Chapter 6 Modular Forms with Multiplicative Coefficients (1

We have seen that

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\,\Delta(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$g_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{4}\, g_2(\tau)$$

$$g_3\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{6}\, g_3(\tau).$$

$\Delta(\tau)$, $g_2(\tau)$, and $g_3(\tau)$ are examples of modular forms.

$$\Delta(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n)\, q^n \qquad (q = e^{2\pi i \tau})$$

It can be shown that $\tau(n)$ is a multiplicative function

i.e. $\tau(mn) = \tau(m)\,\tau(n)$ for $(m,n) = 1$.

$$g_2(\tau) = \frac{4\pi^4}{3}\left\{1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n\right\}$$

$$\sigma_k(n) = \sum_{\substack{d \mid n \\ (d>0)}} d^k$$

$$g_3(\tau) = \frac{8\pi^6}{27}\left\{1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, q^n\right\}$$

The coefficients $\sigma_3(n)$, $\sigma_5(n)$ are also multiplicative functions.

Modular forms of weight k $(k \in \mathbb{Z})$

A function $f(\tau)$ is an <u>entire modular form of weight k</u> if

(1) f is analytic on H

(2) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k} f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

(3) $f(\tau)$ has an expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n)\, q^n, \qquad |q|<1,\ q = e^{2\pi i \tau}$$

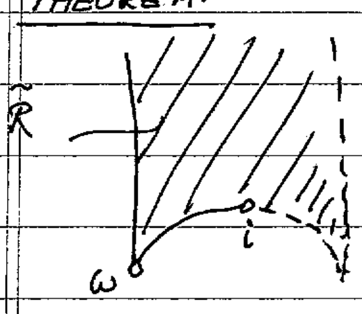
$(\operatorname{Im}\tau > 0)$.

A form $f(\tau)$ is a <u>cusp form</u> if $a(0) = 0$.
The zero function is a modular form of weight k for any k.
We will show that space of ~~...~~ entire modular forms of weight k, $\mathcal{M}_k$ is a finite dimensional vector space (over $\mathbb{C}$) and $\dim \mathcal{M}_k \geq 1$ iff k is even and $k \geq 4$, or $k=0$.

For $P \in H$, let $v_p(f)$ = order of zero of f at $\tau = P$
Let $v_{i\infty}(f)$ = smallest r s.th. $a(r) \neq 0$ in the expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n.$$

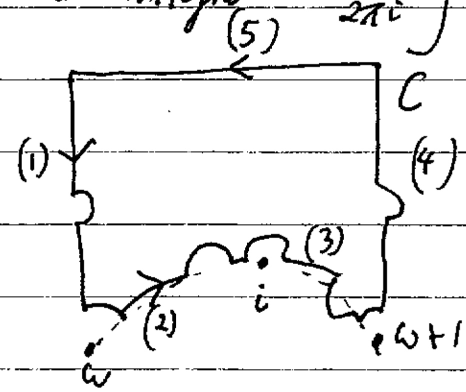
THEOREM:



Suppose $f(\tau)$ is not identically zero, and $f(\tau)$ is an entire modular form. Then

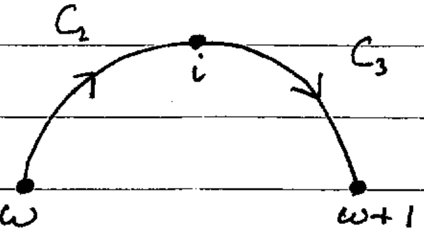
$$v_{i\infty}(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_i(\omega) + \sum_{P \in \tilde{R}} v_p(f) = \frac{k}{12}$$

Proof: The proof proceeds analogous to the case when $k=0$ (modular functions). As in the proof the the modular function case we consider the integral $\frac{1}{2\pi i} \int \frac{f'(\tau)}{f(\tau)} d\tau$



$f(\tau+1) = f(\tau)$ for all τ

So $\int_{C_1} + \int_{C_4} = 0$.



Let $S(\tau) = -1/\tau$.
If $\tau(\theta)$ parametrizes C_2
then $S(\tau(\theta))$ parametrizes $-C_3$.

$$f(S(\tau)) = f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$$

$$f'(S(\tau))\, S'(\tau) = \tau^k f'(\tau) + k\tau^{k-1} f(\tau)$$

$$\frac{f'(S(\tau))\, S'(\tau)}{f(S(\tau))} = \frac{f'(\tau)}{f(\tau)} + \frac{k}{\tau}$$

$$\int_{C_2} \frac{f'(\tau)}{f(\tau)}\, d\tau = \int_{C_2} \frac{f'(S(\tau))\, S'(\tau)}{f(S(\tau))}\, d\tau - \int_{C_2} \frac{k}{\tau}\, d\tau$$

$$\int_{C_2} \frac{f'(\tau)}{f(\tau)}\, d\tau = -\int_{C_3} \frac{f'(\tau)\, d\tau}{f(\tau)} - k\int_{\omega}^{i} \frac{1}{\tau}\, d\tau$$

$$\int_{C_2} + \int_{C_3} = -k \log \tau \Big]_{\omega}^{i}$$

$$= -k\,(\log i - \log \omega)$$

$$= -k\left(\frac{\pi i}{2} - \frac{2\pi i}{3}\right)$$

$$= -k\pi i\left(\frac{3-4}{6}\right) = \frac{k\pi i}{6}$$

So

$$\frac{1}{2\pi i}\int_{C_2} + \frac{1}{2\pi i}\int_{C_3} = \frac{k}{12}$$

As before we find

$$\sum_{p\in\tilde{R}} v_p(f) = -v_{i\infty}(f) - \frac{1}{2} v_i(f) - \frac{1}{3} v_\omega(f) + \frac{k}{12}$$

Hence

$$v_{i\infty}(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\omega(f) + \sum_{p \in \tilde{R}} v_p(f) = \frac{k}{12}.$$

<u>Theorem</u> :

(a) The only entire modular form of weight $k = 0$ are the constant functions.

