Chapter 2 The Modular Group & Modular Functions

The Modular Group Γ

is the set of all Möbius transformations

$$\tau' = \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{Z}$, $ad-bc=1$.

This group is identified with $SL_2(\mathbb{Z}) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : \det(A)=1\}$ provide we identify A with $-A$ since it reps. the same Möbius transf. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write

$$A\tau = \frac{a\tau+b}{c\tau+d}.$$

Theorem The modular group Γ is generated by two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T\tau = \tau+1$$

& $$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S\tau = -1/\tau. \qquad (S^2 = -I)$$

That is every ~~any~~ $A \in \Gamma$ can be expressed

$$A = T^{n_1} S T^{n_2} \cdots S T^{n_k}$$

where n_i are integers. This rep. is not unique.

Proof: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we may assume $c \geq 0$. We use induction on c.

If $c=0$, $ad = 1$ so $a=d=\pm 1$,

$$A = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} = T^{\pm b}.$$

The result is true for $c=0$.

If $c=1$, $ad-b=1$, $b = ad-1$

$$A = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$\frac{a\tau + ad-1}{\tau + d} = \frac{a(\tau+d)-1}{(\tau+d)+0}$$

$$= \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$A = T^a S T^d$ $\underline{\underline{ie}}$ $\dfrac{az + ad - 1}{z + d} = \dfrac{a(z+d) - 1}{1(z+d) + 0}$

$$= \frac{-1}{(z+d)} + a$$

Now assume the result holds for all $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $0 < c' < c$. $c \geqslant 2$.

Since $ad - bc = 1$, $(c, d) = 1$.

$d = cq + r$ when $0 < r < c$.

$$AT^{-q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -aq + b \\ c & -cq + d \end{pmatrix}$$

$$= \begin{pmatrix} a & -aq + b \\ c & r \end{pmatrix}$$

$$AT^{-q}S = \begin{pmatrix} a & -aq + b \\ c & r \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} & \\ r & \end{pmatrix}$$

Since $0 < r < c$ This matrix is ~~generated~~ can be written as a product of the desired form ~~and~~ (by induction hypoth) & Hence A can. $\square$

Fundamental Regions

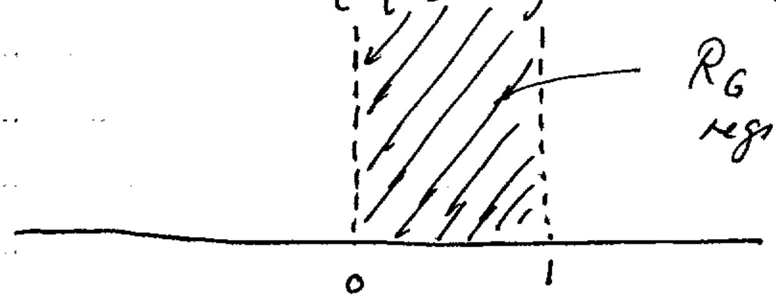
Let $G < \Gamma$. Two points τ and $\tau' \in H$ are said to be <u>equivalent</u> under G if

$$\tau' = A\tau \quad \text{for some } A \in G.$$

This is equivalence rel. since G is a group.
This equiv. rel. divides the upper half plane H into disjoint equivalence classes called <u>orbits</u>.
Let $\tau \in H$.
The orbit $\quad G\tau = \{ A\tau : A \in G \}$.

<u>Definition</u>: Let $G < \Gamma$. An open subset $R_G \subset H$ is called a <u>fundamental region</u> if
(a) No two distinct points of R_G are equivalent under G.
(b) If $\tau \in H$ there is a point τ' in $\overline{R_G}$ such that $\tau' \sim \tau$ under G.

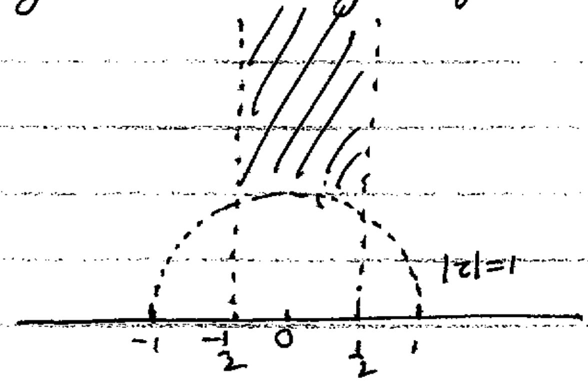
<u>Example</u>
Let $\quad G = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$; $\tau' = \tau + n$ (translations)



R_G is a fundamental region of G.

<u>Theorem</u>

$R_\Gamma = \{\tau \in H : |\tau| > 1 \text{ and } |\tau + \bar{\tau}| < 1\}$
is a fundamental region for the modular group Γ.



