Chapter 4 Congruences for the coefficients of the modular function j

Let $j(\tau) = 12^3 J(\tau) = 12^3 \dfrac{g_2^3(1,\tau)}{\Delta(1,\tau)}$.

Then

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} c(n) q^n \qquad (q = e^{2\pi i \tau})$$

where $c(n)$ are integers. (See Thm 1.20 p. 21)

We will prove
$$c(2n) \equiv 0 \pmod{2^{11}}$$
$$c(3n) \equiv 0 \pmod{3^5}$$
$$c(5n) \equiv 0 \pmod{5^2}$$
$$c(7n) \equiv 0 \pmod{7^1}.$$
for $n \geq 1$.

The subgroup $\Gamma_0(N)$

Let N be any positive integer.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

Thm: Let $S\tau = -1/\tau$, $T\tau = \tau + 1$ be the (usual) generators of Γ.
Let p be any prime.
Then for every $V \in \Gamma$, $V \notin \Gamma_0(p)$
$\exists\, P \in \Gamma_0(p)$ and an integer k, $0 \leq k < p$
such that

$$V = P\, S T^k.$$

Note: $[\Gamma : \Gamma_0(p)] = p+1$ The right cosets
$$\Gamma = \Gamma_0(p) \cup \bigcup_{k=0}^{p-1} \Gamma_0(p)\, S T^k$$

<u>Proof:</u> Suppose we are given

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad C \not\equiv 0 \pmod{p}.$$

We wish to find $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$ $(c \equiv 0 \pmod{p})$ and an integer k, $0 \le k < p$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ST^k$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} kA - B & A \\ kC - D & C \end{pmatrix}$$

We want $kC \equiv D \pmod{p}$. This has a soln since $C \not\equiv 0 \pmod{p}$ &

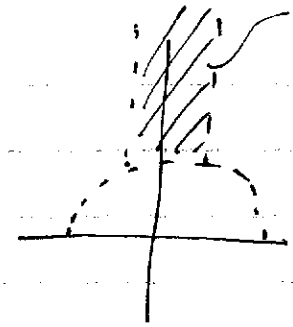
$$k \equiv C^{-1} D \pmod{p}.$$

Choose such a k, $0 \le k < p$, then let

$c = kC - D$, $a = kA - B$, $b = A$, $d = C$.

Note $ST^k = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix} \notin \Gamma_0(p)$.

<u>EX</u> Show cosets are distinct. $\square$



Fundamental Region of $\Gamma_0(p)$

Thm Let p be prime, then

$$R = R_r \cup \bigcup_{k=0}^{p-1} ST^k(R_r)$$

is a fundamental region of $\Gamma_0(p)$.

Proof: Let R denote the set

$$R = R_r \cup \bigcup_{k=0}^{p-1} ST^k(R_r).$$

We show that

(i) $\tau \in H$, then $\exists V \in \Gamma_0(p)$ such that $V\tau \in \overline{R}$.

(ii) no two distinct points of R are ~~equivent~~ equivalent under $\Gamma_0(p)$.

(i) Let $\tau \in H$, choose $\tau_1 \in \overline{R_r}$, $A \in \Gamma$ s.t.

$$A\tau = \tau_1.$$

$$\Gamma = \Gamma_0(p) \cup \bigcup_{k=0}^{p-1} \Gamma_0(p) ST^k.$$

Then

$$A^{-1} = PW \quad \text{where} \quad P \in \Gamma_0(p)$$

& $W = I$, or ST^k, $0 \leq k < p-1$.

Then
$$P = A^{-1}W^{-1},$$
$$P^{-1} = WA,$$

$\overline{A \cup B} = \overline{A} \cup \overline{B}$

$f(\overline{A}) \subset \overline{f(A)}$

f cts

let
$$V = P^{-1} \in \Gamma_0(p) \text{ and}$$
$$V\tau = WA\tau = W\tau_1 \in \overline{R_r} \text{ or } ST^k(\overline{R_r}) \text{ } 0 \leq k \leq p-1$$

so $V\tau \in \overline{R}$.

(ii) Suppose $\tau_1, \tau_2 \in R$ $V\tau_1 = \tau_2$, $V \in \Gamma_0(p)$.

<u>Case 1</u> $\tau_1, \tau_2 \in R_\Gamma \implies \tau_1 = \tau_2$ since $V \in \Gamma_0(p) \subset \Gamma$.

<u>Case 2</u> $\tau_1 \in R_\Gamma \quad \tau_2 \in ST^k(R_\Gamma)$

$$\tau_2 = ST^k \tau_3, \quad \tau_3 \in R_\Gamma$$

$$V\tau_1 = \tau_2 = ST^k\tau_3$$

$$\tau_1 = V^{-1}ST^k\tau_3$$

$\implies V^{-1}ST^k = I$ since $\tau_1, \tau_3 \in R_\Gamma$

$$\implies V = ST^k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix} \notin \Gamma_0(p) \ \#$$

<u>Case 3</u> $\tau_1 \in (ST^{k_1})(R_\Gamma), \ \tau_2 \in (ST^{k_2})(R_\Gamma)$

$\tau_1 = ST^{k_1}\tau_1', \quad \tau_2 = ST^{k_2}\tau_2',$

$\tau_1', \tau_2' \in R_\Gamma.$

$$V\tau_1 = \tau_2$$

$$VST^{k_1}\tau_1' = ST^{k_2}\tau_2'$$

$$\implies \cancel{VST^{k_1}} \ (ST^{k_2})^{-1}VST^{k}\tau_1' = \tau_2'$$

$$\implies (ST^{k_2})^{-1}VST^{k_1} = I$$

$$\implies V\,ST^{k_1} = ST^{k_2}$$

$$VS = ST^{k_2-k_1}$$

$$V = ST^{k_2-k_1}S$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & k_2-k_1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} & \\ k_2-k_1 & \end{pmatrix} \in \Gamma_0(p)$$

$\implies k_2 \equiv k_1 \pmod p$

$\implies k_1 = k_1$ since $0 \le k_i < p$.

