Chapter 3 The Dedekind eta function

For $\operatorname{Im}\tau > 0$,

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right)$$

$$= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i\tau}.$$

Theorem (Complex Analysis, A.V. Ahlfors, p192)
The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely (and has non zero limit) iff the series $\sum_{n=1}^{\infty}|a_n|$ converges.

$\sum_{n=1}^{\infty}|q|^n$ converges if $|q|<1$. So the eta-product converges for $\operatorname{Im}\tau > 0$ & $\eta(\tau)\neq 0$. The convergence is uniform for sets $\operatorname{Im}\tau > M$ and so $\eta(\tau)$ is analytic on H.

$$\eta(\tau+1) = e^{\pi i/12}\,\eta(\tau)$$

Theorem For $\tau\in H$,

(*) $$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau)$$

where $\sqrt{\ }$ is any branch of the square root function where $\sqrt{z}>0$ for $z>0$.

Note: $\tau = x+iy$, $-i\tau = y-ix$
$= re^{i\theta}$, $r>0$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$
$\sqrt{-i\tau} = \sqrt{r}\,e^{i\theta/2}$.

Proof: We prove (*) for $\operatorname{Re}\tau = 0$. The general result will then follow by analytic continuation.

[illegible]

Let $\tau = i\frac{a}{\pi}$, $a > 0$.

Then $-\frac{1}{\tau} = i\frac{\pi}{a} = i\frac{b}{\pi}$ where $ab = \pi^2$.

$$\eta(\tau) = e^{+\pi i\left(\frac{ia}{\pi 12}\right)} \prod_{n=1}^{\infty}\left(1 - e^{+2\pi i\left(i\frac{a}{\pi}\right)n}\right)$$

$$= e^{-a/12} \prod_{n=1}^{\infty}\left(1 - e^{-2an}\right)$$

$$\eta\left(-\frac{1}{\tau}\right) = e^{\frac{\pi i}{12}\left(-\frac{1}{\tau}\right)} \prod_{n=1}^{\infty}\left(1 - e^{2\pi i\left(-\frac{1}{\tau}\right)n}\right)$$

$$= e^{-b/12} \prod_{n=1}^{\infty}\left(1 - e^{-2bn}\right)$$

$$\sqrt{-i\tau} = \sqrt{\frac{a}{\pi}} = \frac{\sqrt{a}}{\sqrt{\pi}} = \frac{\sqrt{a}}{(ab)^{\frac{1}{4}}} = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}$$

(*) is equivalent to

$$a^{1/4} e^{-a/12} \prod_{n=1}^{\infty}\left(1 - e^{-2an}\right) = b^{\frac{1}{4}} e^{-b/12} \prod_{n=1}^{\infty}\left(1 - e^{-2bn}\right)$$

where $ab = \pi^2$.

Taking logarithms

$$\frac{1}{4}\log a - \frac{a}{12} + \sum_{n=1}^{\infty}\log\left(1 - e^{-2an}\right) = \frac{1}{4}\log b - \frac{b}{12} + \sum_{n=1}^{\infty}\log\left(1 - e^{-2bn}\right)$$

$$\log(1 - z) = -\sum_{n=1}^{\infty}\frac{z^n}{n} \qquad (|z| < 1)$$

$$\log\left(1 - e^{-2an}\right) = -\sum_{m=1}^{\infty}\frac{\left(e^{-2an}\right)^m}{m}$$

$$\sum_{n=1}^{\infty}\log\left(1 - e^{-2an}\right) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{e^{-2anm}}{m}$$

$$= -\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=1}^{\infty}e^{-2anm} = -\sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-2am}}{1 - e^{-2am}}$$

$$= -\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{\left(e^{2am} - 1\right)}$$

So $(*)$ is equivalent to

$$(2)\quad \sum_{n=1}^{\infty} \frac{1}{n\,(e^{2an}-1)} - \frac{1}{4}\log a + \frac{a}{12}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n\,(e^{2bn}-1)} - \frac{1}{4}\log b + \frac{b}{12}.$$

$$\coth z = \frac{e^{z}+e^{-z}}{e^{z}-e^{-z}} = \frac{e^{2z}+1}{e^{2z}-1}$$

$$\coth z - 1 = \frac{(e^{2z}+1)-(e^{2z}-1)}{e^{2z}-1}$$

$$= \frac{2}{e^{2z}-1}.$$

$$(3)\quad \text{So } \frac{1}{e^{z}-1} = \frac{1}{2}\coth\frac{z}{2} - \frac{1}{2}.$$

So (2) can be rewritten as

$$(4)\quad \sum_{n=1}^{\infty} \frac{1}{n}\left(\coth(an) - \coth(bn)\right) = \frac{1}{2}\log\left(\frac{a}{b}\right) - \frac{1}{6}(a-b).$$

We need the partial fraction expansion of

$$\coth(\pi z) = \frac{1}{\pi z} + \frac{2z}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2+z^2}$$

$$\coth(z) = \frac{1}{z} + \frac{2z}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2+\left(\frac{z}{\pi}\right)^2}$$

$$= \frac{1}{z} + 2z\sum_{m=1}^{\infty}\frac{1}{\pi^2 m^2 + z^2}$$

$$\frac{1}{n}\left(\coth(an) - \coth(bn)\right)$$

$$= \frac{1}{n}\Bigg(\frac{1}{an} + 2an\sum_{m=1}^{\infty}\frac{1}{\pi^2 m^2 + a^2 n^2}$$

$$- \frac{1}{\cdots} - 2bn\sum^{\infty}\frac{1}{\cdots}\Bigg)$$

$$= \frac{1}{n^2}\left(\frac{1}{a}-\frac{1}{b}\right) + 2a\sum_{m=1}^{\infty}\frac{1}{abm^2+a^2n^2}$$

$$- 2b\sum_{m=1}^{\infty}\frac{1}{abm^2+b^2n^2}$$

$$= \frac{1}{n^2}\left(\frac{1}{a}-\frac{1}{b}\right) + 2\sum_{m=1}^{\infty}\left(\frac{1}{m^2b+n^2a} - \frac{1}{m^2a+n^2b}\right)$$

So (4) can be written as

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{a}-\frac{1}{b}\right) + 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m^2b+n^2a} - \frac{1}{m^2a+n^2b}\right)$$

$$= \frac{1}{2}\log(a\cancel{/}b) - \frac{1}{6}(a-b)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{\pi^2}{6}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{ab}{6}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{6}(b-a)$$

So (4) can be written as

$$(5)\quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m^2b+n^2a} - \frac{1}{m^2a+n^2b}\right) = \frac{1}{4}\log\frac{a}{b}.$$

Now set $a = \pi e^{\gamma}$, $b = \pi e^{-\gamma}$ so

$$\frac{a}{b} = e^{2\gamma},\quad \log\left(\frac{a}{b}\right) = 2\gamma$$

and (5) is equivalent to

$$(6)\quad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m^2e^{-\gamma}+n^2e^{\gamma}} - \frac{1}{m^2e^{\gamma}+n^2e^{-\gamma}}\right) = \frac{\pi\gamma}{2}$$

for any real γ.

Now we prove (6).