<u>Lemma</u> Let $\omega_2'/\omega_1' \notin \mathbb{R}$,

$$\Omega = \{m\omega_1' + n\omega_2' : m, n \in \mathbb{Z}\}.$$

Then $\exists$ a fundamental pair $(\omega_1, \omega_2) \sim (\omega_1', \omega_2')$
such that
$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}, \quad ad - bc = 1,$$

and $|\omega_2| \geq |\omega_1|$, $|\omega_1 + \omega_2| \geq |\omega_2|$, $|\omega_1 - \omega_2| \geq |\omega_2|$.

<u>Proof:</u>
We arrange the elements of Ω in a sequence according to increasing distance from the origin

$$\Omega = \{0, W_1, W_2, \dots\}$$

where $0 < |W_1| \leq |W_2| \leq \cdots$

and $\arg W_n < \arg W_{n+1}$ if $|W_n| = |W_{n+1}|$.

Let $\omega_1 = W_1$. Let ω_2 be the first member of Ω that is not a multiple of ω_1.

Hence the triangle $0, \omega_1, \omega_2$ contains no elt. of Ω except the vertices. So (ω_1, ω_2) is a fundamental pair which also spans Ω.

Hence there are integers $a, b, c, d \in \mathbb{Z}$, $ad-bc = \pm 1$ s.th

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}.$$

If $ad-bc=-1$, replace c by $-c$, d by $-d$ and ω_1 by $-\omega_1$. Hence same eqn holds but $ad-bc=1$.

Therefore $|\omega_2| \geq |\omega_1|$ by construction.

$\omega_1 \pm \omega_2$ is not a multiple of ω_1, and so it occurs later in seq. &

$$|\omega_1 \pm \omega_2| \geq |\omega_2|$$

by construction. □

<u>Theorem</u> If $\tau' \in H$ there exists a complex number $\tau \in H$ equivalent under Γ such that

$$|\tau| \geq 1, \quad |\tau+1| \geq |\tau| \quad \text{and} \quad |\tau-1| \geq |\tau|.$$

Proof: Let $\omega_1' = 1$, $\omega_2' = \tau'$.

$$\Omega = \{ m + n\tau' : m, n \in \mathbb{Z}\}.$$

Then $\exists$ a fundamental pair (ω_1, ω_2) with

$$|\omega_2| \geq |\omega_1|, \quad |\omega_1 \pm \omega_2| \geq |\omega_2|.$$

Let $\tau = \frac{\omega_2}{\omega_1}$. Then $\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} =$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$$

So $\tau = \frac{\omega_2}{\omega_1} = \frac{a\tau'+b}{c\tau'+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau'$, with $ad-bc=1$.

So $|\tau| = \left|\frac{\omega_2}{\omega_1}\right| \geq 1$, $\left|1 \pm \frac{\omega_2}{\omega_1}\right| \geq \left|\frac{\omega_2}{\omega_1}\right|$, $|1 \pm \tau| \geq |\tau|$.

NOTE: Let $\tau \in H$. $|\tau \pm 1| \geq |\tau| \iff |\tau + \bar{\tau}| \leq 1$.

<u>Proof</u>: ($\Leftarrow$) Suppose $|\tau \pm 1|^2 \geq |\tau|^2 \Rightarrow (\tau \pm 1)(\bar{\tau} \pm 1) \geq \tau\bar{\tau}$
$\Rightarrow \tau\bar{\tau} \pm (\tau+\bar{\tau}) + 1 \geq \tau\bar{\tau} \Rightarrow \pm(\tau+\bar{\tau}) \geq -1$
$\Rightarrow \mp(\tau+\bar{\tau}) \leq 1 \Rightarrow |\tau+\bar{\tau}| \leq 1.$ □

<u>Theorem</u>

The open set

$$R_\Gamma = \{z \in H : |z| > 1,\ |z+\bar{z}| < 1\}$$

is a fundamental region for Γ. Moreover if $A \in \Gamma$ and if $Az = z$ for some $z \in R_\Gamma$ then $A = I$.

<u>Proof:</u> Previous thm shows that if $z' \in H$ $\exists z \in \overline{R_\Gamma}$ such that $z' \sim z$ under Γ.

<u>wts</u> no two distinct pts of R_Γ are equivalent under Γ. Suppose $z' \sim z$. ~~[illegible]~~

$$z' = Az \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\operatorname{Im} z' = \frac{\operatorname{Im} z}{|cz+d|^2}$$

~~Suppose~~ <u>Claim</u> $z \in R_\Gamma$ & $c \neq 0 \Rightarrow \operatorname{Im} z' < \operatorname{Im} z$.

Suppose $z \in R_\Gamma$ & $c \neq 0$. Then

$$|cz+d|^2 = (cz+d)(c\bar{z}+d)$$
$$= c^2 z\bar{z} + cd(z+\bar{z}) + d^2$$
$$\cancel{[illegible]} > c^2 - |c||d| + d^2$$

If $d = 0$, then $|cz+d|^2 \geq c^2 \geq 1$ & $\operatorname{Im} z' < \operatorname{Im} z$.