$\Longrightarrow \quad V = SIS = S^2 = I$

& $\quad \tau_1 = \tau_2. \quad \square$

<u>Modular functions for $\Gamma_0(p)$</u> (automorphic function under $\Gamma_0(p)$)

f is <u>automorphic under $\Gamma_0(p)$</u> if

(a) f is meromorphic on H

(b) $f(Az) = f(z)$ for all $z \in H$, $A \in \Gamma_0(p)$

(c)
$$f(z) = \sum_{n=-m}^{\infty} a_n q^n \qquad (q = e^{2\pi i z})$$
for $|q| < \delta$.

$*$ (d) $f\left(\frac{-1}{pz}\right) = \sum_{n=-k} b_n q^n \quad (|q| < \delta')$

<u>Theorem:</u> f automorphic under $\Gamma_0(p)$ and bounded on H then f is a constant.

Let $A_k = ST^k$, $A_p = I$

Then let $T_k = \Gamma_0(p)(ST^k)$, $0 \le k \le p-1$,
$T_p = \Gamma_0(p)$.

$\boxed{T_k = \Gamma_0(p) A_k \text{ where } A_k = \begin{cases} ST^k & k<p \\ I & k=p \end{cases}}$

Then T_k, $0 \le k \le p$ are the right cosets of $\Gamma_0(p)$ (in Γ).

For each k, choose any $V_k \in T_k$ and define
$$f_k(\tau) = f(V_k \tau).$$

$$f_p(\tau) = f(\tau) \qquad \text{can take } V_p = I \in \Gamma_0(p).$$

<u>Note</u> The value $f_k(\tau)$ does not depend on elt V_k chosen in T_k.

[Suppose $A, B \in T_k$. Then $A = P_1 A_k$ $\quad B = P_2 A_k$

$P_1^{-1} A = P_2^{-1} B$

$A = P B \qquad P \in \Gamma_0(p)$

$$f(A\tau) = f(PB\tau) = f(B\tau) \quad \text{since } P \in \Gamma_0(p).]$$

Let $V \in \Gamma$, then

$$f_k(V\tau) = f(V_k V\tau)$$

$V_k V \in \Gamma \Rightarrow V_k V \in T_m$ some m

$$V_k V = Q A_m \qquad Q \in \Gamma_0(p).$$

$$f_k(V\tau) = f(V_k V\tau) = f(Q A_m \tau)$$
$$= f(A_m \tau) \qquad \text{since } Q \in \Gamma_0(p)$$

$$= f_m(\tau) \qquad \text{since } A_m \in T_m.$$

For $k = 0, 1, \dots, p$ $V_k V$ are in distinct cosets ($\because V_k V = P V_{k'} V \Rightarrow V_k = P V_{k'} \Rightarrow k = k'$).
As k runs thro integers $0, 1, 2, \dots, p$ so does m.
Hence V induces a permutation σ of $\{0, 1, 2, \dots, p\}$ s.t.

$$f_k(V\tau) = f_{\sigma(k)}(\tau)$$

Let $w \in H$.

$$\varphi(\tau) = \prod_{k=0}^{p} \left(f_k(\tau) - f(w) \right)$$

$$\varphi(V\tau) = \prod_{k=0}^{p} \left(f_k(V\tau) - f(w) \right)$$

$$= \prod_k \left(f_{\sigma(k)}(\tau) - f(w) \right) = \varphi(\tau)$$

so $\varphi(\tau)$ is modular on Γ & bdd

So φ is constant. But $\varphi(w) = 0$.

Hence $\varphi(\tau) \equiv 0$.

$$\varphi(i) = 0 = \prod_{k=0}^{p} \left(f_k(i) - f(w)\right)$$

$\Rightarrow$ f only takes the values $f_0(i), \ldots, f_p(i)$

~~$\Rightarrow$ f finite values so f must be cons~~

$\Rightarrow$ f is a constant (since f is holomorphic). $\square$

Construction of functions for $\Gamma_0(p)$.

Let $f(\tau)$ be a modular function for Γ.

Let p be a prime. Define

$$f_p(\tau) = \frac{1}{p} \sum_{\lambda=0}^{p-1} f\left(\frac{\tau+\lambda}{p}\right)$$

(Note: Some books define an operator $U_p(f) = f_p(\tau)$.)

Suppose

$$f(\tau) = \sum_{n=-m}^{\infty} a(n)\, q^n \qquad (|q| < \delta)$$

$$= \sum_{n=-m}^{\infty} a(n)\, e^{2\pi i n \tau}$$

$$f\left(\frac{\tau+\lambda}{p}\right) = \sum_{n=-m}^{\infty} a(n)\, e^{2\pi i n\left(\frac{\tau+\lambda}{p}\right)}$$

$$= \sum_{n=-n}^{\infty} a(n)\, e^{\frac{2\pi i n\lambda}{p}}\, e^{2\pi i n(\tau/p)}$$

$$= \sum_{n=-m}^{\infty} a(n)\, \zeta^{n\lambda}\, e^{2\pi i n(\tau/p)}$$

where $\zeta = e^{2\pi i/p}$ (primitive p-th root of unity)

$$\zeta^p = 1, \quad \zeta^p - 1 = (\zeta - 1)(\zeta^{p-1} + \zeta^{p-2} + \cdots + \zeta + 1)$$

So $1 + \zeta + \cdots + \zeta^{p-1} = 0$.

In fact,

$$\sum_{k=0}^{p-1} \zeta^{mk} = \begin{cases} 0 & \text{if } m \not\equiv 0 \pmod{p} \\ p & \text{if } m \equiv 0 \pmod{p}. \end{cases}$$

$$f_p(\tau) = \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(\frac{\tau+\lambda}{p}\right)$$

$$= \frac{1}{p}\sum_{\lambda=0}^{p-1}\sum_{n=-m}^{\infty} a(n)\, \zeta^{n\lambda} e^{2\pi i n(\tau/p)}$$

$$= \sum_{n=-m}^{\infty}\left(\frac{1}{p}\sum_{\lambda=0}^{p-1} \zeta^{n\lambda}\right) a(n)\, e^{2\pi i n(\tau/p)}$$

$$= \sum_{\substack{n=-m \\ n\equiv 0 \pmod p}}^{\infty} a(n)\, e^{2\pi i n(\tau/p)}$$

$$= \sum_{n=-\left[\frac{m}{p}\right]}^{\infty} a(np)\, e^{2\pi i n\tau} \qquad (np \geq -m$$

$$= \sum_{n=-\left[\frac{m}{p}\right]}^{\infty} a(np)\, q^{n}.$$

Move this next to $f_p(-\frac{1}{p\tau})$ part to finish

Theorem: If f is a modular function for Γ then f_p is a modular function for $\Gamma_0(p)$.

