\omega|^\alpha} \leqslant 8 \sum_{\substack{\omega \neq 0 \\ \omega \in T}} \frac{1}{|\omega|^\alpha}$$

$$= 8 \sum_{m=1}^{M} \sum_{n=0}^{m} \frac{1}{|m+in|^\alpha} = 8 \sum_{m=1}^{M} \sum_{n=0}^{m} \frac{1}{(m^2+n^2)^{\alpha/2}}$$

$$\leqslant 8 \sum_{m=1}^{M} \sum_{n=0}^{m} \frac{1}{m^\alpha} = 8 \sum_{m=1}^{M} \frac{m+1}{m^\alpha}$$

$$\leqslant 16 \sum_{m=1}^{M} \frac{1}{m^{\alpha-1}} < \infty \quad \text{for } \alpha > 2.$$

General Case:

Let $\omega_1 = a_1 + ib_1$

$\omega_2 = a_2 + ib_2$.

Then $\omega = m\omega_1 + n\omega_2 = (ma_1 + na_2) + i(mb_1 + nb_2)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = A\begin{pmatrix} m \\ n \end{pmatrix}$$

where $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$. $\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_2 = \sqrt{x^2+y^2}$.

(EX) Then $|\omega| = \left\| A \begin{pmatrix} m \\ n \end{pmatrix}\right\|_2 \le \|A\|_F \left\|\begin{pmatrix} m \\ n \end{pmatrix}\right\|_2 = \|A\|_F \sqrt{m^2+n^2}$

where $\|A\|_F = \left(\sum\sum a_{ij}^2\right)^{\frac{1}{2}}$

So similarly, $\left\|\begin{pmatrix} m \\ n \end{pmatrix}\right\|_2 = \left\|A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right\|_2$ where $\omega = x+iy$

$$\le \|A^{-1}\|_F \, |\omega| \qquad \sqrt{m^2+n^2} \le \|A^{-1}\| \, |\omega|$$

$$\frac{1}{|\omega|} \le \frac{1}{\|A^{-1}\|_F} \frac{1}{\sqrt{m^2+n^2}}$$

$$\frac{1}{|\omega|^\alpha} \le \frac{1}{\|A^{-1}\|_F^\alpha} \frac{1}{(m^2+n^2)^{\alpha/2}}$$

Since $\sum_{(m,n)\neq(0,0)} \frac{1}{(m^2+n^2)^{\alpha/2}}$ converges for $\alpha > 2$,

it follows that $\sum_{\omega \neq 0} \frac{1}{|\omega|^\alpha}$ converges for $\alpha > 2$.

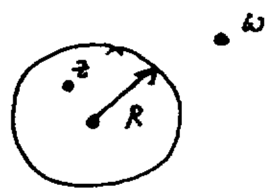
EX: Let $\omega_2/\omega_1 \notin \mathbb{R}$. Show $\Omega(\omega_1, \omega_2)$ can not have arbitrary small elements.

ie $\exists\, \delta = \delta(\omega_1, \omega_2)$: $\omega \in \Omega$ & $\omega \neq 0$

$\Longrightarrow \quad |\omega| > \delta$.

<u>Lemma</u> Suppose $\alpha > 2$ and $R > 0$.
The series $$\sum_{\substack{|\omega| > R \\ \omega \in \Omega}} \frac{1}{(z-\omega)^\alpha}$$
converges absolutely and uniformly in the disk $|z| \leq R$.

<u>Proof:</u>



For $\omega \in \Omega$, $|\omega| > R$, choose ω so that $|\omega| = R + d$ minimal, $d > 0$. (<u>EX</u> Show ω exists)

If $|z| \leq R$, $\omega \in \Omega$ and $|\omega| > R$ then
$$|\omega| \geq R + d,$$
and we have
$$\left|\frac{z-\omega}{\omega}\right| = \left|1 - \frac{z}{\omega}\right| \geq 1 - \left|\frac{z}{\omega}\right| \geq 1 - \frac{R}{R+d}.$$

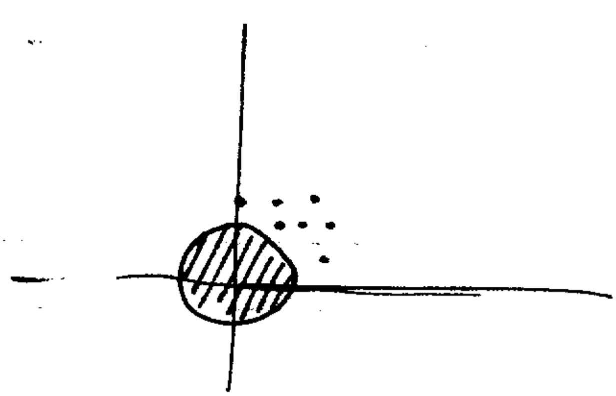
Hence $$\left|\frac{z-\omega}{\omega}\right|^\alpha \geq \left(1 - \frac{R}{R+d}\right)^\alpha = \frac{1}{M},$$

where $M = \left(1 - \frac{R}{R+d}\right)^{-\alpha}$.

So for $|z| \leq R$, $$\frac{1}{|z-\omega|^\alpha} \leq \frac{M}{|\omega|^\alpha}$$

Hence $$\sum_{\substack{|\omega| > R \\ \omega \in \Omega}} \frac{1}{|z-\omega|^\alpha} \leq \sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \frac{M}{|\omega|^\alpha} < \infty.$$

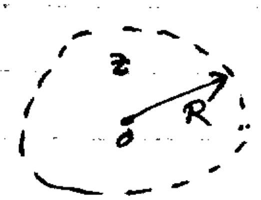
It follows that the convergence is absolute and <u>uniform</u>.