Let $\quad a_{mn} = \dfrac{1}{m^2 e^{-\gamma} + n^2 e^{\gamma}} - \dfrac{1}{m^2 e^{\gamma} + n^2 e^{-\gamma}}.$

Then

$$a_{mn} = - a_{nm}.$$

Hence

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} = \lim_{n \to \infty} \sum_{\nu=1}^{n} \left(\sum_{\mu=1}^{n} + \sum_{\mu=n+1}^{\infty} a_{\mu\nu} \right)$$

(7) $$= \lim_{n \to \infty} \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} a_{\mu\nu}.$$

Now,

$$\frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}} < \int_{\mu-1}^{\mu} \frac{dx}{x^2 e^{-\gamma} + \nu^2 e^{\gamma}} < \frac{1}{(\mu-1)^2 e^{-\gamma} + \nu^2 e^{\gamma}}$$

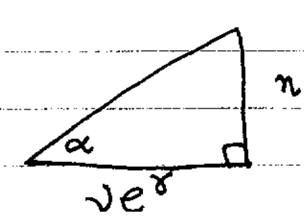
$$\sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}} < \int_{n}^{\infty} \frac{dx}{x^2 e^{-\gamma} + \nu^2 e^{\gamma}} < \sum_{\mu=n}^{\infty} \frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}}$$

$$0 < \int_{n}^{\infty} \frac{dx}{x^2 e^{-\gamma} + \nu^2 e^{\gamma}} - \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}} < \frac{1}{n^2 e^{-\gamma} + \nu^2 e^{\gamma}} = \frac{e^{\gamma}}{n^2 + \nu^2 e^{2\gamma}}$$

$$< \frac{e^{\gamma}}{n^2}$$

$$\int_{n}^{\infty} \frac{dx}{x^2 e^{-\gamma} + \nu^2 e^{\gamma}} = e^{\gamma} \int_{n}^{\infty} \frac{dx}{x^2 + \nu^2 e^{2\gamma}}$$

$$= \frac{e^{\gamma}}{\nu e^{\gamma}} \tan^{-1}\left(\frac{x}{\nu e^{\gamma}}\right)\Bigg]_{n}^{\infty} = \frac{1}{\nu}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{n}{\nu e^{\gamma}}\right)\right)$$



$\tan\alpha = \dfrac{n}{\nu e^{\gamma}}$

$$= \frac{1}{\nu} \tan^{-1}\left(\frac{\nu e^{\gamma}}{n}\right)$$

So

$$0 < \sum_{\nu=1}^{n} \frac{1}{\nu} \tan^{-1}\left(\frac{\nu e^{\gamma}}{n}\right) - \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}} < \frac{e^{\gamma}}{n}$$

Let $x_\nu = \frac{\nu e^{\gamma}}{n}$, $\nu = 1, \dots, n$

$\Delta x_\nu = \frac{e^{\gamma}}{n}$, $\frac{\Delta x_\nu}{x_\nu} = \frac{1}{\nu}$

Letting $n \to \infty$ we find

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}} = \int_0^{e^{\gamma}} \frac{\tan^{-1} x}{x}\, dx$$

Replacing γ by $-\gamma$

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \frac{1}{\mu^2 e^{\gamma} + \nu^2 e^{-\gamma}} = \int_0^{e^{-\gamma}} \frac{\tan^{-1} x}{x}\, dx$$

So

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} = \lim_{n\to\infty} \sum_{\nu=1}^{n} \sum_{\mu=n+1}^{\infty} \left(\frac{1}{\mu^2 e^{-\gamma} + \nu^2 e^{\gamma}} - \frac{1}{\mu^2 e^{\gamma} + \nu^2 e^{-\gamma}} \right)$$

$$= \int_{e^{-\gamma}}^{e^{\gamma}} \frac{\tan^{-1} x}{x}\, dx$$

$x = e^t$
$dx = e^t\, dt$
$\frac{dx}{x} = dt$

$$= \int_{-\gamma}^{\gamma} \tan^{-1}(e^t)\, dt$$

$$= \frac{1}{2} \int_{-\gamma}^{\gamma} \tan^{-1}(e^t) + \tan^{-1}(e^{-t})\, dt$$

$$= \frac{1}{2} \int_{-\gamma}^{\gamma} \pi/2\, dt = \frac{\pi}{2}\gamma. \quad \square$$

<u>Infinite product rep. for $\Delta(\tau)$</u>

Recall, $\Delta(\tau) = \Delta(1,\tau) = g_2^3(1,\tau) - 27 g_3^2(1,\tau)$.

<u>Thm</u> For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\Delta(A\tau) = (c\tau+d)^{12}\,\Delta(\tau).$$

In particular,
$$\Delta(\tau+1) = \Delta(\tau)$$
$$\Delta\left(-\frac{1}{\tau}\right) = \tau^{12}\,\Delta(\tau).$$

<u>Proof</u>

$\Delta(\omega_1,\omega_2)$ is a homog. function of degree -12.

Let $\tau = \omega_2/\omega_1$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

$$\Delta(\omega_1,\omega_2) = \Delta(\omega_1, \tau\omega_1) = \omega_1^{-12}\,\Delta(1,\tau) = \omega_1^{-12}\,\Delta(\tau).$$

Let $\omega_1 = 1$, $\omega_2 = \tau$.

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

Then $\dfrac{\omega_2'}{\omega_1'} = \dfrac{a\tau+b}{c\tau+d}$ and

$$\Omega(\omega_1,\omega_2) = \Omega(\omega_1',\omega_2').$$

So

$$\Delta(\tau) = \Delta(\omega_1,\omega_2) = \Delta(\omega_1',\omega_2')$$
$$= \Delta(c\tau+d,\ a\tau+b)$$
$$= \Delta\left(c\tau+d,\ \left(\frac{a\tau+b}{c\tau+d}\right)c\tau+d\right)$$
$$= (c\tau+d)^{-12}\,\Delta\left(1, \frac{a\tau+b}{c\tau+d}\right),$$

and
$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\,\Delta(\tau).$$

Theorem: If $\tau \in H$, $q = e^{2\pi i \tau}$, we have

$$\Delta(\tau) = (2\pi)^{12} \eta^{24}(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24},$$

and so

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad \text{for } |q|<1.$$

Proof:

$$\eta(\tau+1) = e^{\pi i/12} \eta(\tau)$$
$$\eta^{24}(\tau+1) = e^{2\pi i} \eta^{24}(\tau) = \eta^{24}(\tau).$$
$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau)$$
$$\eta^{24}(-1/\tau) = (-i\tau)^{12} \eta^{24}(\tau) = \tau^{12} \eta^{24}(\tau).$$

Let $f(\tau) = \dfrac{\Delta(\tau)}{\eta^{24}(\tau)}$.

Then $f(\tau+1) = f(\tau)$, $f(-1/\tau) = f(\tau)$

so f is invariant under every transf. in Γ.

In Ch 2 we proved that any nonconst. mod. fct must have a pole in $H \cup \{\infty\}$

$f(\tau)$ is analytic and nonzero on H

because $\Delta(\tau)$, $\eta^{12}(\tau)$ are analytic and nonzero on H.

Hence $f(\tau)$ is a modular function with no zeros or poles in H. It follows by a Thm in Ch 2 that $f(\tau)$ must be constant. ($v_{i\infty}(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\omega(f) + \sum_{\substack{p \in R_\Gamma \\ p \neq i, \omega}} v_p(f) = 0$

this no poles in $H \cup \{i\infty\}$

so $v_{i\infty}(f) = 0$, f must be constant since f ~~does~~ omits the value 0).

f is cnst.

$$\Delta(\tau) = (2\pi)^{12} (q + \cdots)$$
$$\eta^{12}(\tau) = q + \cdots$$
$$f(\tau) = (2\pi)^{12} + \cdots$$

and $f(\tau) = (2\pi)^{12}$. $\square$

Dedekind's Functional Equation

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $c>0$. Let $\tau \in H$.
Then

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)\sqrt{-i(c\tau+d)}\;\eta(\tau)$$

where

$$\varepsilon(a,b,c,d) = \exp\left\{\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right\}$$

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$$

(Dedekind sum).

Note: $\tau = x+iy$ $(y>0)$.

$c\tau+d = (cx+d)+iy$

$-i(c\tau+d) = y - i(cx+d) = re^{i\theta}$, $r>0$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$\sqrt{-i(c\tau+d)} = \sqrt{r}\,e^{i\theta/2}$.

Properties of Dedekind Sums

assume k is a positive integer & $(h,k)=1$.

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$$

Define

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

$((x+1)) = ((x))$ for all x.

note Let $n < x < n+1 \quad (n \in \mathbb{Z})$

so $((x)) = x - n - \frac{1}{2}$.

$-n-1 < -x < -n$.

so $((-x)) = -x - (-n-1) - \frac{1}{2} = -x + n + \frac{1}{2}$

$= -((x))$.