Proof: $f_p(\tau)$ has required q-expansion and is clearly meromorphic on H.

We need

Lemma: Let $V \in \Gamma_0(p)$, $0 \leq \lambda \leq p-1$,

$$T_\lambda \tau = \left(\frac{\tau+\lambda}{p}\right) \leftrightarrow \begin{pmatrix} 1 & \lambda \\ 0 & p \end{pmatrix}$$

Then $\exists$ an integer μ, $0 \leq \mu \leq p-1$ and a $W_\mu \in \Gamma_0(p^2)$ such that $T_\lambda V = W_\mu T_\mu$

Moreover, as λ runs thro a complete residue system $\pmod p$ so does μ.

__Note:__ This means

$$\cancel{f_p(N\tau)}\ f_p(V\tau) = \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(\frac{V\tau+\lambda}{p}\right)$$

$$= \frac{1}{p}\sum_{\lambda=0}^{p-1} f\left(T_\lambda(V\tau)\right)$$

$$= \frac{1}{p}\sum_{\mu=0}^{p-1} f\left(W_\mu(T_\mu \tau)\right)$$

$$= \frac{1}{p}\sum_{\mu=0}^{p-1} f(T_\mu \tau) \qquad \text{since } W_\mu \in \Gamma_0(p^2) \subset \Gamma.$$

$$\cancel{= \frac{1}{p}\sum_{\mu=0}^{p-1} f(T_\mu \tau)}$$

$$= f_p(\tau) \qquad \text{for } V \in \Gamma_0(p)$$

and f_p is a modular fn for $\Gamma_0(p)$.

__Proof of Lemma:__

Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c \equiv 0 \pmod p$.

We want $W_\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(p^2)$

s.t.

$$T_\lambda V = W_\mu T_\mu$$

ie

$$\begin{pmatrix} 1 & \lambda \\ 0 & p \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & p \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & p \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\mu/p \\ 0 & 1/p \end{pmatrix}$$

$$= \begin{pmatrix} a+\lambda c & b+\lambda d \\ pc & pd \end{pmatrix} \begin{pmatrix} 1 & -\mu/p \\ 0 & 1/p \end{pmatrix}$$

$$= \begin{pmatrix} a+\lambda c & \dfrac{-a\mu-\lambda c\mu+b+\lambda d}{p} \\ pc & -c\mu+d \end{pmatrix}$$

We want

$$a\mu+\lambda c\mu \equiv b+\lambda d \mod p$$

$\underset{=}{\text{ie}}$ $\mu(a+\lambda c) \equiv b+\lambda d \mod p$

$\underset{=}{\text{ie}}$ $\mu a \equiv b+\lambda d \mod p$ (since $c\equiv 0 \pmod p$)

($ad-bc=1 \Rightarrow ad \equiv 1 \pmod p$ since $c\equiv 0 \pmod p$)

infact $\mu \equiv bd+\lambda d^2 \pmod p$

It follows that $W_\mu \in \Gamma_0(p^2)$ since $pc \equiv 0 \pmod{p^2}$

$\det(W_\mu)=+1$ & entries of W_μ are integers.

Clearly μ runs thru a complete residue system as λ does since $d \not\equiv 0 \pmod p$, $\square$

and $a \not\equiv 0 \pmod p$.

<u>The behavior of $f_p(z)$ under the generators of Γ</u>

The generators of Γ are $Tz=z+1$, $T\in\Gamma_0(p)$

and $Sz=-1/z$.

Since $T\in\Gamma_0(p)$, $f_p(Tz)=f_p(z)$.

Lemma

Let $T_\lambda \tau = \dfrac{\tau+\lambda}{p}$, $\quad 1 \le \lambda \le p-1$.

For each such λ, $\exists\ \mu = \mu(\lambda)$ and a transf $V = V_\mu \in \Gamma_0(p)$ s.th.

$$T_\lambda S = V T_\mu .$$

Moreover, as λ runs thro $1, \dots, p-1 \pmod p$ so does μ.

Proof. We want $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t.

$$\begin{pmatrix} 1 & \lambda \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \mu \\ 0 & p \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\mu/p \\ 0 & 1/p \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & -1 \\ p & 0 \end{pmatrix}\begin{pmatrix} 1 & -\mu/p \\ 0 & 1/p \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & \frac{-\mu\lambda-1}{p} \\ p & -\mu \end{pmatrix}$$

Choose We want $\quad \lambda\mu \equiv -1 \pmod p$

This has a unique soln since μ has a mult. inverse mod p. Clearly as λ runs thro a reduced residue system so does μ. $\square$

(13)

__Theorem__. Suppose f is a modular funct for Γ and p is a prime, then

$$f_p\left(-\frac{1}{\tau}\right) = f_p(\tau) + \frac{1}{p} f(p\tau) - \frac{1}{p} f\left(\frac{\tau}{p}\right)$$

__Proof__:

$$p f_p(-1/\tau) = p f_p(S\tau)$$

$$= \sum_{\lambda=0}^{p-1} f\left(\frac{S\tau+\lambda}{p}\right) = f\left(\frac{S\tau}{p}\right) + \sum_{\lambda=1}^{p-1} f(T_\lambda S\tau)$$

$$= f\left(\frac{-1}{\tau p}\right) + \sum_{\mu=1}^{p-1} f(V_\mu T_\mu \tau)$$

$$= f\left(-\frac{1}{\tau p}\right) + \sum_{\mu=1}^{p-1} f(T_\mu \tau) \qquad \text{since } V_\mu \in \Gamma_0(p) \subset \Gamma$$

$$f\left(-\frac{1}{\tau}\right) = f(\tau)$$

$$\text{so } f\left(\frac{-1}{\tau p}\right) = f(p\tau)$$

$$= f(p\tau) + \sum_{\mu=0}^{p-1} f(T_\mu \tau) - f(\tau/p)$$

$$= f(p\tau) + p f_p(\tau) - f(\tau/p)$$

$$f_p(-1/\tau) = f_p(\tau) + \frac{1}{p} f(p\tau) - \frac{1}{p} f(\tau/p). \quad \square$$