If $d \neq 0$,

$$|cz+d|^2 > |c|^2 - |c||d| + |d|^2$$
$$= (|c| - |d|)^2 + |cd| \geq |cd| \geq 1$$

& $\operatorname{Im} z' < \operatorname{Im} z$. This proves the claim. □

Hence every elt. of A of Γ with $c \neq 0$ decreases the imag. part of each pt z of R_Γ.

Suppose $z' \sim z$ & $z, z' \in R_\Gamma$.

$$z' = Az, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$z = A^{-1} z', \quad A^{-1} = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}$$

If $c \neq 0$, it follows that

$$\operatorname{Im} \tau' < \operatorname{Im} \tau \ \& \ \operatorname{Im} \tau < \operatorname{Im} \tau'$$

which is impossible. Therefore,

$c = 0$, $\quad ad = 1$, $\quad a = d = \pm 1$, and

$$A = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = T^{\pm b}$$

$$\tau - \tau' = \pm b \implies b = 0 \ \& \ \tau = \tau'!$$

Hence no two distinct pts of R_Γ are equivalent under Γ.

Finally, suppose

$$A\tau \equiv \tau$$

for some τ in R_Γ. Let $\tau' = \tau$. By the argument above $c = 0$, $b = 0$ & $A = I$. So only the identity has fixed points in R_Γ. $\square$

Modular Functions

Definition $f : H \to \mathbb{C}$ is a modular function if

(a) f is meromorphic on H,

(b) $f(A\tau) = f(\tau)$ for every $A \in \Gamma$, $\tau \in H$.

(c) f has a Fourier expansion

$$f(\tau) = \sum_{n=-m}^{\infty} a(n)\, e^{2\pi i n \tau} = \sum_{n=-m}^{\infty} a(n)\, q^n$$

Note (1) $q = e^{2\pi i \tau} = 0$ if $\tau = i\infty$

(2) if $m > 0$ & $a(-m) \neq 0$ we say f has a pole of order m at $i\infty$.

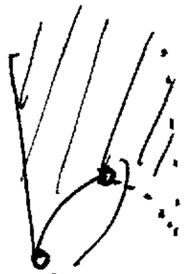
(3) if $m \leq 0$, f is analytic at $i\infty$.

EXAMPLE $J(\tau)$ is a modular function & has a pole of order 1 at $i\infty$.

For $P \in H$ let $v_P(f)$ = order of zero at P of f (or − order of pole)

Let $v_{i\infty}(f)$ = power of q of first non-vanishing term in q-exp of f.

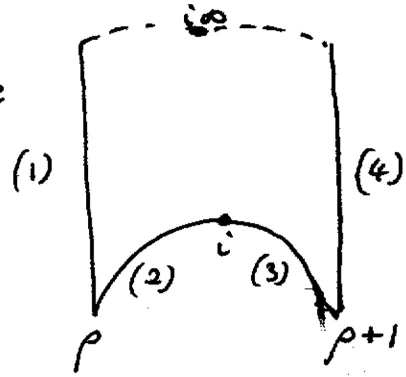
Then $v_{i\infty}(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{p \in \tilde{R}} v_p(f) = 0$



Theorem Suppose f is a modular function that is not identically zero. Then in $\overline{R_\Gamma}$ the number of zeros of f is equal to the number of ~~po~~ poles of f, counted as follows:

$\tilde{R}$ = ~~[illegible]~~ $= \Gamma \backslash H$ (set of Γ-equivalence ... H)

In the interior R_Γ zeros and poles are counted as usual.



$\frac{-1}{i} = i \qquad \frac{-1}{\rho} = \rho + 1$

Any zero or pole on (1), (2) corresponds to a zero/pole on (3), (4). Only zero/poles on (1), (2) are counted.

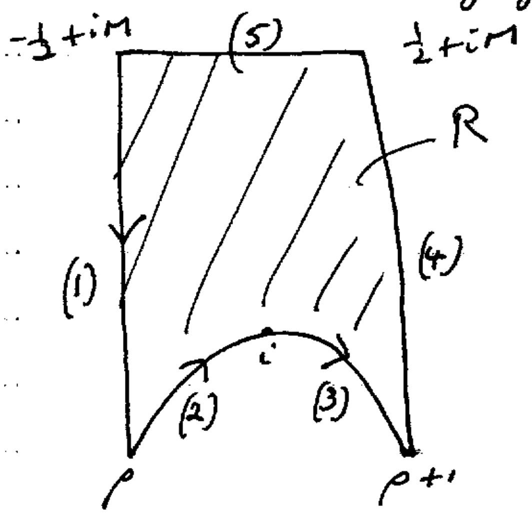
$= e^{i\theta}$

$= e^{(x-\theta)i}$

The order of zero/pole at ρ is divided by 3. The order at i is divided by 2. The order at $i\infty$ is counted.

Proof:

Case 1 f has no zero or poles on the finite part of the boundary of R_Γ.



Choose M large enough so region includes all zeros and poles of f, except at $i\infty$.