So $((x))$ is ~~an~~ a odd function.

$$\sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right) = \sum_{r=0}^{k-1} \left(\left(\frac{r}{k}\right)\right)$$

~~since~~ since $((x))$ has period 1 the sum can be defined over any set of residues $(\text{mod } k)$.

As r runs thro a set of complete residues $(\text{mod } k)$ so does $-r$.

so $$\sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right) = \sum_{r \pmod k} \left(\left(\frac{-r}{k}\right)\right)$$

$$= - \sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right)$$

&

$$\sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right) = 0.$$

Let $(h,k)=1$. As r runs over a complete set of residues mod k so does hr.
(Since $hr_1 \equiv hr_2 \pmod k$ iff $r_1 \equiv r_2 \pmod k$).
Hence,

$$\sum_{r \pmod k} \left(\left(\frac{hr}{k}\right)\right) = 0 \quad \text{if } (h,k)=1.$$

$$\sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right)$$

$$= \sum_{r=1}^{k-1} \left(\frac{r}{k} - \frac{1}{2}\right)\left(\left(\frac{hr}{k}\right)\right)$$

$$= \sum_{r=1}^{k-1} \frac{r}{k}\left(\left(\frac{hr}{k}\right)\right) - \frac{1}{2}\sum_{k=1}^{k-1}\left(\left(\frac{hr}{k}\right)\right)$$

$$= \sum_{r=1}^{k-1} \frac{r}{k}\left(\left(\frac{hr}{k}\right)\right).$$

Hence

$$s(h,k) = \sum_{r \pmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right).$$

<u>Theorem</u> Let $(h,k)=1$, $k>0$.

(a) If $h' \equiv \pm h \pmod k$ then $s(h',k) = \pm s(h,k)$.
(b) If $h\bar{h} \equiv \pm 1 \pmod k$ then $s(\bar{h},k) = \pm s(h,k)$
(c) If $h^2+1 \equiv 0 \pmod k$ then $s(h,k) = 0$.

<u>Proof</u>
(a) Let $h' \equiv \pm h \pmod k$.
So $h' = \pm h + mk$

$$\left(\left(\frac{h'r}{k}\right)\right) = \left(\left(\frac{\pm hr}{k} + mr\right)\right) = \left(\left(\frac{\pm hr}{k}\right)\right) = \pm\left(\left(\frac{hr}{k}\right)\right)$$

& result follows

(b) Suppose $h\bar{h} \equiv \pm 1 \pmod{k}$

~~Then~~

Let r run thro a complete residue syst mod k then so does $\bar{h} r$.

So

$$s(h,k) = \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right)$$

$$= \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r\bar{h}}{k}\right)\right)\left(\left(\frac{h\bar{h}r}{k}\right)\right)$$

$$= \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r\bar{h}}{k}\right)\right)\left(\left(\frac{\pm r}{k}\right)\right)$$

$$= \pm \sum_{r \,(\mathrm{mod}\, k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{\bar{h} r}{k}\right)\right)$$

$$= \pm\, s(\bar{h}, k).$$

(c) Suppose $h^2 + 1 \equiv 0 \pmod{k}$

So $h^2 \equiv -1 \pmod{k}$

$h\bar{h} \equiv -1 \pmod{k}$ where $\bar{h} \equiv h \pmod{k}$

Hence by (b)

$s(h,k) = -s(h,k)$ & $s(h,k) = 0$. $\square$

The Reciprocity Law for Dedekind Sums

Suppose $h, k > 0$ & $(h,k) = 1$.

Then

$$12hk\, s(h,k) + 12kh\, s(k,h) = h^2 + k^2 - 3hk + 1.$$

Proof:

$$\sum_{r \pmod k} \left(\left(\frac{hr}{k}\right)\right)^2 = \sum_{r=1}^{k-1} \left(\left(\frac{r}{k}\right)\right)^2$$

$$= \sum_{r=1}^{k-1} \left(\frac{r}{k} - \frac{1}{2}\right)^2$$

also

$$\sum_{r \pmod k} \left(\left(\frac{hr}{k}\right)\right)^2 = \sum_{r=1}^{k-1} \left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)^2$$

$$= \sum_{r=1}^{k-1} \left(\frac{h^2r^2}{k^2} + \left[\frac{hr}{k}\right]^2 + \frac{1}{4} - \frac{hr}{k} + \left[\frac{hr}{k}\right] - \frac{2hr}{k}\left[\frac{hr}{k}\right]\right)$$

$$= 2h \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$$

$$+ \sum_{r=1}^{k-1} \left[\frac{hr}{k}\right]\left(\left[\frac{hr}{k}\right] + 1\right)$$

$$- \frac{h^2}{k^2} \sum_{r=1}^{k-1} r^2 + \frac{1}{4}\sum_{r=1}^{k-1} 1.$$

$$= 2h\, s(h,k) + \sum_{r=1}^{k-1} \left[\frac{hr}{k}\right]\left(\left[\frac{hr}{k}\right] + 1\right)$$

$$- \frac{h^2}{k^2} \sum_{r=1}^{k-1} r^2 + \frac{1}{4}\sum_{r=1}^{k-1} 1 = \sum_{r=1}^{k-1} \left(\frac{r^2}{k^2} - \frac{r}{k} + \frac{1}{4}\right)$$

Hence,

(14)

$$2hks(h,k) + \sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]\left(\left[\frac{hr}{k}\right]+1\right)$$
$$= \frac{h^2+1}{k^2}\sum_{r=1}^{k-1} r^2 - \frac{1}{k}\sum_{r=1}^{k-1} r.$$

Since $0<r<k$, $0<\frac{hr}{k}<h$
and
$$\left[\frac{hr}{k}\right] = v-1 \quad \text{when } v=1,2,\dots,h.$$

For a given $v=1,2,\dots,h$, let
$$N(v) = \#\left\{r : \left[\frac{hr}{k}\right]=v-1\right\}.$$

$$\left[\frac{hr}{k}\right]=v-1 \quad \text{iff} \quad v-1<\frac{hr}{k}<v$$

(note, $(h,k)=1 \Rightarrow k\nmid hr$)
$$\text{iff } \frac{k}{h}(v-1) < r < \frac{vk}{h} \quad (*)$$

$$\cancel{\text{iff } \left[\frac{k(v-1)}{h}\right]+1 \le r \le \left[\frac{kv}{h}\right] < \frac{vk}{h} \text{ if } v<1}$$

~~Hence~~

$$\cancel{N(v) = \left[\frac{kv}{h}\right] - \left[\frac{k(v-1)}{h}\right] \text{ if } 1\le v\le h-1}$$

note: $(h,k)=1$ so $h\mid vk$ iff $h\mid v$ iff $h=v$

If $1\le v<h-1$, $(*) \Leftrightarrow$
$$\left[\frac{k(v-1)}{h}\right]+1 \le r \le \left[\frac{kv}{h}\right] < \frac{vk}{h} < k$$

$$\& \quad N(v) = \left[\frac{kv}{h}\right] - \left[\frac{k(v-1)}{h}\right].$$

If $v=h$, $\quad \frac{k}{h}(h-1) < r < k$

$$k-\frac{k}{h} < r < k$$

$$\text{iff } \left[\frac{k(h-1)}{h}\right]+1 \le r \le k-1$$

Hence
$$N(h) = k - 1 - \left[\frac{k(h-1)}{h}\right].$$

Therefore,
$$\sum_{r=1}^{k-1} \left[\frac{hr}{k}\right]\left(\left[\frac{hr}{k}\right]+1\right)$$
$$= \sum_{v=1}^{h} v(v-1)\; N(v)$$
$$= \sum_{v=1}^{h} v(v-1)\left(\left[\frac{kv}{h}\right] - \left[\frac{k(v-1)}{h}\right]\right) - h(h-1)$$
$$= \sum_{v=1}^{h-1} v(v-1)\left[\frac{kv}{h}\right] - \sum_{v=1}^{h-1} v(v+1)\left[\frac{kv}{h}\right]$$
$$- h(h-1) + k\,h(h-1)$$
$$= -2\sum_{v=1}^{h-1} v\left[\frac{kv}{h}\right] + h(h-1)(k-1)$$