<u>The function $\varphi(\tau) = \dfrac{\Delta(N\tau)}{\Delta(\tau)}$</u>

<u>Theorem</u>: For a fixed integer N, let

$$\varphi(\tau) = \frac{\Delta(N\tau)}{\Delta(\tau)}, \quad \text{for } \tau \in H.$$

Then φ is a ~~modular~~ automorphic function for $\Gamma_0(N)$. Moreover its Fourier expansion has the form

$$\varphi(\tau) = q^{N-1}\Big(1 + \sum_{n=1}^{\infty} b_n q^n\Big), \quad q = e^{2\pi i \tau},$$

where the b_n are integers.

<u>Proof:</u> $\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) q^n$

$$\Delta(N\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) q^{Nn}$$

$$\varphi(\tau) = \frac{\Delta(N\tau)}{\Delta(\tau)} = \frac{q^N + \tau(2) q^{2N} + \cdots}{q + \tau(2) q^2 + \cdots}$$

$$= q^{N-1} \frac{(1 + \tau(2) q^N + \cdots)}{(1 + \tau(2) q + \cdots)}$$

$$= q^{N-1} \sum_{n=1}^{\infty} b_n q^n \quad \text{where } b_n \text{ are integers since the } \tau(n) \text{ are integers.}$$

$\varphi(\tau)$ is clearly meromorphic on H since $\Delta(\tau)$ is analytic on H. Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, so $N \mid c$.

$$\varphi(V\tau) = \frac{\Delta(NV\tau)}{\Delta(V\tau)}$$

$$\Delta(V\tau) = (c\tau + d)^{12} \Delta(\tau) \quad \text{(by a previous result.)}$$

$$NV\tau = N\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$= \frac{a(N\tau)+bN}{c\tau+d}$$

$$= \frac{a(N\tau)+bN}{\frac{c}{N}(N\tau)+d}$$

$= W(N\tau)$ where $W = \begin{pmatrix} a & bN \\ \frac{c}{N} & d \end{pmatrix} \in \Gamma$

because $N \mid c$ & $\det W = \det V = 1$.

So $\Delta(NV\tau) = \Delta(W(N\tau))$

$$= \left(\frac{c}{N}(N\tau)+d\right)^{12} \Delta(N\tau)$$

$$= (c\tau+d)^{12}\,\Delta(N\tau).$$

Hence $\varphi(V\tau) = \dfrac{\Delta(NV\tau)}{\Delta(V\tau)} = \dfrac{(c\tau+d)^{12}\,\Delta(N\tau)}{(c\tau+d)^{12}\,\Delta(\tau)}$

$$= \varphi(\tau),$$

and φ is a modular fcn for $\Gamma_0(N)$. $\square$

<u>NOTE</u>: $v_{i\infty}(\Delta) = 1$

$$v_{i\infty}(\varphi) = v_{i\infty}\left(\frac{\Delta(N\tau)}{\Delta(\tau)}\right) = N-1$$

φ has no zeros, or poles in H because $\Delta(\tau)$ does not.

<u>Theorem</u> :

$$\varphi\left(-\frac{1}{N\tau}\right) = \frac{1}{N^{12}\varphi(\tau)}$$

<u>Proof</u>: $\varphi\left(-\frac{1}{N\tau}\right) = \dfrac{\Delta\left(\frac{-N}{N\tau}\right)}{\Delta\left(-\frac{1}{N\tau}\right)} = \dfrac{\Delta\left(-\frac{1}{\tau}\right)}{\Delta\left(-\frac{1}{N\tau}\right)}$

$$\Delta\left(-\frac{1}{\tau}\right) = \tau^{12}\Delta(\tau)$$
$$\Delta\left(-\frac{1}{N\tau}\right) = N^{12}\tau^{12}\Delta(N\tau)$$

$$\varphi\left(-\frac{1}{N\tau}\right) = \frac{\tau^{12}\Delta(\tau)}{N^{12}\tau^{12}\Delta(N\tau)} = \frac{1}{N^{12}}\frac{1}{\varphi(\tau)}. \quad \square$$

φ has a zero at $i\infty$ so in some sense φ has a pole at 0.

<u>The univalent function $\Phi(\tau)$</u>

$$\varphi(\tau) = \frac{\Delta(N\tau)}{\Delta(\tau)} = \frac{q^{N-1}\prod_{n=1}^{\infty}(1-q^{Nn})^{24}}{\prod_{n=1}^{\infty}(1-q^{n})^{24}}$$

We seek a function that has no zeros in H and a zero of order 1 at $i\infty$ (and no zero ~~near~~ at 0).
If $N-1 \mid 24$, let $\alpha = \frac{1}{N-1}$, $r = 24\alpha = \frac{24}{N-1}$, so that $24 = (N-1)r$.

Define $\Phi(\tau) = \varphi^{\alpha}(\tau) = q\prod_{n=1}^{\infty}\dfrac{(1-q^{Nn})^{r}}{(1-q^{n})^{r}}$

$$= \left(\underline{\eta(N\tau)}\right)^{r}$$

Then $\Phi(\tau) = \Phi_N(\tau)$ is analytic on H, nonzero on H,

$v_{i\infty}(\Phi) = 1$.

<u>Theorem</u> For $N = 2, 3, 5, 7$ or 13, let

$$r = \frac{24}{N-1}.$$

N	2	3	5	7	13
r	24	12	6	4	2

Then the function $\Phi(\tau) = \left(\dfrac{\eta(N\tau)}{\eta(\tau)}\right)^r$

is a modular function for $\Gamma_0(N)$.

<u>Proof:</u> If $N=2$, $r=24$ & $\Phi(\tau) = \dfrac{\Delta(2\tau)}{\Delta(\tau)} = \varphi(\tau)$ which is a modular function for $\Gamma_0(2)$.