<u>Theorem</u> Suppose $\omega_2/\omega_1 \notin \mathbb{R}$ & $\Omega = \Omega(\omega_1, \omega_2)$.
Define
$$f(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3} \quad \text{for } z \notin \Omega.$$

Then $f(z)$ is a elliptic function with periods ω_1, ω_2 and a pole of order 3 at each period $\omega \in \Omega$.

<u>Proof</u>:
Let $R>0$. Suppose $|z|<R$ and $z \notin \Omega$.



$$f(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$
$$= \underbrace{\sum_{\substack{|\omega| \le R \\ \omega \in \Omega}} \frac{1}{(z-\omega)^3}}_{\text{finite sum}} + \underbrace{\sum_{\substack{|\omega| > R \\ \omega \in \Omega}} \frac{1}{(z-\omega)^3}}_{\substack{\text{analytic function of } z \\ \text{since convergence} \\ \text{is uniform}}}$$

Hence $f(z)$ is meromorphic for $|z|<R$.
But R was arbitrary, $f(z)$ is meromorphic and has a pole of order 3 at each $\omega \in \Omega$.

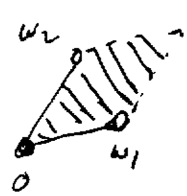
$$f(z+\omega_1) = \sum_{\omega \in \Omega} \frac{1}{(z+\omega_1-\omega)^3}$$
$$= \sum_{\omega \in \Omega} \frac{1}{(z-(\omega-\omega_1))^3}$$
$$= \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3} \quad \left(\begin{array}{l}\text{since } \omega-\omega_1 \in \Omega \\ \text{iff } \omega \in \Omega \text{ \& by} \\ \text{absolute convergence}\end{array}\right)$$
$$= f(z).$$

Similarly, $f(z+\omega_2) = f(z)$ & $f(z)$ is doubly periodic. $\square$

<u>Definition</u> Suppose $\omega_2/\omega_1 \notin \mathbb{R}$ & $\Omega = \Omega(\omega_1, \omega_2)$. The Weierstrass $\wp$ function is defined by

$$\wp(z;\Omega) := \wp(z) := \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}.$$

for $z \notin \Omega$.



<u>Theorem</u> The function $\wp(z)$ has periods ω_1, ω_2. It is analytic except for double poles at each of $\omega \in \Omega$. Moreover, $\wp(z)$ is an even function.

<u>Proof:</u> Let $R > 0$. Suppose $|z| \leq R$, ~~and~~ If $|\omega| > R$, $\omega \in \Omega$ then

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{\omega^2 - (z-\omega)^2}{\omega^2 (z-\omega)^2} \right|$$

$$= \left| \frac{z(2\omega - z)}{\omega^2 (z-\omega)^2} \right|.$$

Choose M in Lemma with $\alpha = 2$. So

$$\frac{1}{|z-\omega|^2} \leq \frac{M}{|\omega|^2} \qquad \text{(Here } M = M(R, \Omega)\text{)}$$

and

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \leq \frac{MR(2|\omega| + R)}{|\omega|^4}$$

$$= \frac{MR\left(2 + \frac{R}{|\omega|}\right)}{|\omega|^3} \leq \frac{3MR}{|\omega|^3}.$$

Hence the series

$$\sum_{\substack{\omega \in \Omega \\ |\omega| > R}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\} \text{ converges absolutely}$$

and uniformly on the disk $|z| \leq R$ and thus is analytic.

~~So~~ The remaining terms give a pole of order 2 at each $\omega \in \Omega$, $|\omega| \leq R$. Since R was arbitrary, $\wp(z)$ is meromorphic with a pole of order 2 at each period.

<u>Claim</u> $\wp(z)$ is even.

~~Suppose~~ Suppose $z \notin \Omega$.

$$\wp(-z) = \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \left\{ \frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2} \right\}$$

$$= \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \left\{ \frac{1}{(z-(-\omega))^2} - \frac{1}{(-\omega)^2} \right\}$$

$$= \frac{1}{z^2} + \sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$$ (since $-\omega$ is a ~~nonzero~~ period iff ω is).

$$= \wp(z).$$

<u>Claim</u> ω_1, ω_2 are periods of $\wp(z)$.

By uniform convergence,

$$\wp'(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

which is elliptic with periods ω_1, ω_2. Let $\omega \in \Omega$.

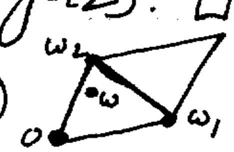
Then $\wp'(z+\omega) = \wp'(z)$ for $z \notin \Omega$.

Hence $\wp(z+\omega) - \wp(z) = c$ (constant).

$(\omega \neq 0)$ $(\omega = \omega_1, \omega_2)$ Let $z = -\omega/2$. $\wp(\omega/2) - \wp(-\omega/2) = 0 = c$ since $\wp(z)$ is even.

Hence the $\omega \in \Omega$ are periods of $\wp(z)$. $\square$ If ω is a period in T the ω is a pole of $\wp$ #.

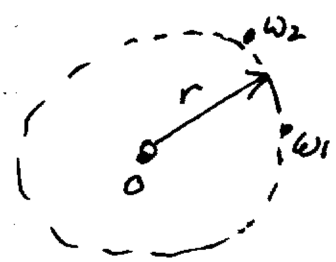
<u>NOTE</u>: Ω = Set of periods of $\wp(z; \Omega)$



<u>The Laurent expansion of $\wp(z)$ for z near $z=0$.</u>

Let $\omega_2/\omega_1 \notin \mathbb{R}$, $\Omega = \Omega(\omega_1, \omega_2)$.

<u>Theorem</u> Let $r = \min_{\substack{\omega \in \Omega \\ \omega \neq 0}} |\omega|$



Then for $0 < |z| < r$,

$$\wp(z) = \wp(z;\Omega)$$
$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}\, z^{2n}$$

where $G_n = G_n(\Omega) = \sum_{\substack{\omega \in \Omega \\ \omega \neq 0}} \frac{1}{\omega^n}$, $(n \geq 3)$.