NOTE: Let $\tau = x + iy$

$|q| = |e^{2\pi i \tau}| = e^{-2\pi y}$

neighborhood of $i\infty$ are sets of the form $\{x+iy : y > c\}$.

So if $f(\tau) = \sum a(n) q^n, 0 < |q| < \delta$

Then f is holo for Im $\tau > \left(-\frac{1}{2\pi} \ln \delta\right)$.

Let ∂R be the boundary of R. Let N, P denote the # of zeros & poles of f in R (counted with multiplicity)

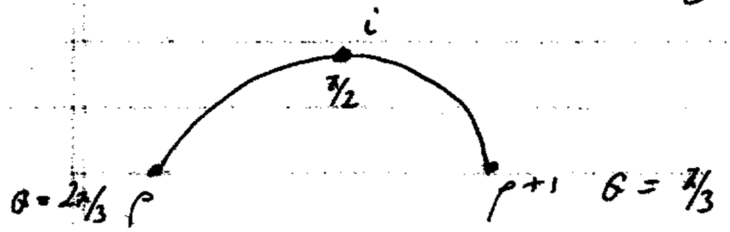
Then

$$N - P = \frac{1}{2\pi i} \int_{\partial R} \frac{f'(z)}{f(z)}\, dz$$

$$= \frac{1}{2\pi i} \left\{ \int_{(1)} + \int_{(2)} + \int_{(3)} + \int_{(4)} + \int_{(5)} \right\}$$

$\int_{(1)} + \int_{(4)} = 0$ since $f(z+1) = f(z)$ & $f'(z+1) = f'(z)$.

Let $\tau = e^{i\theta}$. Then $-\frac{1}{\tau} = -e^{-i\theta} = e^{i(\pi-\theta)}$



Let $S(z) = -1/z$. $\quad f(S(z)) = f(z)$

$$f'(z) = f'(S(z))\, S'(z)$$

$$\int_{(2)} \frac{f'(z)}{f(z)}\, dz = \int_{2\pi/3}^{\pi/2} \frac{f'(\tau(\theta))}{f(\tau(\theta))}\, \tau'(\theta)\, d\theta$$

$$= \int_{2\pi/3}^{\pi/2} \frac{f'(S(\tau(\theta)))\, S'(\tau(\theta))\, \tau'(\theta)\, d\theta}{f(S\,\tau(\theta))}$$

$$= \int_{2\pi/3}^{\cancel{\pi/2}}$$

If $\tau(\theta)$ parametrizes (2), $S(\tau(\theta))$ parametrizes $-(3)$

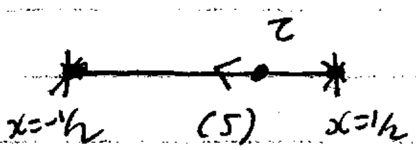
so $\int_{(2)} = -\int_{(3)}$ & $\int_{(2)} + \int_{(3)} = 0$.

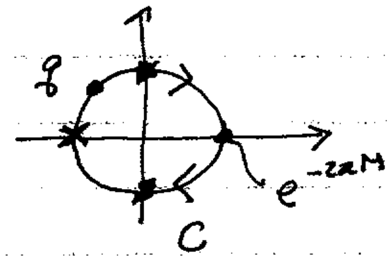
Hence

$$N - P = \frac{1}{2\pi i} \int_{(S)} \frac{f'(\tau)}{f(\tau)}\, d\tau$$

Let $q = e^{2\pi i \tau}$ at $\tau = x + iM$

$q = e^{2\pi i x}\, e^{-2\pi M}$





Let $F(q) = \sum a(n)\, q^n$

so that $F(e^{2\pi i \tau}) = f(\tau)$

Let $q = q(\tau) = e^{2\pi i \tau}$

$$f'(\tau) = F'(q)\, \frac{dq}{d\tau}$$

$$f'(\tau)\, d\tau = F'(q)\, dq$$

$$\frac{f'(\tau)\, d\tau}{f(\tau)} = \frac{F'(q)}{F(q)}\, dq$$

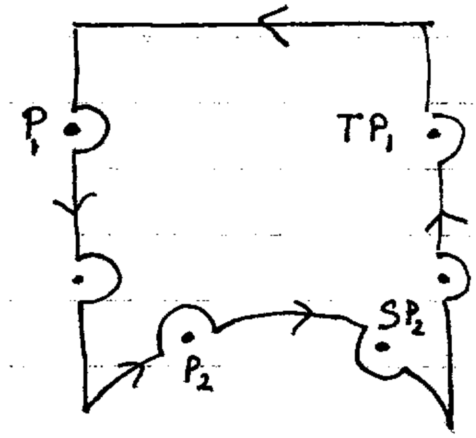
So $\displaystyle \frac{1}{2\pi i} \int_{(S)} \frac{f'(\tau)}{f(\tau)}\, d\tau = \frac{1}{2\pi i} \int_{C} \frac{F'(q)}{F(q)}\, dq$

$= \cancel{R_{i\infty}} = - v_{i\infty}(f)$

The only possible zero or pole of $F(q)$ is at $q = 0$.
The order of this zero or pole corresponds to the order ~~pole~~ of $f(\tau)$ at $i\infty$.