Assume $N \geq 3$. Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Then $ad - bc = 1$ & $c \equiv 0\ (N)$. We suppose w.l.o.g. $c \geq 0$.

If $c = 0$ then $ad = 1$ $a = d = \pm 1$ & $V = T^{\pm b}$.

~~[illegible]~~ $\eta(\tau+1) = \exp\left(\frac{\pi i}{12}\right)\eta(\tau)$, $\eta(\tau+m) = \exp\left(\frac{\pi i m}{12}\right)\eta(\tau)$

$$\Phi(\tau+1) = \left(\frac{\eta(N\tau+N)}{\eta(\tau+1)}\right)^r$$

$$= \exp\left(\frac{\pi i}{12}(N-1)r\right)\left(\frac{\eta(N\tau)}{\eta(\tau)}\right)^r$$

$$= \exp(2\pi i)\left(\frac{\eta(N\tau)}{\eta(\tau)}\right)^r = \Phi(\tau).$$

Assume $c > 0$ & $c = Nm$, $m > 0$.

$$\eta(V\tau) = \varepsilon(V)\left(-i(c\tau+d)\right)^{\frac{1}{2}}\eta(\tau)$$

where

$\varepsilon(V) = \exp\left(\pi i\left(\frac{a+d}{\ldots}\right.\right.$ [illegible]

$$\eta(NVz) = \eta\left(\frac{Naz+Nb}{cz+d}\right)$$

$$= \eta\left(\frac{a(Nz)+Nb}{m(Nz)+d}\right), \quad V_1 = \begin{pmatrix} a & Nb \\ m & d \end{pmatrix} \in \Gamma$$

$$= \varepsilon(V_1)\left\{-i\left(m(Nz)+d\right)\right\}^{\frac{1}{2}} \eta(Nz)$$

$$= \varepsilon(V_1)\left\{-i\,(cz+d)\right\}^{\frac{1}{2}} \eta(Nz)$$

where

$$\varepsilon(V_1) = \exp\left\{\pi i\left(\frac{a+d}{12m} + s(-d,m)\right)\right\}$$

So

$$\Phi(Vz) = \left(\frac{\eta(NVz)}{\eta(Vz)}\right)^r$$

$$= \left(\frac{\varepsilon(V_1)}{\varepsilon(V)}\right)^r \Phi(z)$$

and $\left(\frac{\varepsilon(V_1)}{\varepsilon(V)}\right)^r = e^{-\pi i r \delta}$ where

$$\delta = \left\{\frac{a+d}{12m} + s(-d,m)\right\} - \left\{\frac{a+d}{12c} + s(-d,c)\right\}$$

It can be shown that $r\delta$ is an even integer using properties of Dedekind sums. So

$$\Phi(Vz) = \Phi(z). \qquad \square$$

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

$$j_p(\tau) = 744 + \sum_{n=1}^{\infty} c(pn) q^n$$

To prove congruences for $c(pn)$ we will express $j_p(\tau)$ as a polynomial in $\Phi(\tau)$ with integer coefficients divisible by some power of p (for $p = 2, 3, 5, 7$).

<u>Theorem</u> Let p be prime, $\tau \in H$. Then

$$(*) \quad j_p\left(-\frac{1}{p\tau}\right) = j_p(p\tau) + \frac{1}{p}\left(j(p^2\tau) - j(\tau)\right)$$

and

$$p\, j_p\left(-\frac{1}{p\tau}\right) = q^{-p^2} - q^{-1} + I(q)$$

where $I(q)$ is a power series in q with integer coefficients.

<u>Proof</u> $(*)$ follows from an earlier Theorem.

$$j(\tau) = \frac{1}{q} + c(0) + c(1) q + c(2) q^2 + \cdots$$

$$j_p(\tau) = c(0) + c(p) q + c(2p) q^2 + \cdots$$

$$p\, j_p(p\tau) = p\, c(0) + p\, c(p) q^p + p\, c(2p) q^{2p} + \cdots$$

$$j(p^2\tau) = q^{-p^2} + c(0) + c(1) q^{p^2} + \cdots$$

Hence,

$$p\, j_p\left(-\frac{1}{p\tau}\right) = p\, j_p(p\tau) + j(p^2\tau) - j(\tau)$$

$$= q^{-p^2} - q^{-1} + I(q). \quad \square$$

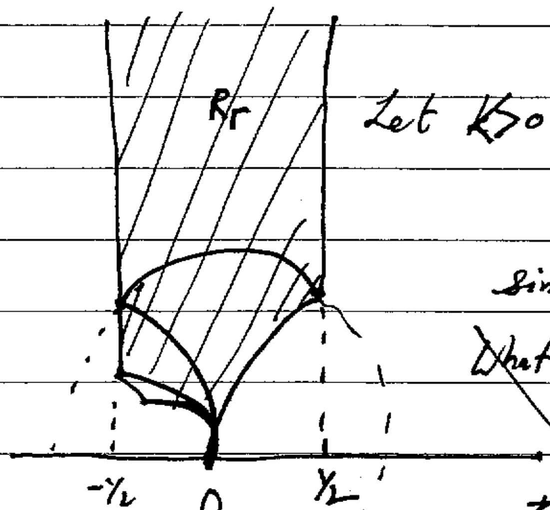
Proposition: Let p be prime. Suppose f is a modular function for $\Gamma_0(p)$, analytic on H, analytic at $i\infty$ and analytic at $\tau = 0$ (ie: $f(-1/\tau)$ is analytic at $i\infty$). Then $f(\tau)$ is constant.

Proof:

A fundamental region for $\Gamma_0(p)$ is

$$\mathcal{F} = R_\Gamma \cup \bigcup_{k=0}^{p-1} ST^k(R_\Gamma).$$

We only need to show f is bounded on $\overline{\mathcal{F}}$.



A nbd of $i\infty$ has the form $\operatorname{Im}\tau > K$.

Let $K > 0$ be arbitrary & fixed.

Hence $f(\tau)$ is bounded for $\operatorname{Im}\tau \geq K$, since f is analytic at $i\infty$ and on H.

Hence f is bounded on R_Γ

What is A nbd of $\tau = 0$?