<u>Proof</u>: Suppose $0 < |z| < r$. Then for $\omega \neq 0$

$$\left|\frac{z}{\omega}\right| = \frac{|z|}{|\omega|} < \frac{r}{r} = 1, \text{ and}$$

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2}\left(\frac{1}{\left(1-\left(\frac{z}{\omega}\right)\right)^2}\right)$$

$$= \frac{1}{\omega^2}\left(1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n\right)$$

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \sum_{n=1}^{\infty} \frac{(n+1)\, z^n}{\omega^{n+2}}.$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$
$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$$
$$= 1 + \sum_{n=1}^{\infty}(n+1)\, z^n$$

Hence $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \left(\sum_{n=1}^{\infty} \frac{(n+1) z^n}{\omega^{n+2}} \right)$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) \Big(\sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \frac{1}{\omega^{n+2}} \Big) z^n \qquad \text{(EX)}$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1)\, G_{n+2}\, z^n$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}\, z^{2n} \quad \text{(since } \wp(z) \text{ is even)}. \ \square$$

<u>DE satisfied by $\wp(z)$</u>

<u>Theorem</u> The function $\wp(z) = \wp(z; \Omega)$ satisfies the non-linear DE

$$[\wp'(z)]^2 = 4\wp^3(z) - 60\, G_4(\Omega)\, \wp(z) - 140\, G_6(\Omega)$$

<u>Proof:</u> Near $z=0$ $\quad 0 < |z| < r = r(\Omega)$.

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots$$

$$\wp'(z) = \frac{-2}{z^3} + 6G_4 z + 20 G_6 z^3 + \cdots$$

$$(\wp'(z))^2 = \frac{4}{z^6} - \frac{24 G_4}{z^2} - 80 G_6 + \cdots$$

$$(\wp(z))^3 = \frac{1}{z^6} + 9G_4 \frac{1}{z^2} + 15 G_6 + \cdots$$

$$4[\wp(z)]^3 = \frac{4}{z^6} + \frac{36 G_4}{z^2} + 60 G_6 + \cdots$$

$$[\wp'(z)]^2 - 4[\wp(z)]^3 = -60\, G_4 \frac{1}{z^2} - 140\, G_6 + \cdots$$

$$[\wp'(z)]^2 - 4[\wp(z)]^3 + 60\, G_4\, \wp(z) = -140\, G_6 + \cdots$$

is elliptic (with periods ω_1, ω_2) and no pole at $z=0$, and hence no poles in a period parallelogram, and so must be constant.

Hence $$[\wp'(z)]^2 = 4\,\wp^3(z) - 60\, G_6\, \wp(z) - 140\, G_6.$$

Let $g_2 = 60\, G_4$, $g_3 = 140\, G_6$. Thus

$$x = \wp(z), \quad y = \wp'(z)$$

provides a parametrization of the elliptic curve

$$y^2 = 4x^3 - g_2 x - g_3.$$

Note: $g_2 = g_2(\Omega)$, $g_3 = g_3(\Omega)$.

__Eisenstein Series__

Let $\omega_2/\omega_1 \notin \mathbb{R}$, $\Omega = \Omega(\omega_1, \omega_2)$. Let $n \geq 3$.

Recall the Eisenstein series of order n

$$G_n = G_n(\Omega) = \sum_{\substack{\omega \neq 0 \\ \omega \in \Omega}} \frac{1}{\omega^n}.$$

Define

$$g_2 = g_2(\Omega) = 60\, G_4, \quad g_3 = g_3(\Omega) = 140\, G_6.$$

__Note__ If n is odd then $G_n = 0$.

<u>Theorem</u> Each Eisenstein series G_n $(n \geq 3)$ can be expressed as a polynomial in g_2, g_3 with positive rational coefficients. In fact, if $b(n) = (2n+1)\, G_{2n+2}$ then $b(1) = g_2/20$, $b(2) = g_3/28$ and

$$(2n+3)(n-2)\, b(n) = 3 \sum_{k=1}^{n-2} b(k)\, b(n-1-k) \quad \text{for } n \geq 3.$$

Equivalently,

$$(2m+1)(m-3)(2m-1)\, G_{2m} = 3 \sum_{r=2}^{m-2} (2r-1)(2m-2r-1)\, G_{2r}\, G_{2m-2r}$$

$b(1) = 3 G_4 = \frac{3}{60} g_2 = \frac{1}{20} g_2$

$b(2) = 5 G_6 = \frac{5}{140} g_3 = \frac{1}{28} g_3$

<u>Proof</u>.

$$(\wp'(z))^2 = 4\wp^3(z) - g_2 \wp(z) - g_3$$

$$2(\wp'(z))\, \wp''(z) = 12\, \wp^2(z)\, \wp'(z) - g_2 \wp'(z)$$

and

$$\wp''(z) = 6\wp^2(z) - \tfrac{1}{2} g_2 .$$

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} b(n) z^{2n}$$

$$= \frac{1}{z^2}\left(1 + \sum_{n=1}^{\infty} b(n) z^{2n+2}\right)$$

$$= \frac{1}{z^2} \sum_{n=0}^{\infty} b(n-1) z^{2n} \quad \left(\begin{array}{l}\text{defining } b(-1) = 1 \\ b(0) = 0\end{array}\right)$$

$$(\wp(z))^2 = \frac{1}{z^4} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b(k-1)\, b(n-k-1)\right) z^{2n}$$

$$= \frac{1}{z^4} + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} b(k-1)\, b(n-k-1)\right) z^{2n-4}.$$