Also,
$$\cancel{2h\, s(k,h) = 2h\sum_{v=1}^{h-1} v(}$$
$$2h\, s(k,h) = 2h \sum_{v \,(\mathrm{mod}\, h)} \left(\left(\frac{v}{h}\right)\right)\left(\left(\frac{kv}{h}\right)\right)$$
$$= 2h \sum_{v=1}^{h-1} \frac{v}{h}\left(\frac{kv}{h} - \left[\frac{kv}{h}\right] - \frac{1}{2}\right)$$
$$= -2\sum_{v=1}^{h-1} v\left[\frac{kv}{h}\right] + \frac{2k}{h}\sum_{v=1}^{h-1} v^2 - \sum_{v=1}^{h-1} v$$

Hence

$$\sum_{r=1}^{k-1} \left[\frac{hr}{k}\right]\left(\left[\frac{hr}{k}\right]+1\right)$$

$$= 2h\, s(k,h) - \frac{2k}{h}\sum_{v=1}^{h-1} v^2 + \sum_{v=1}^{h-1} v + h(h-1)(k-1).$$

Hence

$$2h\, s(h,k) + 2h\, s(k,h)$$

$$\cancel{= \left(\frac{h^2+1}{k^2} + \frac{2k}{h}\right)\sum_{r=1}^{k-1} r^2 + \left(\frac{1}{k}-1\right)\sum_{r=1}^{k-1} k}$$

$$\cancel{- h(h-1)(k-1)}$$

$$= \frac{h^2+1}{k^2}\sum_{r=1}^{k-1} r^2 - \frac{1}{k}\sum_{r=1}^{k-1} r + \frac{2k}{h}\sum_{v=1}^{h-1} v^2 - \sum_{v=1}^{h-1} v$$

$$- h(h-1)(k-1)$$

So $12kh\, s(h,k) + 12kh\, s(k,h)$

$$= (h^2+1)(k-1)(2k-1) - 3k(k-1) + 2k^2(h-1)(2h-1)$$

$$- 3kh(h-1) - 6hk(k-1)(h-1)$$

$$= h^2 + k^2 + 1 - 3kh. \quad \square$$

$$\sum_{r=1}^{N} r^2 = 1/6\, N(N+1)(2N+1)$$

$$TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(TS)^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$ST^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

So

$$\begin{aligned} TSTS &= ST^{-1} \\ TST &= STS \\ TS &= STST^{-1} \\ T &= STST^{-1}S \end{aligned}$$

<u>Theorem</u>: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c > 0$, then

$$\eta\left(\frac{az+b}{cz+d}\right) = \eta(Az) = \varepsilon(A)\left(-i(cz+d)\right)^{1/2}\eta(z)$$

where $\varepsilon(A) = \exp\left(\pi i\left(\frac{a+d}{12c} - s(d,c)\right)\right)$, $\eta \in H$.

<u>Lemma 1</u>: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $c > 0$, then

$$\varepsilon(AT^m) = e^{\pi i m/12}\,\varepsilon(A)$$

for $m \in \mathbb{Z}$.

<u>Proof</u>:

$$AT^m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & am+b \\ c & cm+d \end{pmatrix}.$$

$$\varepsilon(AT^m) = \exp\left(\pi i\left(\left(\frac{a+cm+d}{12c}\right) - s(cm+d,c)\right)\right)$$

$$= \exp\left(\pi i\left(\left(\frac{a+d}{12c}\right) - s(d,c)\right)\exp\left(\frac{\pi i m}{12}\right)\right.$$

(since $cm+d \equiv d \bmod c$)

$$= e^{\pi i m/12}\,\varepsilon(A).$$

Lemma

$$\varepsilon(AS) = \begin{cases} e^{-\pi i/4}\,\varepsilon(A) & \text{if } d>0 \\ e^{\pi i/4}\,\varepsilon(A) & \text{if } d<0 \end{cases}$$

Proof:

$$AS = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}.$$

If $d>0$ then

$$\varepsilon(AS) = \exp\left(\pi i\left(\frac{b-c}{12d} - s(-c,d)\right)\right)$$

$$= \exp\left(\pi i\left(\frac{b-c}{12d} + s(c,d)\right)\right).$$

By Rec. Law,

$$12cd\, s(d,c) + 12cd\, s(c,d) = d^2+c^2-3cd+1$$

$$s(c,d) + s(d,c) = \frac{c}{12d} + \frac{d}{12c} - \frac{1}{4} + \frac{1}{12cd}$$

$$= \frac{c}{12d} + \frac{d}{12c} - \frac{1}{4} + \frac{ad-bc}{12cd}$$

$$= \frac{c}{12d} + \frac{d}{12c} - \frac{1}{4} + \frac{a}{12c} - \frac{b}{12d}$$

Hence,

$$\frac{b-c}{12d} + s(c,d) = \frac{a+d}{12c} - s(d,c) - \frac{1}{4}$$

so

$$\varepsilon(AS) = \exp\left(\pi i\left(\frac{a+d}{12c} - s(d,c) - \frac{1}{4}\right)\right)$$
$$= e^{-\pi i/4}\,\varepsilon(A).$$

If $d<0$ then $AS = -\begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$

and $\varepsilon(AS) = \exp\left(\pi i\left(\frac{-b+c}{-12d} - s(c,-d)\right)\right).$

Now by Rec. Law,

$$-12cd\, s(c,-d) - 12cd\, s(-d,c)$$
$$= d^2 + c^2 + 3cd + 1$$

$$s(c,-d) + s(-d,c) = -\frac{d}{12c} - \frac{c}{12d} - \frac{1}{4} - \frac{1}{12cd}$$

$$s(c,-d) - s(d,c) = -\frac{d}{12c} - \frac{c}{12d} - \frac{1}{4} - \frac{(ad-bc)}{12cd}$$

$$s(c,-d) - s(d,c) = -\frac{d}{12c} - \frac{c}{12d} - \frac{1}{4} - \frac{a}{12c} + \frac{b}{12d}$$

$$\frac{a+d}{12c} - s(d,c) + \frac{1}{4} = -s(c,-d) + \frac{b-c}{12d}$$

Hence $\varepsilon(AS) = e^{\pi i/4}\,\varepsilon(A).$ □

<u>Lemma</u>

If $\eta(A\tau) = \varepsilon(A)\sqrt{-i(c\tau+d)}\;\eta(\tau)$ $(*)$
for some $A \in \Gamma$ with $c>0$ then it is also satisfied
for AT^m & AS.

<u>Proof:</u>

$$\eta(AT^m\tau) = \eta(A(T^m\tau))$$

$$= \varepsilon(A)\sqrt{-i(cT^m\tau+d)}\;\eta(T^m\tau)$$

$$= \varepsilon(A)\sqrt{-i(c(\tau+m)+d)}\;e^{\pi i m/12}\,\eta(\tau) = \varepsilon(A)\sqrt{-i(c\tau+cm+d)}\;e^{\pi i m/12}\,\eta(\tau)$$

(since $\eta(\tau+m) = e^{\pi i m/12}\eta(\tau)$)

$$= \varepsilon(AT^m)\sqrt{-i(c\tau+cm+d)}\;\eta(\tau)$$

& $(*)$ holds for AT^m since $AT^m = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1\end{pmatrix}$

$$= \begin{pmatrix} & \\ c & cm+d\end{pmatrix}$$

$$\eta(AS\tau) = \eta(A(S\tau))$$

$$= \varepsilon(A)\sqrt{-i(cS\tau+d)}\;\eta(S\tau)$$

$$= \varepsilon(A)\sqrt{-i(cS\tau+d)}\;\sqrt{-i\tau}\;\eta(\tau).$$

$$cS\tau+d = -\frac{c}{\tau}+d = \frac{d\tau-c}{\tau}$$

$$-i(cS\tau+d) = \frac{-i(d\tau-c)}{-i\tau}\cdot e^{-\pi i/2}$$

Let $K = \{\tau : \operatorname{Re}\tau > 0\}$

<u>Case 1</u> $d > 0$

Let $\tau = x + iy \in H$, $(y > 0)$

$-i\tau = y - ix = r_1 e^{i\theta_1} \in K \quad r_1 > 0, \ -\frac{\pi}{2} < \theta_1 < \pi/2$