Case 2 f has a zero or pole on the boundary of R_Γ but not at vertex $(\rho, i, \rho+1)$.

We deform path :

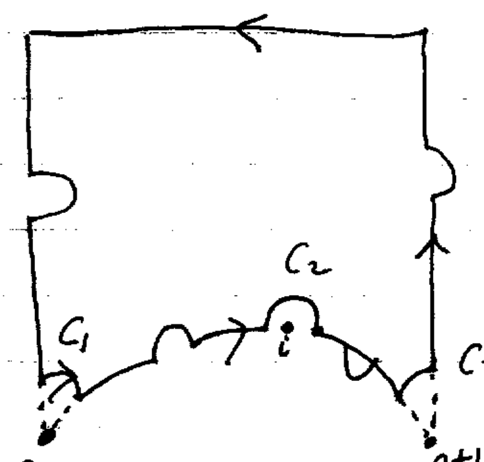


EX For $A \in \Gamma$

$v_{P_1}(A) = v_{AP}(f) = v_P(f)$

Integrals along equivalent edges cancel as before. Every pole/zero on boundary is counted correctly and result follows as before.

Case 3 f has a zero, pole at ρ or i.
We modify path similarly.



Note $\int_{C1}$ does not cancel $\int_{C3}$ etc

near the vertex ρ we write

$f(z) = (z-\rho)^k g(z)$, $g(\rho) \neq 0$

$f'(z) = k(z-\rho)^{k-1} g(z) + (z-\rho)^k g'(z)$

$$\frac{f'(z)}{f(z)} = \frac{k}{(z-\rho)} + \frac{g'(z)}{g(z)}$$

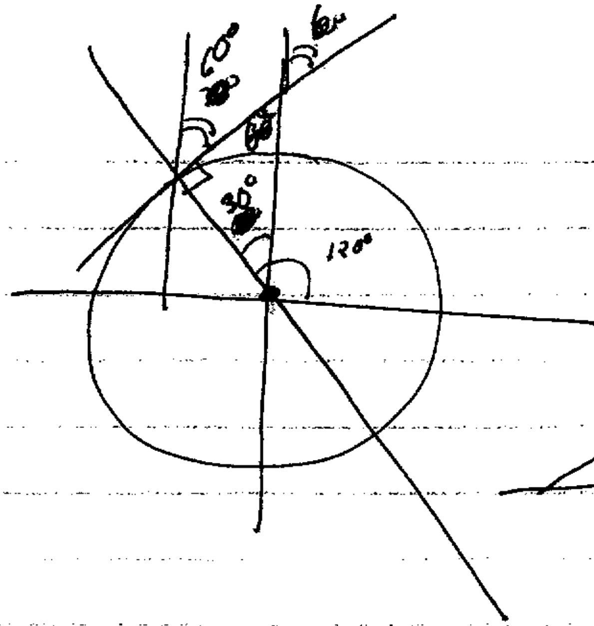
for z near ρ. NOTE $k = v_\rho(f)$

On $C_{1,r}$ (circular arc) let radius $= r$.

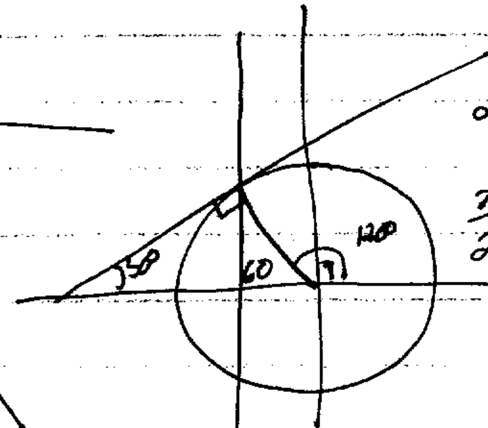
So C_1 : $z = \rho + re^{i\theta}$, $\alpha(r) = \alpha \leq \theta \leq \pi/2$

$$\frac{1}{2\pi i} \int_{C_{1,r}} \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_{C_{1,r}} \frac{k}{z-\rho}\, dz + \frac{1}{2\pi i} \int_{C_{1,r}} \frac{g'(z)}{g(z)}\, dz$$

$$dz = rie^{i\theta}\, d\theta, \quad \frac{1}{2\pi i} \int_{C_{1,r}} \frac{k}{(z-\rho)}\, dz = \frac{1}{2\pi i} \int_{\pi/2}^{\alpha} \frac{kire^{i\theta}}{re^{i\theta}}\, d\theta = \frac{k}{2\pi}\left(\alpha - \pi/2\right)$$
$$= -\frac{k}{2\pi}\left(\pi/2 - \alpha\right)$$