Can take $n=2$ since $n=1 \Rightarrow 0$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{n=1}^{\infty} 2n\, b(n) z^{2n-1}$$

$$\wp''(z) = \frac{6}{z^4} + \sum_{n=1}^{\infty} 2n(2n-1)\, b(n)\, z^{2n-2}$$

$$(\wp(z))^2 = \frac{1}{z^4} + \sum_{n=2}^{\infty}\Big(b(n-1) + \sum_{k=2}^{n} b(k-1)\,b(n-k-1)\Big) z^{2n-4}$$

$$= \frac{1}{z^4} + \sum_{n=1}^{\infty}\Big(b(n) + \sum_{k=2}^{n+1} b(k-1)\,b(n-k)\Big) z^{2n-2}$$

$$= \frac{1}{z^4} + \sum_{n=1}^{\infty}\Big(b(n) + \sum_{k=1}^{n} b(k)\,b(n-k-1)\Big) z^{2n-2}$$

Hence, for $n \geq 3$

$$2n(2n-1)\, b(n) = 6\, b(n) + 6 \sum_{k=1}^{n} b(k)\, b(n-k-1)$$

$$n(2n-1)\, b(n) = 3\, b(n) + 3 \sum_{k=1}^{n} b(k)\, b(n-k-1)$$

$$n(2n-1)\, b(n) = 6\, b(n) + 3 \sum_{k=1}^{n-2} b(k)\, b(n-k-1)$$

$$(2n^2 - n - 6)\, b(n) = 3 \sum_{k=1}^{n-2} b(k)\, b(n-k-1) \qquad \text{(since } b(0)=0,\ b(-1)=1\text{)}$$

$$(2n+3)(n-2)\, b(n) = 3 \sum_{k=1}^{n-2} b(k)\, b(n-k-1),$$

for $n \geq 3$. □

Example: $n=3$ $\quad 9\, b(3) = 3\,(b(1))^2 \quad \Rightarrow 7 G_8 = \frac{1}{3} b(1)^2,\ G_8 = \frac{g_2^2}{7 \cdot 3 \cdot 400}$

<u>The numbers e_1, e_2, e_3</u>

<u>Defn.</u> $\quad e_1 := \wp(\omega_1/2), \quad e_2 := \wp(\omega_2/2), \quad e_3 := \wp\left(\frac{\omega_1+\omega_2}{2}\right)$

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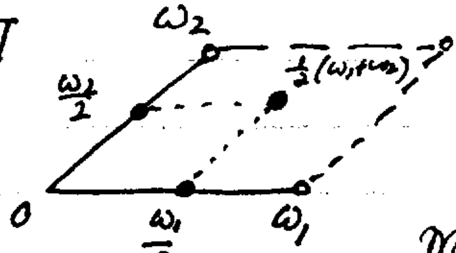


<u>Theorem</u>:

$$4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

Moreover, the roots e_1, e_2, e_3 are distinct and hence $g_2^3 - 27g_3^2 \neq 0$.

<u>Proof</u> Since $\wp(z)$ is even $\wp'(z)$ is odd.

$$\wp'(z) = -2\sum_{\omega\in\Omega} \frac{1}{(z-\omega)^3} \qquad (z \notin \Omega).$$

By periodicity,

$$\wp'(-\tfrac{1}{2}\omega_0) = \wp'(-\tfrac{1}{2}\omega_0 + \omega_0) = \wp'(\tfrac{1}{2}\omega_0)$$

But $\wp'(-\tfrac{1}{2}\omega_0) = -\wp'(\tfrac{1}{2}\omega_0)$.

Hence $\omega_0/2$ is either a zero or a pole of $\wp'(z)$.

The only poles of $\wp'(z)$ in Π occur at 0.

Hence $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ are zeros of $\wp'(z)$.

The only pole of $\wp'(z)$ in Π is a pole of order 3 at $z=0$. Hence each zero $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ is a simple zero and $\wp'(z)$ has no other zeros in Π.

Let

$$F(z) = (\wp'(z))^2 - 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3).$$

Then $F(z)$ is elliptic with periods ω_1, ω_2 and zero for $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ (double zero)

But $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$.

Hence

$$F(z) = c_1\wp^2(z) + c_2\wp(z) + c_3$$

and the pole at $z=0$ is at most order 4. This implies $F(z)$ is constant and hence zero.

Note, $\wp(z)-e_1$ has a <u>double zero</u> at z, i.e. $\wp'()=0$.

Therefore

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3).$$

<u>Claim</u> The e_1, e_2, e_3 are distinct.

The elliptic function

$$\wp(z) - e_1$$

has a zero at $z = \frac{1}{2}\omega_1$ and this is atleast a double zero since $\wp'(z) = 0$ at $z = \frac{1}{2}\omega_1$.

If $e_1 = e_2$ then the function

$$\wp(z) - e_1$$

would have atleast double zeros at $\frac{\omega_1}{2}, \frac{\omega_2}{2}$ which is impossible since the only pole in Π of $\wp(z) - e_1$ is a pole of order 2 at $z = 0$.

Hence $e_1 \neq e_2$ & similarly $e_1 \neq e_3$, $e_2 \neq e_3$.

<u>Ex</u>: The discriminant of the polynomial

$f(x) = 4(x - x_1)(x - x_2)(x - x_3)$ is

$16(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 = a^3 - 27b^2$

if $f(x) = 4x^3 - ax - b$.