$d\tau - c = dx - c + idy$

$-i(d\tau - c) = dy - i(dx - c) = r_2 e^{i\theta_2} \quad r_2 > 0, \ -\frac{\pi}{2} < \theta_2 < \pi/2$
$\in K$

$-i(cS\tau + d) = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1 - \pi/2)}$

and $\quad -\pi < \theta_2 - \theta_1 < \pi$

$\quad -3\pi/2 < \theta_2 - \theta_1 - \pi/2 < \pi/2$

But $S\tau \in H$ so $S\tau = x' + iy' \quad (y' > 0)$

$c(S\tau) + d = c(x' + iy') + d$

$= cx' + d + ciy'$

$-i(cS\tau + d) = cy' - i(cx' + d) \in K$

Hence

$$-\pi/2 < \theta_2 - \theta_1 - \pi/2 < \pi/2$$

&

$\sqrt{-i(cS\tau + d)} = \sqrt{\frac{r_2}{r_1}}\, e^{i\left(\left(\frac{\theta_2 - \theta_1}{2}\right) - \pi/4\right)}$

$\sqrt{-i\tau} = \sqrt{r_1}\, e^{i(\theta_1/2)}$

$\sqrt{-i(cS\tau + d)}\ \sqrt{-i\tau} = \sqrt{r_2}\, e^{i\theta_2/2}\, e^{-\pi i/4}$

$= e^{-\pi i/4} \sqrt{-i(d\tau - c)}$

Hence

$\eta(AS\tau) = \varepsilon(A)\, e^{-\pi i/4} \sqrt{-i(d\tau - c)}\ \eta(\tau)$

$= \varepsilon(AS) \sqrt{-i(d\tau - c)}\ \eta(\tau)$

& (*) holds for AS since

$$AS = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} d & -c \end{pmatrix}$$

<u>Case II $d<0$</u>

Let $\tau = x+iy \in H$ $(y>0)$

$-i\tau = y - ix = r_1 e^{i\theta_1} \in K$, $r_1>0$, $-\frac{\pi}{2}<\theta_1<\frac{\pi}{2}$.

$$-i(cS\tau+d) = -i\left(\frac{c}{-\tau}+d\right)$$

$$= -i\left(\frac{-d\tau+c}{-\tau}\right) = -i\frac{(-d\tau+c)}{-i\tau}\cdot(i)$$

$$= -i\left(\frac{-d\tau+c}{-i\tau}\right)e^{\pi i/2}$$

$$-i(-d\tau+c) = -i(-dx+c-diy)$$

$$= -dy - i(c-dx) = r_2 e^{i\theta_2}$$

$r_2>0$, $-\frac{\pi}{2}<\theta_2<\frac{\pi}{2}$

So $-i(cS\tau+d) = \frac{r_2}{r_1}e^{i(\theta_2-\theta_1+\pi/2)}$

$$-\pi < \theta_2 - \theta_1 < \pi$$

$$-\pi/2 < \theta_2 - \theta_1 + \pi/2 < 3\pi/2$$

But as before,

$$-i\,(cSz+d) \in K \quad \text{so}$$

$$-\pi/2 < \theta_2 - \theta_1 + \pi/2 < \pi/2$$

&

$$\sqrt{-i(cSz+d)} = \sqrt{\frac{r_2}{r_1}}\, e^{i\left(\frac{\theta_2-\theta_1}{2} + \pi/4\right)}$$

$$\sqrt{-iz} = \sqrt{r_1}\, e^{i\theta_1/2}$$

$$\sqrt{-i(cSz+d)}\ \sqrt{-iz}\ \cancel{\eta(z)} = \sqrt{r_2}\, e^{i\theta_2/2}\, e^{\pi i/4}$$

$$= \sqrt{-i(-dz+c)}\ e^{\pi i/4}$$

&

$$\eta(ASz) = \varepsilon(A\cancel{S})\ \sqrt{-i(-dz+c)}\ e^{\pi i/4}\,\eta(z)$$

$$= \varepsilon(AS)\sqrt{-i(-dz+c)}\ \eta(z). \quad \square$$

$$\eta(z) = e^{2\pi i z/24} \prod_{n=1}^{\infty} (1-q^n)$$

NOTE: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ & $c = 0$

then $ad = 1$ & $A = T^m$ &

$$\eta(Az) = \eta(T^m z) = e^{\pi i m/12} \eta(A).$$

Theorem:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $c \geq 0$

(1) If $c = 0$ then $\eta(Az) = e^{\pi i m^2/12} \eta(A)$.

(2) If $c > 0$ then $\eta(Az) = \varepsilon(A)\sqrt{-i(cz+d)}\; \eta(z)$

where $\varepsilon(A) = \exp\left\{\pi i \left(\frac{a+d}{12c} - s(d,c)\right)\right\}$.

Proof: Every element of $A \in \Gamma$ can be written as

$$A = ST^{m_1} S \cdots ST^{m_k}$$

for some integers $m_1, \ldots, m_k$. We proceed by induction on k.

$\boxed{k=1}$ We know

$$\eta(Sz) = \sqrt{-iz}\; \eta(z), \text{ \& and result holds.}$$

$$ST^{m_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & m_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & m_1 \end{pmatrix}$$

if $m_1 \neq 0$ the result holds by Lemma 3.

if $m_1 = 0$ result holds since $ST^{m_1} = S$.

Suppose result holds for $k=n$ (some $n \geq 1$).

Let

$$A = \underbrace{ST^{m_1}S \cdots ST^{m_k}}_{B} S T^{m_{k+1}}$$

Let $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

Case 1 $c'=0$. Then $B = T^{m}$ some m.

$$\eta(B\tau) = e^{\pi i m/12}\eta(\tau).$$

$$\eta(B(S\tau)) = e^{\pi i m/12}\eta(S\tau)$$

$$= e^{\pi i m/12}\sqrt{-i\tau}\;\eta(\tau)$$

$$T^m S = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}$$

$$\varepsilon(T^m S) = \exp\left(\pi i\left(\frac{m}{12} + 0\right)\right) = e^{\pi i m/12}.$$

So $\eta(BS\tau) = \varepsilon(BS)\,\sqrt{-i\tau}\;\eta(\tau)$

$\cancel{T^m S T^{m'} = \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & m' \\ 0 & 1 \end{pmatrix}}$

& result holds for BS. By Lemma 3 it holds for $BST^{m_{k+1}}$.

Case 2. $c' > 0$ $d' \neq 0$.
Result holds.

Case 3. $c' > 0$ $d' = 0$.

$$B = \begin{pmatrix} a' & b' \\ c' & 0 \end{pmatrix} \quad \& \quad c'b' = -1, \; c' = 1, \; b' = -1$$

$$= \begin{pmatrix} a' & -1 \\ 1 & 0 \end{pmatrix} = T^{a'}S, \quad A = T^{a'}S^2T^{m_{k+1}}$$

$$= T^{a'+m_{k+1}}$$

Congruence Properties of Dedekind Sums

Thm Let $k>0$, $(h,k)=1$. Then

$$6k\, s(h,k) \in \mathbb{Z}.$$

If $\theta = (3,k)$ then

(a) $12hk\, s(h,k) \equiv 0 \pmod{\theta k}$, (if $h>0$)

(b) $12hk\, s(h,k) \equiv h^2+1 \pmod{\theta k}$.