$\pi/2 - \alpha_0 = 60° = 2\pi/6$



$\alpha_0 = 30° = \frac{\pi}{6}$

$\frac{\pi}{2} - \alpha_0 = \frac{3\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

So $$\lim_{r\to 0} \frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)}\, dz = -\frac{k}{6}$$

and similarly
$$\lim_{r\to 0} \frac{1}{2\pi i} \int_{C_3} \frac{f'(z)}{f(z)}\, dz = -\frac{k}{6}$$
$$\lim_{r\to 0} \frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_3}\right) = -\frac{k}{3} = -\frac{1}{3}\, v_\rho(f)$$

Near i, suppose
$$f(z) = (z-i)^{l}\, h(z), \qquad h(i) \neq 0.$$
We find
$$\lim_{r\to 0} \frac{1}{2\pi i}\left(\int_2 \frac{f'(z)}{f(z)}\right) = -\frac{l}{2} = -\frac{1}{2}\, v_i(f)$$

$\frac{l}{2\pi}(-\pi)$

Hence
$$\sum_{p\in \tilde{R}} v_p(f) = -v_{i\infty}(f) - \frac{1}{2} v_i(f) - \frac{1}{3} v_\rho(f)$$

& $$v_{i\infty}(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\omega(f) + \sum_{p\in \tilde{R}} v_p(f) = 0. \quad \square$$

$(\omega = \rho)$.

<u>Theorem</u> If f is a nonconstant modular function then for every complex constant c, $f - c$ has the same number of zeros as poles in $\overline{R_\Gamma}$ (under the conventions of the valence Thm). In other words, f takes on every value equally often in the closure of R_Γ.

<u>Proof:</u> If f is a modular function then so is $f - c$.

Let d be a value of f, i.e. $d = f(\tau_0)$ for some $\tau_0 \in H$. Then # of solutions of

$$f(\tau) - d = 0$$

is equal to the # of zeros of $f - d$

$= \#$ of poles of $f - d$

$= \#$ of poles of f (constant & does not depend on d). ∎

~~Let α be any complex~~

~~Let α be any complex no.~~

~~# of poles of f = # of poles of f~~

Since τ_0 is a zero $f(\tau) - d = f(\tau) - f(\tau_0)$

# of poles of $f \geq 1$.

Let α be any complex no. Then

$$1 \leq \# \text{ of poles of } f = \# \text{ of poles of } (f - \alpha)$$
$$= \# \text{ of zeros of } (f - \alpha)$$

Hence f must assume the value α in $\overline{R_\Gamma}$ (including $i\infty$).

<u>Corollary</u>: If f is modular and bounded on H then it is constant.

OTE:

$f) + \frac{v_i(f)}{2}$

$\frac{}{3}$

$-\sum v_p(f) - v_{i\infty}(f)$

$\equiv 0 \ (\mathrm{mod}\ 3)$

$f) \equiv 0 \ (\mathrm{mod}\ 2)$

of poles $>$

# of poles ≥ 1

Special values of J

Theorem: The function $J(\tau)$ takes every value exactly once in $\tilde{R}$ ~~the closure of R_F~~. In particular, at the vertices we have

$$J(\omega) = 0, \qquad J(i) = 1, \qquad J(i\infty) = \infty.$$

Also, $\quad v_\omega(J) = 3, \quad v_{i\infty}(J) = -1$

$$v_i(J-1) = 2.$$

Proof: First we show

$$g_2(\omega) = g_3(i) = 0.$$

Since $\omega^3 = 1$, $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1)$,

$$\omega^2 + \omega + 1 = 0$$

$$\frac{1}{60} g_2(\omega) = \sum_{m,n}{}' \frac{1}{(m+n\omega)^4} \qquad \Omega = \Omega(1,\omega)$$

$$= \sum{}' \frac{1}{(m\omega^3 + n\omega)^4} = \frac{1}{\omega^4} \sum{}' \frac{1}{(m\omega^2 + n)^4}$$

$$= \frac{1}{\omega} \sum{}' \frac{1}{(n - m - m\omega)^4} = \frac{1}{\omega} \frac{1}{60} g_2(\omega)$$

and $g_2(\omega) = 0$.

$$\frac{1}{140} g_3(i) = \sum{}' \frac{1}{(m+ni)^6}$$

$$= \sum{}' \frac{1}{(i(n-im))^6}$$

$$= -\sum{}' \frac{1}{(n-im)^6} = -\frac{1}{140} g_3(i)$$

& $g_3(i) = 0$.

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \tag{15}$$

The only pole of $J(\tau)$ is at $i\infty$ since $\Delta(\tau) \neq 0$ for $\tau \in H$ and $v_{i\infty}(J) = 1$ (see Thm ?)

(since $J(\tau) = \frac{1}{12^3}\frac{1}{q} + \frac{744}{12^3} + \dots$)

Note: $v_\omega(f) \equiv 0\ (3)$ & $v_i(f) \equiv 0\ (2)$

Hence # of zeros of $J(\tau) = 1$.