Hence $4(e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2 = g_2^3 - 27 g_3^2 \neq 0.$ $\square$

<u>The discriminant Δ</u>

Let $\omega_2/\omega_1 \notin \mathbb{R}$.

$$\text{Let } g_2 = g_2(\Omega) = g_2(\omega_1, \omega_2)$$

$$g_3 = g_3(\Omega) = g_3(\omega_1, \omega_2)$$

$$\Delta(\omega_1, \omega_2) = g_2^3 - 27 g_3^2.$$

Suppose $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Let $n \geq 3$

$$G_n(\Omega) = G_n(\omega_1, \omega_2) = \sum_{\omega \neq 0} \frac{1}{\omega^n}$$

$$G_n(\lambda\omega_1, \lambda\omega_2) = \sum_{\omega \neq 0} \frac{1}{(\lambda\omega)^n} \overset{\omega \in \Omega}{=} \frac{1}{\lambda^n} G_n(\omega_1, \omega_2)$$

Hence $G_n(\omega_1, \omega_2)$ is a homogeneous function of degree $-n$.

g_2 is a homogeneous function of degree -4

g_3 .. .. -6

Therefore

$\Delta(\omega_1, \omega_2)$ is a homogeneous function of degree -12.

$$\Delta(\lambda\omega_1, \lambda\omega_2) = \lambda^{-12}\Delta(\omega_1, \omega_2)$$

Note: Let $\lambda = 1/\omega_1$. Then

$$\Delta(1, \omega_2/\omega_1) = \omega_1^{12}\Delta(\omega_1, \omega_2)$$

Similarly,

$$g_2(1, \omega_2/\omega_1) = \omega_1^{4} g_2(\omega_1, \omega_2)$$

$$g_3(1, \omega_2/\omega_1) = \omega_1^{6} g_3(\omega_1, \omega_2).$$

Let $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

For $\tau \in H$ we define

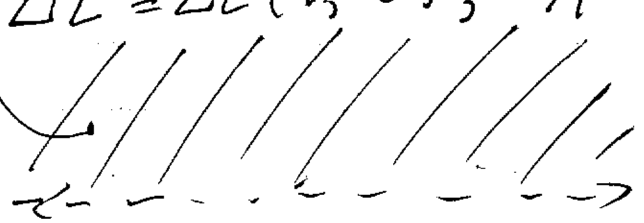
$$g_2(\tau) = g_2(1, \tau) = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4},$$

$$g_3(\tau) = g_3(1, \tau) = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6},$$

$$\Delta(\tau) = \Delta(1, \tau) = g_2^3(\tau) - 27 g_3^2(\tau).$$

NOTE: $\Delta(\tau) \neq 0$ for $\tau \in H$

$\Omega = \Omega(1, \tau)$, $\tau/1 = \tau \notin \mathbb{R}$, $\omega = m\cdot 1 + n\tau$



Klein's modular invariant $J(\tau)$

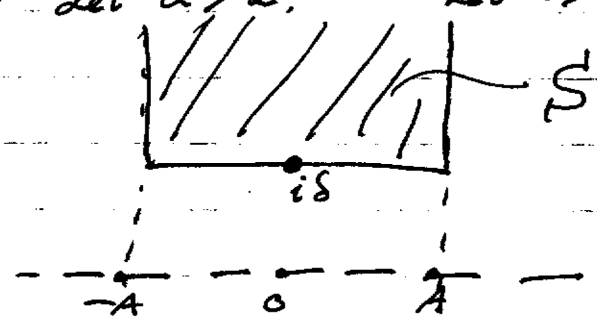
Defn For $\omega_2/\omega_1 \notin \mathbb{R}$, define
$$J(\omega_1,\omega_2) = \frac{g_2^{\,3}(\omega_1,\omega_2)}{\Delta(\omega_1,\omega_2)}$$

$$J(\lambda\omega_1, \lambda\omega_2) = J(\omega_1,\omega_2) \quad \text{for } \lambda \neq 0$$
so $J(1, \frac{\omega_2}{\omega_1}) = J(\omega_1,\omega_2)$

For $\tau \in H$, we define
$$J(\tau) = J(1,\tau).$$

Theorem The functions $g_2(\tau)$, $g_3(\tau)$, $\Delta(\tau)$, $J(\tau)$ are analytic on H.

Proof: Let $\alpha > 2$. Let $\delta > 0$, $A > 0$.



$S = \{x+iy : y \geq \delta, |x| \leq A\}$

Claim The series $\displaystyle\sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^\alpha}$ converges

absolutely and uniformly on any strip S.

It suffices to prove that $\exists K = K(A,\delta) > 0$ s.t.
$$|m+n\tau|^2 > K\,|m+ni|^2 \quad \text{for all } (m,n) \neq (0,0),$$
and $\tau \in H.S(??)$

$$\Leftrightarrow (m+nx)^2 + n^2y^2 > K(m^2+n^2) \qquad (*)$$

$(*)$ holds when $n=0$ for any $0 < K < 1$.

Assume $n \neq 0$, & let $q = m/n$.

$(*) \Leftrightarrow$
$$(**) \qquad \frac{(q+x)^2 + y^2}{1+q^2} > K.$$

Suppose $|x| \leq A$ & $y \geq \delta$.

Case 1. $|g| \leq A + \delta$

Then $(g+x)^2 + y^2 \geq \delta^2$

$1 + g^2 \leq 1 + (A+\delta)^2$

and $\dfrac{(g+x)^2 + y^2}{1+g^2} \geq \dfrac{\delta^2}{1+(A+\delta)^2}$

Case 2 Suppose $|g| > A + \delta$. Then

$$\left|\frac{x}{g}\right| \leq \frac{|x|}{A+\delta} \leq \frac{A}{A+\delta} < 1$$

and $\left|1 + \frac{x}{g}\right| \geq 1 - \left|\frac{x}{g}\right| \geq 1 - \frac{A}{A+\delta} = \frac{\delta}{A+\delta}$

and $|g + x| \geq \dfrac{|g|\,\delta}{A+\delta}$

$$\frac{(g+x)^2 + y^2}{1+g^2} \geq \frac{g^2\delta^2}{(A+\delta)^2} \cdot \frac{1}{1+g^2}$$

$$= \frac{\delta^2}{(A+\delta)^2} \cdot \frac{g^2}{1+g^2}$$

$$> \frac{\delta^2}{(A+\delta)^2} \, \frac{(A+\delta)^2}{1+(A+\delta)^2}$$ (Since $\frac{g^2}{1+g^2} = 1 - \frac{1}{1+g^2}$ is inc.)

$$= \frac{\delta^2}{1+(A+\delta)^2}$$

So $(**)$ holds with $K = \dfrac{\delta^2}{1+(A+\delta)^2}$. ∎

Let $K = \dfrac{\delta^2}{1+(A+\delta)^2}$.