Proof:

$$6k\, s(h,k) = 6k \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$$

$$= \frac{6h}{k}\sum_{r=1}^{k-1} r^2 - 6\sum_{r=1}^{k-1} r\left[\frac{hr}{k}\right] - 3\sum_{r=1}^{k-1} r$$

$$\frac{6h}{k}\sum_{r=1}^{k-1} r^2 = \frac{6h}{k}\cdot\frac{1}{6}\cdot k(k-1)(2k-1) \in \mathbb{Z} \quad \&$$

$$6k\, s(h,k) \in \mathbb{Z}.$$

also $6k\, s(h,k) \equiv h(k-1)(2k-1) \pmod 3$,
and

$$12k\, s(h,k) \equiv h(k-1)(4k-2)$$
$$\equiv h(k-1)(k+1) \pmod 3.$$

If $3 \mid k$ then $3 \nmid h$ & $12k\, s(h,k) \equiv -h \not\equiv 0 \pmod 3$.
If $3 \nmid k$ then $3 \mid (k-1)(k+1)$ &
$$12k\, s(h,k) \equiv 0 \pmod 3$$
Hence $12k\, s(h,k) \equiv 0 \pmod 3$ iff $3 \nmid k$
Thus
$$12h\, s(k,h) \equiv 0 \pmod 3 \text{ iff } 3 \nmid h$$

If $3 = \theta = (3,k)$ then $3 \mid k$ & $3 \nmid h$ since $(h,k)=1$
&

$$12hk\, s(k,h) \equiv 0 \pmod{3k} \equiv 0 \pmod{\theta k}.$$

If $\theta = 1 = (3,k)$

then $$12hk\, s(k,h) = k(12h\, s(k,h)) \equiv 0 \pmod{k}$$
$$\equiv 0 \pmod{\theta k}.$$

Hence (a) holds in all cases.

$\boxed{\text{Suppose } h > 0}$.

$$12hk\, s(h,k) + \overbrace{12hk\, s(k,h)}^{\equiv 0 \pmod{\theta k}} = h^2 + k^2 - 3hk + 1$$
$$= h^2 + 1 + k(k-3h).$$

If $3 \mid k$, $\theta = (3,k) = 3$ &
$$k(k-3h) \equiv 0 \pmod{\theta k}.$$

If $3 \nmid k$, $\theta = (3,k) = 1$ &
$$k(k-3h) \equiv 0 \pmod{\theta k}.$$

Hence
$$12hk\, s(h,k) \equiv h^2 + 1 \pmod{\theta k}.$$

$\boxed{\text{Suppose } h < 0}$

$$-12kh\, s(-h,k) - 12hk\, s(k,-h)$$
$$= h^2 + k^2 + 3hk + 1$$

$$12kh\, s(h,k) - 12hk\, s(k,-h) = h^2 + k^2 + 3hk + 1$$

Similarly, $$12hk\, s(h,k) \equiv h^2 + 1 \pmod{\theta k}. \quad \square$$

<u>Theorem</u>

(1) $$12k\, s(h,k) \equiv (k-1)(k+2) - 4h(k-1) + 4\sum_{r<k/2}\left[\frac{2hr}{k}\right] \pmod 8$$

<u>Proof:</u>

$$12k\, s(h,k) = 2h(k-1)(2k-1) - 12\sum_{r=1}^{k-1} r\left[\frac{hr}{k}\right]$$

$$-3\,k(k-1)$$

$$= -2h(k-1) + 4hk(k-1) - 12\sum_{r=1}^{k-1} r\left[\frac{hr}{k}\right]$$

$$+ k(k-1) - 4k(k-1).$$

Hence

$$12k\, s(h,k) \equiv -2h(k-1) - 4\sum_{r=1}^{k-1} r\left[\frac{hr}{k}\right] + k(k-1) \pmod 8$$

$$\equiv (k-1)(k-2h) - 4\sum_{\substack{r=1 \\ r \text{ odd}}}^{k-1}\left[\frac{hr}{k}\right] \pmod 8$$

$$\left(\begin{matrix} r = 2m+1 \\ 4r \equiv 4 \pmod 8\end{matrix}\right)$$

$$\equiv (k-1)(k-2h) - 4\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right] + 4\sum_{2r<k}\left[\frac{2hr}{k}\right] \pmod 8$$

Now,

$$-4\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right] = 4\sum_{r=1}^{k-1}\left(\left(\frac{hr}{k}\right)\right) - 4\sum_{r=1}^{k-1}\frac{hr}{k} + 2\sum_{r=1}^{k-1} 1$$

Since $[x] = -((x)) + x - 1/2$ when $x \notin \mathbb{Z}$.

$$\sum_{r=1}^{k-1} \left(\left(\frac{hr}{k}\right)\right) = 0, \text{ and so}$$

$$-4\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right] = -\frac{4h}{k}\,\frac{k(k-1)}{2} + 2(k-1)$$

$$= (k-1)(2-2h).$$

$$12k\,s(h,k) \equiv (k-1)(k-2h) + (k-1)(2-2h)$$

$$+4\sum_{r<k/2}\left[\frac{2hr}{k}\right] \pmod 8$$

$$\equiv (k-1)(k+2) - 4h(k-1) + 4\sum_{r<k/2}\left[\frac{2hr}{k}\right] \pmod 8.$$

<u>Cor</u>. If k is odd, then

$$12k\,s(h,k) \equiv (k-1) + 4\sum_{r<k/2}\left[\frac{2hr}{k}\right] \pmod 8.$$

<u>Proof</u> Let k odd. $-4h(k-1) \equiv 0\,(8)$ &
$(k-1)(k+2) = k^2+k-2 \equiv 1+k-2 \equiv k-1 \pmod 8$. $\square$

<u>Theorem</u> If $k = 2^{\lambda}k_1$ where $\lambda > 0$ & k_1 is odd
then for $h \geq 1$ h odd we have

$$12hk\,s(h,k) \equiv h^2+k^2+1+5k$$

$$-4k\sum_{v<h/2}\left[\frac{2kv}{h}\right] \pmod{2^{\lambda+3}}$$

<u>Proof:</u> By Reciprocity,
$$12hk\,s(h,k) = h^2+k^2-3hk+1-12hk\,s(k,h).$$

Since h is odd,

$$12h\, s(k,h) \equiv (h-1) + 4\sum_{v<h/2}\left[\frac{2kv}{h}\right] \pmod 8,$$

and

$$12hk\, s(k,h) \equiv k(h-1) + 4k\sum_{v<h/2}\left[\frac{2kv}{h}\right] \pmod{2^{\lambda+3}}.$$

Hence

$$12hk\, s(h,k) \equiv h^2+k^2-3hk+1-k(h-1)$$

$$-4k\sum_{v<h/2}\left[\frac{2kv}{h}\right] \pmod{2^{\lambda+3}}$$

$$\equiv h^2+k^2+1+k-4hk-4k\sum_{v<h/2}\left[\frac{2kv}{h}\right] \pmod{2^{\lambda+3}}$$

Since h is odd, $4k(h+1) \equiv 0 (2^{\lambda+3})$
& $k-4hk \equiv 5k \pmod{2^{\lambda+3}}$
&

$$12hk\, s(h,k) \equiv h^2+k^2+1+5k-4k\sum_{v<h/2}\left[\frac{2kv}{h}\right] \pmod{2^{\lambda+3}}.$$

$\square$

<u>Theorem</u>

Let $p = 3, 5, 7$ or 13 & let $r = \dfrac{24}{p-1}$.

Given integers a,b,c,d with
$$ad-bc=1$$
such that $c = c_1 p$ with $c_1 > 0$, let

$$\delta = \left\{ s(a,c) - \left(\frac{a+d}{12c}\right)\right\} - \left\{ s(a,c_1) - \left(\frac{a+d}{12c_1}\right)\right\}.$$

Then $r\delta$ is an even integer.