Suppose

It has the form $\operatorname{Im}\left(-\frac{1}{\tau}\right) > k$.

Suppose $\tau = x + iy$ & $\operatorname{Im}(-1/\tau) = k$

$$-1/\tau = -\frac{1}{x+iy} = \frac{-(x-iy)}{x^2+y^2} = \frac{-x}{x^2+y^2} + \frac{y}{x^2+y^2}i$$

So

$$\frac{y}{x^2+y^2} = k$$

$$y = k(x^2+y^2)$$

$$kx^2 + ky^2 - y = 0$$

$$x^2 + y^2 - \frac{1}{k}y = 0$$

$$x^2 + \left(y - \frac{1}{2k}\right)^2 = \left(\frac{1}{2k}\right)^2 \quad \text{circle center } \left(0, \frac{1}{2k}\right)$$

(20A)

Since $f(-\frac{1}{\tau})$ is also analytic at $i\infty$ and on H.
Therefore $f(-\frac{1}{\tau})$ is bounded for $\operatorname{Im}\tau \geq K$
any fixed $K > 0$.
Hence $f(-\frac{1}{\tau})$ is bounded on $T^k(R_r)$
for each k.
Let $\tau \in ST^k(R_r)$. Then $\tau' = S\tau \in S^2T^k(R_r) = T^k(R_r)$,
and $\tau = S\tau'$. So $f(\tau) = f(-\frac{1}{\tau'})$ and
f is bounded on $ST^k(R_r)$ for each k.
Hence f is bounded on $\overline{\mathcal{F}}$, $\overline{\mathcal{F}} \setminus \{0\}$ and H
~~so f is a zero~~ and so f is a constant. $\square$

<u>Theorem</u> Assume $p = 2, 3, 5, 7$ or 13, and let

$$\Phi(\tau) = \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^r \text{ where } r = \frac{24}{p-1}.$$

Then there exist integers $a_1, a_2, \ldots, a_{p^2}$ such that

$$j_p(\tau) = p^{(r/2)-1}\left\{a_1 \Phi(\tau) + a_2 \Phi^2(\tau) + \cdots + a_{p^2} \Phi^{p^2}(\tau)\right\} + c(q).$$

<u>Proof:</u> By Thm 4.10,

$$p\, j_p\left(\frac{-1}{p\tau}\right) = q^{-p^2} - \frac{1}{q} + \cdots$$

We know

$$\varphi\left(-\frac{1}{p\tau}\right) = \frac{1}{p^{12}} \frac{1}{\varphi(\tau)} \quad \text{where } \varphi(\tau) = \frac{\Delta(p\tau)}{\Delta(\tau)}$$

$$\Phi(\tau) = (\varphi(\tau))^\alpha \quad \text{where } \alpha = \frac{1}{p-1}$$

$$= \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^r \quad r = \frac{24}{p-1}$$

See p21A

So

$$\Phi\left(-\frac{1}{p\tau}\right) = \frac{1}{p^{12\alpha}} \frac{1}{\Phi(\tau)} \quad (12\alpha = r/2)$$

$$p^{r/2}\, \Phi\left(-\frac{1}{p\tau}\right) = \frac{1}{\Phi(\tau)} = \frac{1}{q} + \cdots$$

Let $\psi(\tau) = p^{r/2}\, \Phi\left(-\frac{1}{p\tau}\right)$ (has integer coeffs)

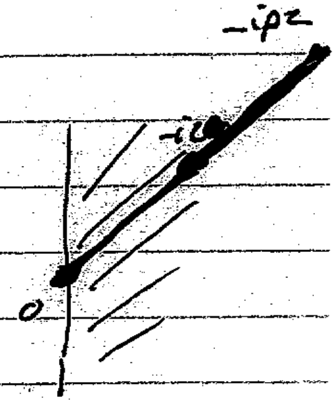
$$p\, j_p\left(-\frac{1}{p\tau}\right) - (\psi(\tau))^{p^2} = \left(q^{-p^2} - q^{-p^2}\right) + b_1 q^{-(p^2)+1} + \cdots$$

has a pole of order $\leq (p^2 - 1)$ at $(\tau = i\infty)$

$$\Phi\left(-\frac{1}{p\tau}\right) = \left(\frac{\eta\left(p\left(-\frac{1}{p\tau}\right)\right)}{\eta\left(-\frac{1}{p\tau}\right)}\right)^{r}$$

$$= \left(\frac{\eta\left(-\frac{1}{\tau}\right)}{\eta\left(-\frac{1}{p\tau}\right)}\right)^{r}$$

$$= \left(\frac{\sqrt{-i\tau}}{\sqrt{-ip\tau}} \; \frac{\eta(\tau)}{\eta(p\tau)}\right)^{r}$$



$\tau = x + iy \quad -i\tau = y - ix$
$\quad = re^{i\theta}$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$

$\sqrt{-i\tau} = \sqrt{r}\, e^{i\theta/2}$
$\sqrt{-ip\tau} = \sqrt{rp}\, e^{i\theta/2}$

$$\frac{\sqrt{-i\tau}}{\sqrt{-ip\tau}} = \frac{1}{\sqrt{p}}$$

Hence

$$\Phi\left(-\frac{1}{p\tau}\right) = \frac{1}{p^{r/2}} \frac{1}{\Phi(\tau)} = \frac{1}{p^{12\alpha}} \frac{1}{\Phi(\tau)}$$

So

$$p\, j_p\left(-\frac{1}{pz}\right) - \left(\psi(z)\right)^{p^2} - b_1\left(\psi(z)\right)^{p^2-1}$$

has a pole of order $\le p^2-2$ at $(z=i\infty)$.