Then

$$|m+n\tau|^2 > K\,|m+ni|^2 \quad \text{for } (m,n) \neq (0,0),\ \tau \in S$$

$$|m+n\tau|^\alpha > K^{\alpha/2}\,|m+ni|^\alpha$$

$$\sum_{(m,n)\neq(0,0)} \frac{1}{|m+n\tau|^\alpha} < \frac{1}{K^{\alpha/2}} \sum_{(m,n)\neq(0,0)} \frac{1}{|m+ni|^\alpha}$$

Hence convergence is absol. & uniform on S & ~~...~~

~~...~~ Hence $F_\alpha(\tau) = \sum' \dfrac{1}{(m+n\tau)^\alpha}$ is analytic for $\alpha \geq 3$ (α integer).

Hence $g_2(\tau)$, $g_3(\tau)$, $\Delta(\tau)$ are analytic on H & $J(\tau)$ is analytic on H since $\Delta(\tau) \neq 0$. $\square$

Invariance of J under unimodular transformation

Let $\omega_2/\omega_1 \notin \mathbb{R}$.

Let $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc = 1 \right\} = SL_2(\mathbb{Z})$.

Each $A \in SL_2(\mathbb{Z})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on pair of periods (ω_1, ω_2) by

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

$$\omega_2' = a\omega_2 + b\omega_1$$
$$\omega_1' = c\omega_2 + d\omega_1$$

$$\Omega(\omega_1, \omega_2) = \Omega(\omega_1', \omega_2').$$

So $\quad g_k(\omega_1, \omega_2) = g_k(\omega_1', \omega_2') \qquad k = 2, 3,$

and so $\quad \Delta(\omega_1, \omega_2) = \Delta(\omega_1', \omega_2')$

& $\quad J(\omega_1, \omega_2) = J(\omega_1', \omega_2')$

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{a\omega_2 + b\omega_1}{c\omega_2 + d\omega_1} = \frac{a\tau + b}{c\tau + d},$$

Möbius or Linear Fractional Transform

where $\tau = \omega_2/\omega_1$.

NOTE: If $\tau \in H$ then $\tau' \in H$

EX

since $\operatorname{Im} \tau' = \dfrac{(ad-bc)}{|c\tau + d|} \operatorname{Im} \tau = \dfrac{\operatorname{Im} \tau}{|c\tau + d|}$.

~~Mark~~ Let Γ be the set of transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

called *unimodular transformations*. Γ forms a group called the unimodular group. Composition of transformations corresponds to matrix multiplication. For $A \in SL_2(\mathbb{Z})$ we write

$$A\tau = \frac{a\tau + b}{c\tau + d}.$$

EX

$$A(B\tau) = (AB)(\tau) \qquad \text{for } A, B \in SL_2(\mathbb{Z}).$$

<u>Theorem</u>: For $A \in \Gamma$,

$$J(A\tau) = J(\tau)$$

for any $\tau \in H$.

<u>Proof</u>:
Let $A \in \Gamma$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau \in H$

Let $\omega_2' = a\tau + b$

$\omega_1' = c\tau + d.$

Then $\Omega(1, \tau) = \Omega(\omega_1', \omega_2').$

$$J(1, \tau) = J(\omega_1', \omega_2')$$
$$= J\left(1, \frac{\omega_2'}{\omega_1'}\right) \quad \text{(Since } J \text{ has degree } 0)$$

But $\dfrac{\omega_2'}{\omega_1'} = \dfrac{a\tau + b}{c\tau + d} = A\tau$, &

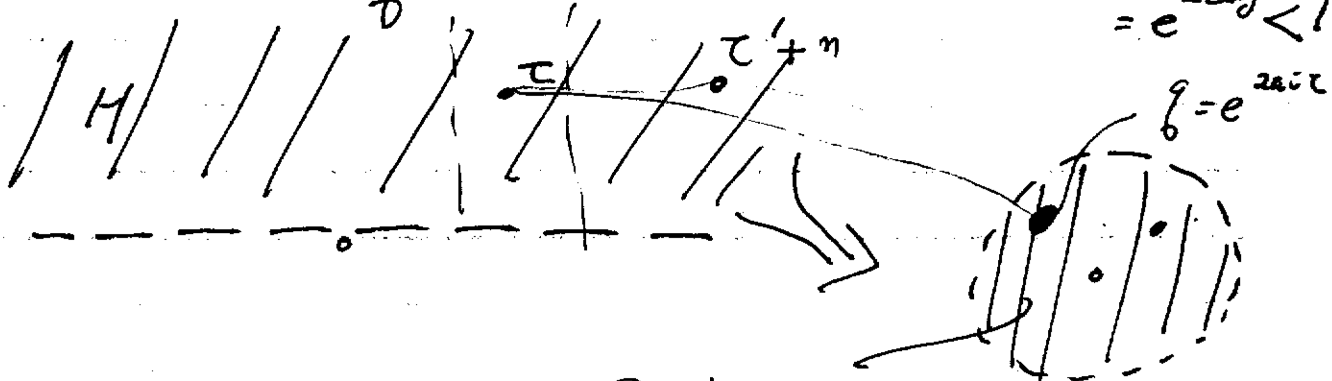
$$J(A\tau) = J(\tau). \quad \square$$

<u>Theorem</u>: For $\tau \in H$, $J(\tau)$ can be represented by absolutely convergent Fourier series

$$J(\tau) = \sum_{n=-\infty}^{\infty} a(n)\, e^{2\pi i n\tau}.$$

<u>Proof</u>: Let $\tau \in H$, then

$q = e^{2\pi i\tau}$ satisfies $|q| = \left|e^{2\pi i(x+iy)}\right| = e^{-2\pi y} < 1.$