We know $v_\omega(J) \geq 3$. By Valence Thm $v_\omega(J) = 3$ and ω is the only zero of $J(\tau)$.

$$J(i) = \frac{g_2^3(i)}{g_2^3(i) - 27 g_3^2(i)} = \frac{g_2^3(i)}{g_2^3(i)} = 1$$

It follows that $v_i(J-1) = 2$ since

$$-1 + \tfrac{1}{3}(v_\omega(J-1)) + \tfrac{1}{2}(v_i(J-1)) + \sum v_p(J-1) = 0$$

& $v_i(J-1) \geq 1$ & $v_p(J-1) \geq 0$ for $p \neq i\infty$.

$J(\tau)$ takes every value exactly once in $\tilde{R}$ since $v_{i\infty}(J) = 1$.

<u>Theorem</u> Any rational function ~~is~~ in J is a modular function. Conversely, any modular function is a rational function in J.

<u>Proof</u> Suppose $f(\tau)$ is a modular function.

$$v_{i\infty}(f) + \tfrac{1}{2} v_i(f) + \tfrac{1}{3} v_\omega(f) + \sum_p v_p(f) = 0$$

$$v_i(f) \equiv 0 \pmod 2 \qquad v_\omega(f) \equiv 0 \pmod 3$$

In $\tilde{R}$ let $m_p = \begin{cases} v_p(f) & p \text{ zero or pole of } f,\ p \in \tilde{R},\ p \neq \omega, i \\ \tfrac{1}{2} v_i(f) & \text{if } p = i \\ \tfrac{1}{3} v_\omega(f) & \text{if } p = \omega. \end{cases}$

Let $P = \{ p \in \tilde{R} : v_p(f) \neq 0 \}$.

Let $g(\tau) = \prod_{p \in P} (J(\tau) - J(p))^{m_p}$

g has zeros and poles at $p \in P$. If $\tau \neq p$ $J(\tau) \neq J(p)$.

For $p \neq i, \omega$ $\quad v_p(J - J(p)) = 1$

So $\quad v_p(g) = m_p = v_p(f)$

$v_i(J - J(i)) = 2$

$v_i(g) = 2m_p = v_i(f)$

$v_\omega(J - J(\omega)) = v_\omega(J) = 3$

$v_\omega(g) = 3m_\omega = v_\omega(f)$.

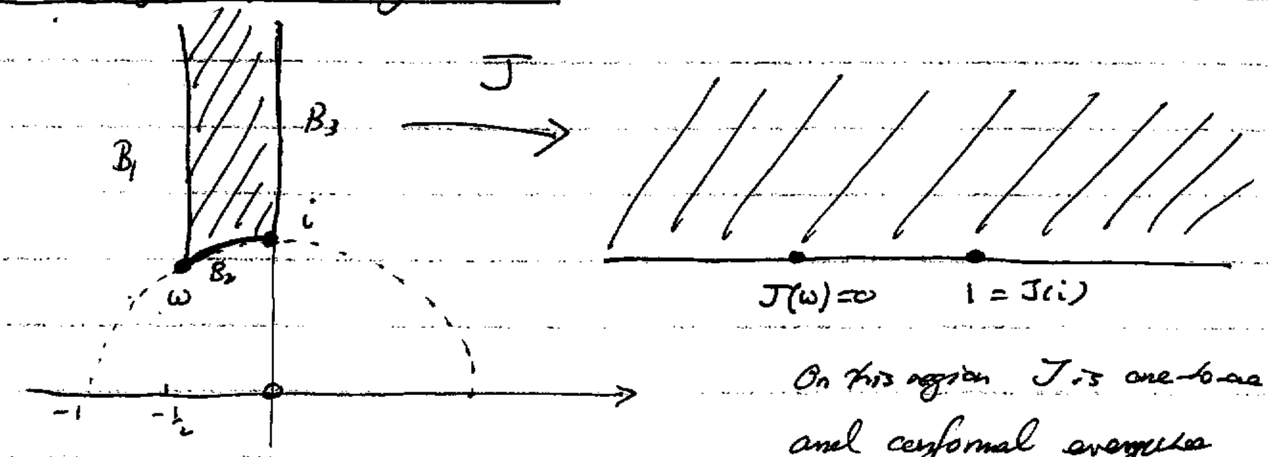
Now $\quad v_p\left(\frac{g}{f}\right) = 0$ for all $p \in P$.

Let $h = \frac{g}{f}$, $\quad v_p(h) = 0$ for all $p \in P$.

It follows that $v_{i\infty}(h) = 0$ by the valence formula since h is a modular function and h must be a constant.

Hence $\quad f(\tau) = C\, g(\tau) = C \prod_{p \in P} (J(\tau) - J(p))^{m_p}$. $\square$

<u>Mapping properties of $J(\tau)$</u>



On this region J is one-to-one and conformal everywhere except at $\tau = i, \omega$

ie J is analytic and $J'(\tau_0) \neq 0$, $\tau_0 \neq i, \omega$.