We may find integers $b_1, b_2, \dots, b_{p^2-1}$ such that

$$f\left(\frac{-1}{pz}\right) = p\, j_p\left(\frac{-1}{pz}\right) - \left(\psi(z)\right)^{p^2} - b_1\left(\psi(z)\right)^{p^2-1}$$

$$\cdots - b_{p^2-1}\left(\psi(z)\right)^{1}$$

is analytic in q (at $z=i\infty$), and has integer coefficients.

$$\psi\left(-\frac{1}{pz}\right) = p^{r/2}\,\Phi\left(\frac{-1}{p\left(\frac{-1}{pz}\right)}\right) = p^{r/2}\,\Phi(z)$$

and

$$f(z) = p\, j_p(z) - \left(p^{r/2}\Phi(z)\right)^{p^2} - b_1\left(p^{r/2}\Phi(z)\right)^{p^2-1}$$

$$\cdots - b_{p^2-1}\left(p^{r/2}\Phi(z)\right)$$

p	r	$\frac{r}{2}-1$
2	24	11
3	12	5
5	6	2
7	4	1
13	2	0

is a modular function for $\Gamma_0(p)$, analytic on H, analytic at $i\infty$ and analytic at $z=0$ by construction.

Hence $f(z)$ is bounded and must be a constant.

$\Phi(z)$ is zero at $i\infty$ so

$$f(z) = f(i\infty) = p\, j_p(i\infty) = p\, c(0).$$

So

$$j_p(z) = \cancel{(p^{r/2})}\, p^{\frac{p^2}{2}r-1}\left(\Phi(z)\right)^{p^2} + b_1\, p^{\frac{(p^2-1)r}{2}-1}\left(\Phi(z)\right)^{p^2-1} + \cdots + b_{p^2-1}\, p^{r/2-1}\,\Phi(z) + c(0). \ \square$$

<u>Corollary</u>: For $p = 2, 3, 5, 7$, let $r = (24/p-1)$ then

$$c(pn) \equiv 0 \pmod{p^{(r/2)-1}} \text{ for } n \ge 1.$$

Computing $J(\tau)$

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240\sum_{k=1}^{\infty} \sigma_3(k)\, q^n\right\}$$

$$\Delta(\tau) = (2\pi)^{12}\sum_{n=1}^{\infty} \tau(n)\, q^n$$

$$= (2\pi)^{12}\, q \prod_{n=1}^{\infty}(1-q^n)^{24}$$

$$= g_2^3(\tau) - 27\, g_3^2(\tau)$$

$$= \frac{64\pi^{12}}{27}\left\{(1+240A)^3 - (1-504B)^2\right\}$$

$$A = \sum_{n=1}^{\infty} \sigma_3(n)\, q^n, \quad B = \sum_{n=1}^{\infty} \sigma_5(n)\, q^n$$

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} = \qquad\qquad 4^3 = (2^3)^2 = 64$$

$$= \frac{1 + 240A}{(1+240A)^3 - (1-504B)^2}$$

$$\Phi(\tau) = \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^r, \qquad r = \frac{24}{p-1}$$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}$$

$$= \sum^{\infty}(-1)^n q^{n(3n-1)/2} + \sum^{\infty}(-1)^n q^{n(3n+1)/2}$$

```
> with(numtheory):
Warning, new definition for order
>
> A:=sum(sigma[3](n)*q^n,n=1..200):
> B:=sum(sigma[5](n)*q^n,n=1..200):
> G2:=1+240*A: G3:=1-504*B:
> J:=G2^3/(G2^3 - G3^2):
> series(J,q,10);
```

$$\frac{1}{1728}q^{-1}+\frac{31}{72}+\frac{1823}{16}q+\frac{335840}{27}q^2+\frac{16005555}{32}q^3+11716352\,q^4+\frac{41650330075}{216}q^5+2460661632\,q^6+\frac{1653962743405}{64}q^7+O(q^8)$$

```
>
>
>
> j:=series(J*12^3,q,200):
> series(j,q,10);
```

$$q^{-1}+744+196884\,q+21493760\,q^2+864299970\,q^3+20245856256\,q^4+333202640600\,q^5+4252023300096\,q^6+44656994071935\,q^7+401490886656000\,q^8+3176440229784420\,q^9+O(q^{10})$$

```
> with(qseries);
************************************************************
*
*  qseries package version 0.5a - Sat May 13 14:12:56 EDT 2000
*  This version runs on Maple V - Release 5
*
*  Please report any problems to frank@math.ufl.edu
*  See
*  http://www.math.ufl.edu/~frank/qmaple/qmaple.html
*  for documentation and help.
*
************************************************************
```

[*aqprod, etamake, etaq, findhom, findhomcombo, findnonhom, findnonhomcombo, findpoly, init, jac2prod, jac2series, jacprod, jacprodmake, prodmake, qbin, qfactor, quinprod, sift*, θ, θ2, θ3, θ4, *tripleprod, version, winquist*]

```
> UP:=proc(F,q,p,T)
>    local x,n:
>    x:=0:
>    for n from 0 to trunc(T/p) do
>      x := x + coeff(F,q,p*n)*q^n:
>    od:
> RETURN(x):
> end:
> j2:=UP(j,q,2,96):
> modp(j2,2^11);
```

744

```
> j3:=UP(j,q,3,196):
> modp(j3,3^5);
```

15

```
> series(j2,q,10);
```

$$744+21493760\,q+20245856256\,q^2+4252023300096\,q^3+401490886656000\,q^4+22567393309593600\,q^5+874313719685775360\,q^6+25497827389410525184\,q^7+593121772421445058560\,q^8+11459912788444786513920\,q^9+O(q^{10})$$

```
> PHI2:=series( q*etaq(q,2,100)^24/etaq(q,1,100)^24,q,100):
> series(j2,q,10);
```

$$744+21493760\,q+20245856256\,q^2+4252023300096\,q^3+401490886656000\,q^4+22567393309593600\,q^5+874313719685775360\,q^6+25497827389410525184\,q^7+593121772421445058560\,q^8+11459912788444786513920\,q^9+O(q^{10})$$

```
> series(j2-PHI2*21493760,q,10);
```

$$744+19730006016\,q^2+4245575172096\,q^3+401434487029760\,q^4+22567003713699840\,q^5+874311452008120320\,q^6+25497815819405492224\,q^7+593121719301250744320\,q^8+11459912564888253235200\,q^9+O(q^{10})$$

```
> series(j2-PHI2*21493760-19730006016*PHI2^2,q,10);;
```

$$744 + 3298534883328\, q^3 + 378231999954944\, q^4 + 22179348555497472\, q^5 + 869335465570861056\, q^6 + 25445423656550268928\, q^7 + 592650137432657756160\, q^8 + 11456179476020053671936\, q^9 + O(q^{10})$$

```
> series(j2-PHI2*21493760-19730006016*PHI2^2-3298534883328*PHI2^3,q,10
> );
```