$D : \{q : |q| < 1, q \neq 0\}.$

For $q \in D$, define
$$f(q) = J(\tau) \quad \text{where } q = e^{2\pi i \tau}$$
is well-defined since
$$q = e^{2\pi i \tau} = e^{2\pi i \tau'} \Rightarrow \tau - \tau' \in \mathbb{Z}$$
$$\tau = \tau' + n \qquad (n \in \mathbb{Z})$$
$$\& \; J(\tau) = J(\tau' + n) = J(\tau') \quad \text{since } \tau' \mapsto \tau' + n \Leftrightarrow \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma.$$
f is analytic on D since
$$J(\tau) = f(e^{2\pi i \tau})$$
$$J'(\tau) = 2\pi i \, e^{2\pi i \tau} f'(e^{2\pi i \tau})$$
$$f'(e^{2\pi i \tau}) = \frac{J'(\tau)}{2\pi i \, e^{2\pi i \tau}}$$

Hence $f(q)$ must have a Laurent series expansion valid for $0 < |q| \leq 1$.
So
$$f(q) = \sum_{-\infty}^{\infty} a(n) q^n$$
absolutely convergent for $q \in D$. Hence
$$J(\tau) = \sum_{-\infty}^{\infty} a(n) e^{2\pi i n \tau}$$
absolutely convergent for $\tau \in H$.

NOTE:

$q \; q$
$: -2\pi y + 2\pi i x$
$\tau = x + iy$
$: \frac{1}{2\pi i} \log q$

Suppose x_1, x_2 are any fixed real nos with $x_2 - x_1 < 1$.
$$q = e^{2\pi i \tau} = e^{2\pi i (x+iy)} = e^{-2\pi y} e^{2\pi i x}$$
$$\log q = 2\pi i \tau, \quad \tau = \frac{1}{2\pi i} \log q$$
$$\mathrm{Log}_\alpha z = \ln|z| + i \arg z \qquad \alpha < \arg z < \alpha + 2\pi$$
$$f(q) = J\left(\frac{1}{2\pi i} \mathrm{Log}_\alpha q\right)$$
(another reason why f is analytic on D)

<u>Fourier expansions of $g_2(\tau)$ & $g_3(\tau)$</u>

<u>Lemma</u>: For $\tau \in H$, $n > 0$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^4} = \frac{8\pi^4}{3} \sum_{r=1}^{\infty} r^3 e^{2\pi i r n \tau}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^6} = -\frac{8\pi^6}{15} \sum_{r=1}^{\infty} r^5 e^{2\pi i r n \tau}$$

<u>Proof</u>: We need the partial fraction decomposition of cot:

$$\pi \cot \pi\tau = \frac{1}{\tau} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{1}{\tau+m} - \frac{1}{m} \right)$$

Let $q = e^{2\pi i \tau}$ where $\tau \in H$ so that $|q| < 1$.

$$\pi \cot \pi\tau = \pi \frac{\cos \pi\tau}{\sin \pi\tau}$$

$$= \pi i \frac{e^{\pi i \tau} + e^{-\pi i \tau}}{e^{\pi i \tau} - e^{-\pi i \tau}} = \pi i \frac{e^{2\pi i \tau} + 1}{e^{2\pi i \tau} - 1}$$

$$= \pi i \left(\frac{q+1}{q-1} \right) = -\pi i \left(\frac{(1-q) + 2q}{1-q} \right)$$

$$= -\pi i \left(\frac{1}{1-q} + \frac{q}{1-q} \right)$$

$$= -\pi i \left((1+q) \sum_{r=0}^{\infty} q^r \right) = -\pi i \left(1 + 2 \sum_{r=1}^{\infty} q^r \right)$$

$$\frac{1}{\tau} + \sum_{m\neq 0}\left(\frac{1}{\tau+m} - \frac{1}{m}\right) = -\pi i\left(1 + 2\sum_{r=1}^{\infty} e^{2\pi i \tau r}\right)$$

Diff'ing we find

$$-\frac{1}{\tau^2} - \sum_{m\neq 0}\frac{1}{(\tau+m)^2} = (-\pi i)\,2\sum (2\pi i r)\, e^{2\pi i \tau r}$$

$$= -(2\pi i)^2 \sum_{r=1}^{\infty} r\, e^{2\pi i \tau r}$$

$$(-)(-2)(-3)\sum_{m}\frac{1}{(\tau+m)^4} = -(2\pi i)^4\sum_{r=1}^{\infty} r^3 e^{2\pi i \tau r}$$

$$\sum_{m}\frac{1}{(\tau+m)^4} = \frac{8\pi^4}{3}\sum_{r=1}^{\infty} r^3 e^{2\pi i \tau r}$$

$$(-4)(-5)\sum_{m}\frac{1}{(\tau+m)^6} = \frac{8\pi^4}{3}(2\pi i)^2\sum r^5 e^{2\pi i \tau r}$$

$$\sum_{m}\frac{1}{(\tau+m)^6} = -\frac{8\pi^6}{15}\sum_{r=1}^{\infty} r^5 e^{2\pi i \tau r}$$

Result follows by replacing τ by $n\tau$. $\square$

<u>Theorem</u> For $\tau \in H$,

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty}\sigma_3(k)\, q^k\right\}$$

$$g_3(\tau) = \frac{8\pi^6}{27}\left\{1 - 504\sum_{k=1}^{\infty}\sigma_5(k)\, q^k\right\}$$

where $q = e^{2\pi i \tau}$, $\sigma_\alpha(k) = \sum_{\substack{d\mid k\\ d>0}} d^\alpha$.