Note if $J'(\tau_0) = 0$ then $v_{\tau_0}(J - J(\omega)) \geq 2$

$J(i) = 1$

Let $\tau = 0 + iy$, $q = e^{-2\pi y}$

$12^3 J(\tau) = \frac{1}{q} + 744 + \cdots$

$J(iy) \approx e^{2\pi y} \to \infty$ as $y \to \infty$

so $J(B_3) = [1, \infty)$.

On B_1 $\tau = -\frac{1}{2} + iy$, $q = e^{2\pi i(-\frac{1}{2} + iy)} = -e^{-2\pi y}$

$J(-\frac{1}{2} + iy) \approx -e^{+2\pi y} \to -\infty$ as $y \to \infty$

and $J(B_1) = (-\infty, 0]$.

The coefficients $c(n)$ are rational integers (and hence real).

So if $q = e^{2\pi i \tau}$, $\bar{q} = e^{-2\pi i \bar{\tau}}$

and $J(\tau) = \overline{J(-\bar{\tau})}$.

On B_2, $\tau\bar{\tau} = 1$, and $-\bar{\tau} = -1/\tau$,

$J(-\bar{\tau}) = J(-1/\tau) = J(\tau)$ (since $\tau \to -1/\tau \in \Gamma$)

$J(\tau) = \overline{J(\tau)}$ & $J(\tau)$ is real on B_2.

So $J(B_2) = [0, 1]$.

Since J is conformal the shaded region is mapped onto the upper halfplane $\tau : \operatorname{Im}\tau \geq 0$.

Inversion problem for Eisenstein series

Recall $\wp(z;\omega_1,\omega_2)$ satisfies

$$[\wp'(z)]^2 = 4\wp^3(z) - g_2\wp(z) - g_3.$$

where $g_2^3 - 27g_3^2 \neq 0$.

Theorem: Given any two complex numbers a_2, a_3 such that $a_2^3 - 27a_3^2 \neq 0$, there exist complex numbers ω_1, ω_2 (with non real ratio) such that

$$g_2(\omega_1,\omega_2) = a_2, \qquad g_3(\omega_1,\omega_2) = a_3.$$

Proof:

Case (I) $a_2, a_3 \neq 0$. Choose complex τ, $\operatorname{Im}\tau > 0$ such that

$$J(\tau) = \frac{a_2^3}{a_2^3 - 27a_3^2}.$$

Note $J(\tau) \neq 0$ since $a_2 \neq 0$, so $g_2(1,\tau) \neq 0$. Also $J(\tau) \neq 1$ since $a_3 \neq 0$ &

$$\frac{J(\tau)-1}{J(\tau)} = \frac{27a_3^2}{a_2^3}$$

$$J-1 = \frac{+27a_3^2}{a_2^3-27a_3^2}$$

$$J(\tau)-1 = \frac{27g_3^2(1,\tau)}{g_2^3(1,\tau)-27g_3^2(1,\tau)}$$

so $g_3(1,\tau) \neq 0$.

Now choose ω_1:

$$\omega_1^2 = \frac{a_2}{a_3}\frac{g_3(1,\tau)}{g_2(1,\tau)},$$

and $\omega_1 \neq 0$.

Let $\omega_2 = \tau\omega_1$.

Then

$$\frac{g_2(\omega_1,\omega_2)}{g_3(\omega_1,\omega_2)} = \frac{g_2(\omega_1,\tau\omega_1)}{g_3(\omega_1,\tau\omega_1)}$$

$$= \frac{\omega_1^{-4}}{\omega_1^{-6}}\frac{g_2(1,\tau)}{g_3(1,\tau)} = \omega_1^2\frac{g_2(1,\tau)}{g_3(1,\tau)} = \frac{a_2}{a_3}$$

$$g_3(\omega_1,\omega_2) = \frac{a_3}{a_2}g_2(\omega_1,\omega_2) \quad (*)$$

$$\frac{J(\tau)-1}{J(\tau)} = \frac{J(\omega_1,\omega_2)-1}{J(\omega_1,\omega_2)}$$

since J has deg. 0

$$= 27\frac{g_3^2(\omega_1,\omega_2)}{g_2^3(\omega_1,\omega_2)} = 27\frac{\left(a_3/a_2\right)^2 g_2^2}{g_2^3} = \frac{27a_3^2}{a_2^2 g_2(\omega_1,\omega_2)}$$

so that

$$g_2(\omega_1,\omega_2) = a_2$$

and $$g_3(\omega_1,\omega_2) = a_3 \quad \text{by } (*).$$

<u>Ex</u> Remaining cases $a_2 = 0$, $a_3 = 0$. $\square$

Properties of the J function can be used to prove

<u>Picard's Theorem</u> Every nonconstant entire function omits at most one value.

<u>Eg</u> e^z omits only the value 0.