<u>Proof</u>. By Thm 3.8(b) with $k=c$, we have

$$12ac\left(s(a,c) - \left(\frac{a+d}{12c}\right)\right) \equiv a^2+1-a(a+d)$$

$$\equiv a^2+1-a^2-ad \equiv -bc \pmod{\theta c}$$

where $\theta = (3,c)$. $\qquad ad-bc=1$

Similarly with $k = q = c/p$

$$12ac_1\left(s(a,c_1) - \left(\frac{a+d}{12c_1}\right)\right) \equiv a^2+1-a(a+d) \bmod (\theta_1 c_1)$$

$$12ac\left(s(a,c_1) - \left(\frac{a+d}{12c_1}\right)\right) \equiv p(a^2-a^2+1-ad) \bmod (\theta_1 c)$$

$$\equiv -pbc \pmod{\theta_1 c}$$

where $\theta_1 = (3,c_1)$, $\theta = (3,c_1p)$ & $\theta_1 | \theta$.
It follows that

(*) $$12acr\delta \equiv r(p-1)bc \pmod{\theta_1 c}.$$

But

$$r(p-1)bc \equiv 24bc \equiv 0 \pmod{\theta_1 c},$$

and

$$12acr\delta \equiv 0 \pmod{\theta_1 c},$$

$$(12c\delta)\cdot a \cdot r \equiv 0 \pmod{\theta_1 c}.$$

note that (1) $12c\delta$ is an integer (from Thm 3.8)
(2) $(a,c)=1$ since $ad-bc=1$
(3) $(a,\theta_1)=1$ since ~~[illegible]~~ if $\theta_1=3$, $3|c$ & $3\nmid a$.
& so (4) $(a,\theta_1 c)=1$ &

(**) $$(12c\delta)\cdot r \equiv 0 \pmod{\theta_1 c}.$$

Claim $\quad 12c\delta r \equiv 0 \pmod{3c}$ $\quad (***)$ $\quad$ 32

Case 1. $p > 3$. $\quad \theta = (3, c) = (3, c_1 p) = (3, c_1) = \theta_1$
and
$$12c\delta r \equiv 0 \pmod{\theta c}.$$
If $\theta = 3$ this gives result. If $\theta = 1$, then $3 \nmid c$ & $3 \nmid c_1$ (since $c = c_1 p$) &
$$12c\,s(a,c) \equiv 0 \pmod 3$$
$$\text{and } 12c_1\,s(a,c_1) \equiv 0 \pmod 3 \quad \text{(by (38))}$$
Hence
$$12cr\delta \equiv r\left(\frac{c}{c_1}(a+d) - (a+d)\right)$$
$$\equiv r(p-1)(a+d) \equiv 24(a+d) \equiv 0 \pmod 3$$
Here?

& by (*)
$$12cr\delta \equiv 0 \pmod{3c}.$$
Case 2. $p = 3$. $\quad r = 12$. $\quad \theta = (3, c) = (3, 3c_1) = 3$.

If $\theta_1 = 3$, (**) gives result.
If $\theta_1 = 1$, then $3 \nmid c_1$ & by (38)
$$12c_1\,s(a,c_1) \equiv 0 \pmod 3$$
and
$$12c\,s(a,c_1) \equiv 0 \pmod 9.$$
$$12c\delta = 12c\,s(a,c) - (a+d) - 12c\,s(a,c_1) + 3(a+d)$$
$$\equiv 12c\,s(a,c) + 2(a+d) \pmod 9.$$
~~Hence~~
$$12ac\delta r \equiv 12rac\,s(a,c) + 2r(a^2 + ad) \pmod 9$$
By Thm 3.8 (i)
$$12ac\,s(a,c) \equiv (a^2+1) \pmod{\theta c}, \quad \& \theta = 3$$
and
$$12ac\,s(a,c) \equiv a^2+1 \pmod 9 \quad \text{since } 3 \mid c$$
So
$$12ac\delta r \equiv r(a^2+1) + 2r(a^2+ad) \pmod 9$$
$$\equiv 3ra^2 + r + 2r(1+bc) \pmod 9$$
$$\equiv 3ra^2 + 3r + 2rbc \pmod 9$$
$$\equiv 0 \pmod 9$$
since $3 \mid r$ & $3 \mid c$.

Now $3 \nmid a$ since $(a,c) = 1$ ($\& 3 \mid c$)

So $(a, 9) = 1$ & $12c\delta r \equiv 0 \pmod 9$

$c = 3c_1$ & $3 \nmid c_1$

Hence $3c = 9c_1$ & $(9, c_1) = 1$.

By (**)

$$12c\,\delta r \equiv 0 \pmod{c_1} \qquad (\text{since } c_1 \mid c)$$

& $12c\,\delta r \equiv 0 \pmod 9$,

& so

$$12c\,\delta r \equiv 0 \pmod{3c}.$$

Claim $12c\,\delta r \equiv 0 \pmod{24c}$.

NOTE This implies δr is an even integer.

Case I c odd.

By Cor.

$$12c\,s(a,c) \equiv c - 1 + 4\,T(a,c) \pmod 8.$$

where $T(a,c) = \sum_{v < c/2} \left[\frac{2av}{c}\right]$

Hence

$$12c\left(s(a,c) - \left(\frac{a+d}{12c}\right)\right) \equiv c - 1 + 4\,T(a,c) - (a+d) \pmod 8.$$

p	r
3	12
5	6
7	4
13	2

Similarly,

$$12c_1\left(s(a,c_1) - \left(\frac{a+d}{c_1}\right)\right) \equiv c_1 - 1 + 4\,T(a,c_1) - (a+d) \pmod 8$$

$c = c_1 p$. &

$$12c\left(s(a,c_1) - \frac{a+d}{c_1}\right) \equiv c - p + 4p\,T(a,c_1) - p(a+d) \pmod 8.$$

$$12cr\delta \equiv r(p-1) + r(a+d)(p-1) \pmod 8$$

(since r is even & $4rT \equiv 0\ (8)$)

$$\equiv 0 \pmod 8$$

Since $r(p-1) = 24$.

Here we have

$$12c\,\delta r \equiv 0 \pmod{3c}$$

& $$12c\,\delta r \equiv 0 \pmod{8}$$

and hence

$$12c\,\delta r \equiv 0 \pmod{24c}$$

since c odd & $(3c, 8) = 1$.

<u>Case 2</u>. c even. write $c = 2^{\lambda}\gamma$ $(\lambda \geq 0, \gamma \text{ odd})$

a is odd since $(a,c) = 1$

assume $a \geq 1$. By Th (3.10)

$$\cancel{12ac\,s(a,c) \equiv a(c-1) + 4a}$$

$$12ac\,s(a,c) \equiv a^2 + c^2 + 1 + 5c - 4c\,T(c,a) \pmod{2^{\lambda+3}}$$

$$12ac\left(s(a,c) - \left(\frac{a+d}{12c}\right)\right)$$

$$\equiv a^2 + c^2 + 1 + 5c - 4c\,T(c,a) - a(a+d) \quad (2^{\lambda+3})$$

$$\equiv c^2 + 5c - 4c\,T(c,a) - bc \quad (2^{\lambda+3}) \qquad (ad - bc = 1,\ -bc = 1 - ad)$$

Similarly,

$c = p\,c_1$ $(p \text{ odd})$ so $c_1 = 2^{\lambda}\gamma'$ $(\gamma' \text{ odd})$

&

$$12ac_1\left(s(a,c_1) - \left(\frac{a+d}{12c_1}\right)\right)$$

$$\cancel{\equiv c_1^2 + 5c_1 - 4c_1\,T(c_1,a)}$$

$$\equiv a^2 + c_1^2 + 1 + 5c_1 - 4c_1\,T(c_1,a) - a(a+d) \quad (2^{\lambda+3})$$

$$\equiv c_1^2 + 5c_1 - 4c_1\,T(c_1,a) - bc \quad (2^{\lambda+3})$$

$c = pc_1 \quad c^2 = pc_1 c$

So

$12ac\left(s(a,c_1) - \left(\frac{a+d}{12c_1}\right)\right)$

$\equiv c_1 c + 5c - 4\overset{c}{\cdot} T(c_1,a) - pbc \quad (2^{\lambda+3})$

r is even so $4cr \equiv 0 \pmod{2^{\lambda+3}}$

&

$12car\delta \equiv r\,cc_1(p-1) + r(p-1)bc \quad (2^{\lambda+3})$

$\equiv 24c(c_1 + b) \quad (2^{\lambda+3})$

$\equiv 0 \pmod{2^{\lambda+3}}.$

Therefore,

$12car\delta \equiv 0 \pmod{2^{\lambda+3}}$ since a is odd.