$$744 + 140737488355328\, q^4 + 13510798882111488\, q^5 + 655273745782407168\, q^6 + 21419119827774078976\, q^7 + 531085860157758898176\, q^8 + 10658696259695517106176\, q^9 + O(q^{10})$$

```
> series(j2-PHI2*21493760-19730006016*PHI2^2-3298534883328*PHI2^3-140737488355328*PHI2^4,q,
  10);
```

$$744 + O(q^{10})$$

```
> series(j2-PHI2*21493760-19730006016*PHI2^2-3298534883328*PHI2^3-140737488355328*PHI2^4,q,
  40);
```

$$744 + O(q^{49})$$

```
> 'j2'=Phi*21493760+19730006016*Phi^2+3298534883328*Phi^3+140737488355328*Phi^4+744;
```

$$j2 = 21493760\,\Phi + 19730006016\,\Phi^2 + 3298534883328\,\Phi^3 + 140737488355328\,\Phi^4 + 744$$

```
> Phi*21493760+19730006016*Phi^2+3298534883328*Phi^3+140737488355328*Phi^4;
```

$$21493760\,\Phi + 19730006016\,\Phi^2 + 3298534883328\,\Phi^3 + 140737488355328\,\Phi^4$$

```
> R:=%/2^11;
```

$$R := 10495\,\Phi + 9633792\,\Phi^2 + 1610612736\,\Phi^3 + 68719476736\,\Phi^4$$

```
> seq(ifactor(coeff(R,Phi,k)),k=1..4);
```

$$(5)\ (2099),\ (2)^{16}\ (3)\ (7)^2,\ (2)^{29}\ (3),\ (2)^{36}$$

```
>
>
> findnonhomcombo(j2,[PHI2],q,4,0);
```

# of terms , 26

matrix is , 6, *x* , 26

-----*possible linear combinations of degree*------, 4

$$\{140737488355328\, X_1^{\ 4} + 3298534883328\, X_1^{\ 3} + 19730006016\, X_1^{\ 2} + 21493760\, X_1 + 744\}$$

```
> coeffac:=proc(POLY,VAR)
>    #factorise integer coeffs of a polynomial
>    local x,k,dg,c,c1:
>    x:=0: dg:=degree(POLY,VAR):
>    for k from 0 to dg do
>        c:=coeff(POLY,VAR,k):
>        if type(c,integer) then
>           c1:=ifactor(c):
>        else
>           c1:=c:
>        fi:
>        x := x + c1*VAR^k:
>    od:
>    RETURN(x):
> end:
> coeffac(pol,X[1]);
```

$$(2)^3\ (3)\ (31) + (2)^{11}\ (5)\ (2099)\, X_1 + (2)^{27}\ (3)\ (7)^2\, X_1^{\ 2} + (2)^{40}\ (3)\, X_1^{\ 3} + (2)^{47}\, X_1^{\ 4}$$

```
>
> PHI3:=series( q*etaq(q,3,100)^12/etaq(q,1,100)^12,q,200):;
> findnonhomcombo(j3,[PHI3],q,9,-10);
```

# of terms , 21

matrix is , 11, *x* , 21

-----*possible linear combinations of degree*------, 9

$$\{193832456676800198967967 23\, X_1^{\ 9} + 287159195076741035508099 6\, X_1^{\ 8} + 171350136979948354521294\, X_1^{\ 7}$$

$+ 525391261393615723148 4 X_1^6 + 87382873045686266037 X_1^5 + 762254724599300520 X_1^4 + 3074562801976176 X_1^3$
$+ 4241651700456 X_1^2 + 864299970 X_1 + 744\}$

```
> pol3:=(op(%));
```

$pol3 := 19383245667680019896796723 X_1^9 + 28715919507674103550809 96 X_1^8 + 1713501369799483545212 94 X_1^7$
$+ 5253912613936157231484 X_1^6 + 87382873045686266037 X_1^5 + 762254724599300520 X_1^4 + 3074562801976176 X_1^3$
$+ 4241651700456 X_1^2 + 864299970 X_1 + 744$

```
> coeffac(pol3,X[1]);
```

$(2)^3\ (3)\ (31) + (2)\ (3)^5\ (5)\ (355679) X_1 + (2)^3\ (3)^{16}\ (109)\ (113) X_1^2 + (2)^4\ (3)^{20}\ (7)\ (7873) X_1^3$
$+ (2)^3\ (3)^{28}\ (5)\ (7)^2\ (17) X_1^4 + (3)^{33}\ (11)\ (1429) X_1^5 + (2)^2\ (3)^{37}\ (2917) X_1^6 + (2)\ (3)^{45}\ (29) X_1^7 + (2)^2\ (3)^{50} X_1^8$
$+ (3)^{53} X_1^9$

```
>
```

Conclusions:

$$j_2(\tau) = 744 + 2^{11}\cdot 5\cdot 2099\, \Phi(\tau) + 2^{27}\cdot 3\cdot 7^2\, \Phi^2(\tau) + 2^{40}\cdot 3\cdot \Phi^3(\tau) + 2^{47}\, \Phi^4(\tau),$$

where $\Phi(\tau) = \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{24}$

$$j_3(\tau) = 744 + 3^5\cdot 2\cdot 5\cdot 355679\, \Phi(\tau) + 3^{16}\cdot 2^3\cdot 109\cdot 113\, \Phi^2(\tau) + 3^{20}\cdot 2^4\cdot 7\cdot 7873\, \Phi^3(\tau) + 3^{28}\cdot 2^3\cdot 5\cdot 7^2\cdot 17\, \Phi^4(\tau) + 3^{33}\cdot 11\cdot 1429\cdot \Phi^5(\tau) + 3^{37}\cdot 2^2\cdot 2917\, \Phi^6(\tau) + 3^{45}\cdot 2\cdot 29\cdot \Phi^7(\tau) + 3^{50}\cdot 2^2\cdot \Phi^8(\tau) + 3^{53}\, \Phi^9(\tau).$$