Proof:

$$g_2(\tau) = 60 \sum{}' \frac{1}{(m+n\tau)^4} \qquad L = L(1,\tau)$$

$$= 60\left\{ \sum_{m\neq 0} \frac{1}{m^4} + \sum_{n=1}^{\infty}\left(\sum_{m=-\infty}^{\infty} \frac{1}{(m+n\tau)^4} + \frac{1}{(m-n\tau)^4} \right)\right.$$

$$= 60\left\{ 2\zeta(4) + 2\sum_{n=1}^{\infty}\sum_{-\infty}^{\infty} \frac{1}{(m+n\tau)^4} \right\}$$

$$= 60\left\{ 2\frac{\pi^4}{90} + \frac{2\cdot 8\pi^4}{3}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} r^3 q^{nr} \right\} \qquad (q = e^{2\pi i\tau})$$

$$= 60\left\{ \frac{2\pi^4}{90} + \frac{16\pi^4}{3}\sum_{k=1}^{\infty}\left(\sum_{r\mid k} r^3\right) q^k \right\}$$

$$= \frac{120}{90}\pi^4\left\{ 1 + \frac{16}{3}\cdot\frac{90}{2}\sum_{k=1}^{\infty} \sigma_3(k)\, q^k \right\}$$

$$= \frac{4\pi^4}{3}\left\{ 1 + 240\sum_{k=1}^{\infty} \sigma_3(k)\, q^k \right\}. \qquad \square$$

Fourier expansions of $\Delta(\tau)$ & $J(\tau)$

Theorem: If $\tau \in H$,

$$\Delta(\tau) = (2\pi)^{12}\sum_{n=1}^{\infty} \tau(n)\, q^n \qquad (q = e^{2\pi i\tau})$$

where $\tau(n)$ are integers, $\tau(1) = 1$, $\tau(2) = -24$.

Note: $\tau(n)$ is called Ramanujan's tau-function.

Proof: Let $q = e^{2\pi i\tau}$, $A = \sum_{n=1}^{\infty} \sigma_3(n) q^n$,

$$B = \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

Then

$$\Delta(\tau) = g_2^3(\tau) - 27 g_3^2(\tau)$$

$$= \frac{2^6 \pi^{12}}{3^3}(1 + 240A)^3$$

$$- 27 \frac{2^6 \pi^{12}}{3^6}(1 - 504B)^2$$

(*) $$= \frac{64 \pi^{12}}{27}\left((1+240A)^3 - (1-504B)^2\right)$$

$$(1+240A)^3 - (1-504B)^2$$

$$= 1 + 720A + 3(240)^2A^2 + (240)^3A^3$$

$$-(1 - 1008B + (504)^2B^2)$$

$$= 12^2(5A + 7B) + 12^3(100A^2 + 8000A^3 - 147B)$$

$4 = 2^3 \cdot 3^2 \cdot 7$

$^2 = 2^6 \cdot 3^4 \cdot 7^2$

$= (2^2 \cdot 3)^3 \cdot 7^2 \cdot 3$

$$5A + 7B = \sum_{n=1}^{\infty} (5\sigma_3(n) + 7\sigma_5(n))\, q^n$$

$d \equiv 0, \pm 1\ (3)$

$$5d^3 + 7d^5 = d^3(5 + 7d^2)$$

$$\equiv \begin{cases} d^3(d^2-1) \equiv 0 & \bmod 3 \\ d^3(1-d^2) \equiv 0 & \bmod 4 \end{cases}$$

$$\equiv 0 \pmod{12}.$$

Hence,

$$\Delta(\tau) = \frac{64\pi^{12}}{27}\left\{ 12^3 \sum_{n=1}^{\infty} \tau(n) q^n \right\}$$

where $\tau(n)$ are integers

$$= (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) q^n.$$

$\square$

In Ch3 we will show

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

where $q = e^{2\pi i z}$.

Ramanujan proved or conjectured many properties of $\tau(n)$.

(1) $\tau(n)$ is multiplicative; i.e.

$$\tau(mn) = \tau(m)\,\tau(n)$$

if m, n are relatively prime.

(2) $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$

(3) $\tau(n) = O(n^6)$ (i.e. $\frac{\tau(n)}{n^6}$ is bounded).

(4) Ramanujan conjectured that

$$|\tau(n)| \leqslant n^{11/2}\, \sigma_0(n) \quad \text{for } n \geqslant 1.$$

(Proved by Deligne (1973) as a consequence of the Weil conjectures for algebraic varieties over finite fields).

(5) <u>Lehmer's Conjecture</u> $\tau(n) \neq 0$ for all $n \geqslant 1$.

<u>Theorem</u>: For $\tau \in H$,

$$12^3 J(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

$$(q = e^{2\pi i \tau})$$

where $c(n)$ are integers.

$$c(1) = 196884$$

<u>Proof</u>:

$$g_2(\tau) = \frac{4\pi^4}{3}(1 + 240q + \cdots)$$

$$g_3(\tau) = \frac{8\pi^6}{27}(1 - 504q + \cdots)$$

and each series in ( ) has integer coefficients.

$$\Delta(\tau) = \frac{64\pi^{12}}{27}\left(12^3 \sum \tau(n) q^n\right)$$

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} = \frac{\frac{2^6\pi^{12}}{27}(1 + 240q + \cdots)^3}{\frac{64\pi^{12}}{\cancel{27}} 12^3 q (1 - 24q + \cdots)}$$

$$= \frac{1}{12^3} \frac{(1 + 240q + \cdots)^3}{q(1 - 24q + \cdots)}$$

and $12^3 J(\tau) = \frac{1}{q} \frac{(1 + 240q + \cdots)^3}{(1 - 24q + \cdots)}$

has integer co-effs.

$$12^3 J(\tau) = \frac{1}{q}(1 + 720q + \cdots)(1 + 24q + \cdots)$$

$$= \frac{1}{q}(1 + 744q + \cdots)$$

$$= \frac{1}{q} + 744 + \cdots \qquad \Box$$