By (***) $\quad 12cr\delta \equiv 0 \pmod{3.\underbrace{2^{\lambda}.\gamma}_{c}}$ $\quad \gamma$ odd

& $12cr\delta \equiv 0 \pmod{3.\gamma}$

& hence

$12cr\delta \equiv 0 \pmod{3.2^{\lambda+3}.\gamma}$

since $(2^{\lambda+3}, 3\gamma) = 1$

ie $12cr\delta \equiv 0 \pmod{24c}$.

Now assume $a < 0$.

Choose $t \geq 1$: $a' = a + tc > 0$.

So $s(a',c) = s(a,c)$ & $s(a',c_1) = s(a,c_1)$.

~~Let $\delta' = \ldots$~~

$ad - bc = 1 \quad d(a' - tc) - bc = 1$

$a'd - \cancel{bc} - tcd = 1$

$a'd - \underbrace{(b+td)}_{b'}\,c = 1$

~~So~~ $r\delta'$ is even where

$$\delta' = \left(s(a',c) - \frac{a'+d}{12c}\right) - \left(s(a',c_1) - \frac{a'+d}{12c_1}\right)$$

$$= \left(s(a,c) - \frac{(a+tc+d)}{12c}\right) - \left(s(a,c_1) - \left(\frac{a+tc+d}{12c_1}\right)\right)$$

$$= \delta + \frac{tc}{12c_1} - \frac{tc}{12c}$$

$$= \delta + \frac{t}{12}(p-1)$$

$$r\delta' = r\delta + t\,\frac{r(p-1)}{12} = r\delta + 2t$$

& $r\delta$ is even. $\square$.

SUPPLEMENT

<u>Jacobi's Triple Product Identity</u> Suppose $|q|<1$ and $z \neq 0$. Then

$$\prod_{n=1}^{\infty}(1-zq^{n})(1-z^{-1}q^{n-1})(1-q^{n})$$
$$= \sum_{m=-\infty}^{\infty} (-1)^m z^m q^{m(m+1)/2}$$

<u>Proof</u>: Let $L(z) = \prod_{n=1}^{\infty}(1-zq^{n})(1-z^{-1}q^{n-1})(1-q^{n})$

$$L(zq) = \prod_{n=1}^{\infty}(1-zq^{n+1})(1-z^{-1}q^{n-2})(1-q^{n})$$
$$= (1-zq^2)(1-zq^3)\cdots$$
$$(1-z^{-1}q^{-1})(1-z^{-1})(1-z^{-1}q)\cdots$$
$$(1-q)(1-q^2)(1-q^3)\cdots$$
$$= \frac{(1-z^{-1}q^{-1})}{(1-zq)} L(z)$$
$$= -z^{-1}q^{-1}\frac{(1-zq)}{(1-zq)} L(z) = -z^{-1}q^{-1}L(z).$$

$L(z)$ has a Laurent expansion

$$L(z) = \sum_{m=-\infty}^{\infty} l_m(q) z^m \qquad 0<|z|<\infty$$

$$L(zq) = \sum_{m=-\infty}^{\infty} l_m(q) z^m q^m$$

$$-z^{-1}q^{-1}L(z) = \sum_{m=-\infty}^{\infty} l_m(q)(-q^{-1}) z^{m-1}$$
$$= -q^{-1}\sum_{m=-\infty}^{\infty} l_{m+1}(q) z^m$$

So $l_m(q) q^m = -q^{-1} l_{m+1}(q)$

So $l_{m+1}(q) = -q^{m+1} l_m(q)$ for all m (2)

$l_m(q) = -q^{m} l_{m-1}(q)$

$l_m(q) = (-q^{m})(-q^{m-2})(\,)\cdots(-q^{1})\,l_0(q)$

$= (-1)^m q^{m(m+1)/2} l_0(q)$ for $m \geq 0$.

$l_0(q) = (-q^{0})\, l_{-1}(q)$

$= (-q^{0})(-q^{-2})\cdots(-q^{-m+1})\, l_{-m}(q)$

$l_{-m}(q) = (-1)^m q^{m(m-1)/2} l_0(q)$

$= (-1)^{-m} q^{(-m)(-m+1)/2} l_0(q)$.

So

$l_m(q) = (-1)^m q^{m(m+1)/2} l_0(q)$ for all m

Need to show $l_0(q) = 1$

$$\prod_{n=1}^{\infty}(1 + zq^n)(1+z^{-1}q^{n-1}) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{-\infty}^{\infty} (-z)^m l_m(q).$$

Coeff of $z^0 q^N$ = # of pairs of integer strings

$n_1 > n_2 > \cdots > n_k \geq 1$

$m_1 > m_2 > \cdots > m_k \geq 0$

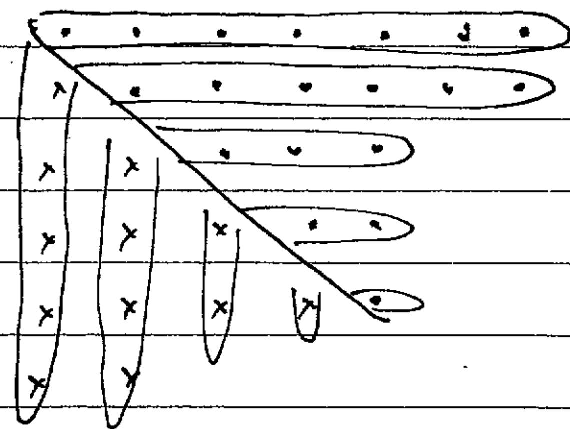
such that $n_1 + \cdots + n_k + m_1 + \cdots + m_k = N$

Claim Coeff of $z^0 q^N = p(N)$ the number of partitions of N.

Example:

$7 \geq 6 \geq 3 \geq 2 \geq 1$

$5 \geq 4 \geq 2 \geq 1 \geq 0$



This corresponds to the partition

$7+7+5+5+5+2$

So $$\frac{l_0(q)}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{N \geq 0}^{\infty} p(N) q^N = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

$$l_0(q) = 1$$

Corollary: $$\prod_{n=1}^{\infty}(1-q^n) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}.$$

Proof In J.T.P let $q \to q^3$, $z = q^{-1}$,

$$\prod_{n=1}^{\infty}(1-q^{3n-1})(1-q^{3n-3})(1-q^{3n})$$

$$= \sum_m (-1)^m q^{3m(m+1)/2 - m} = \sum_m (-1)^m q^{m(3m-1)/2}$$

Corollary: $$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2}.$$

Proof:
$$\prod_{n=1}^{\infty}(1-zq^{n-1})(1-z^{-1}q^n)(1-q^n)$$
$$= \sum_{m=-\infty}^{\infty}(-1)^m z^{-m} q^{m(m+1)/2}.$$
$$= \sum_{m=0}^{\infty}(-1)^m z^{-m} q^{m(m+1)/2} + \sum_{m=-1}^{-\infty}(-1)^m z^{-m} q^{m(m+1)/2}$$

let $m=-n-1$
ie $n=-m-1$

$$= \sum_{m=0}^{\infty}(-1)^m z^{-m} q^{m(m+1)/2} + \sum_{n=0}^{\infty}(-1)^{-n-1} z^{n+1} q^{(-n-1)(-n)/2}$$
$$= \sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2}(z^{-m} - z^{m+1})$$
$$= \sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2} z^{-m}(1-z^{2m+1})$$

$$\prod_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^n)(1-q^n)$$
$$= \sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2} z^{-m}(1+z+\cdots+z^{2m})$$

Then let $z=1$. $\square$