(b) If k is odd, if $k < 0$ or if $k = 2$, the only entire modular form of weight k is the zero function.

(c) Every nonconstant entire modular form has weight $k \geq 4$, where k is even.

(d) The only entire cusp form of weight $k < 12$ is the zero function.

Proof

(a) In Ch2 we proved that the only modular function analytic on H was the constant function.

(b) $v_{i\infty}(f)$, $v_i(f)$, $v_\omega(f)$, $v_p(f)$ are all nonnegative integers. If f is constant then $k = 0$. If f is not zero then

$$12\, v_{i\infty}(f) + 6\, v_i(f) + 4\, v_\omega(f) + 12 \sum_{p \in \tilde{R}} v_p(f) = k$$

This implies k must be even and $k \geq 0$.

If $k > 0$ then $k \geq 4$. and not zero fcn

If f is a cusp form then $v_{i\infty}(f) \geq 1$ then $k \geq 12$.

So the only entire cusp form of weight $k < 12$ is the zero function. □

<u>Representation of entire forms in terms of G_4 & G_6</u>

Recall

$$G_k(1,\tau) = G_k(\tau) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(m+n\tau)^k} \quad \text{for } k \geq 3$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ then

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

$$\omega_1 = a\tau + b$$

$$\omega_2 = c\tau + d$$

$$\Omega(1,\tau) = \Omega(c\tau+d, a\tau+b)$$

$$= \Omega\left((c\tau+d)\,1, (c\tau+d)\left(\frac{a\tau+b}{c\tau+d}\right)\right)$$

$$G_k(c\tau+d, a\tau+b) = G_k(1,\tau)$$

$$G_k(c\tau+d, a\tau+b) = (c\tau+d)^{-k}\, G_k\left(1, \frac{a\tau+b}{c\tau+d}\right)$$

So

$$G_k\left(1, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, G_k(1,\tau)$$

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, G_k(\tau)$$

We know G_4, G_6 are analytic at $i\infty$. Thm 1.13 (p.12) implies $G_{2n}(\tau)$ $(n \geq 2)$ is analytic at $i\infty$ since it is a polynomial in $G_4(\tau)$, $G_6(\tau)$.

<u>NOTE</u>: $G_k(\tau) \equiv 0$ for $k \geq 3$, k odd.

So each $G_{2n}(\tau)$ $(n \geq 2)$ is an entire modular form of weight $2n$.

<u>Notation</u>: Let M_k denote the vector space of entire modular forms of weight k.

Let S_k – space of cusp forms of weight . . . .

$k \neq 2$

<u>Theorem</u> : Let $k \geq 0$ be even. Define $G_0(z) = 1$, (6)

$z \in H$. M_k is f.d. and a basis is formed by

$$G_{k-12r}\,\Delta^r, \quad 0 \leq r \leq \left[\tfrac{k}{12}\right], \quad k - 12r \neq 2.$$

<u>Proof</u>:

<u>Case 1</u> $k < 12$.

$k = 0$ $M_0 = \mathbb{C}$ & $G_0 = 1$ is a basis.

$k = 4, 6, 8, 10$. $G_k \in M_k$. Suppose $f \in M_k$.

$G_k(i\infty) = 2\zeta(k) \neq 0$. Let $g(z) = f(z) - \frac{f(i\infty)}{G_k(i\infty)} G_k(z)$

Then $g \in M$ & $g(i\infty) = 0$, so $g \in S_k = \{0\}$

Hence $f(z) = c\, G_k(z)$ $c \in \mathbb{C}$, $\dim M_k = 1$ & $\{G_k\}$ is a basis for M_k.

<u>Case 2</u> $k \geq 12$. Suppose result is true for $k' < k$.

Again suppose $f \in M_k$. Since $G_k \in M_k$ & $G_k(i\infty) \neq 0$.

Let $g(z) = f(z) - \frac{f(i\infty)}{G_k(i\infty)} G_k(z)$, then $g \in S_k$

$\Delta \in S_{12}$ and Δ does not vanish in H & $v_{i\infty}(\Delta) = 1$

So, $h = \frac{g}{\Delta} \in M_{k-12}$

$$h = \sum_{r=0}^{[(k-12)/12]} b_r\, G_{k-12-12r}\, \Delta^r \qquad (b_r \in \mathbb{C})$$

$$\text{So} \quad g = \sum_{\substack{r=1 \\ k-12r \neq 2}}^{[k/12]} b_{r-1}\, G_{k-12r}\, \Delta^r$$

$$\text{and} \quad f(z) = g(z) + c\, G_k(z) = \sum_{\substack{r=0 \\ k-12r \neq 2}}^{[k/12]} b_{r-1}\, G_{k-12r}\, \Delta^r \quad \text{where } b_{-1} = c.$$

So the $G_{k-12r}\Delta^r$ span M_k by induction for all $k \geq 0$, $k \neq$ ... k even.

The $G_{k-12r}\,\Delta^r$ are linearly indept. Since

$$v_{i\infty}\left(G_{k-12r}\,\Delta^r\right) = r. \qquad \square$$

$$\Delta = g_2^3(z) - 27\, g_3^2(z) = (60\, G_4(z))^3 - 27\, (140\, G_6(z))^2.$$

In Theorem 1.14 (pp. 14–15) we proved that each G_n is a polynomial in $g_2(z), g_3(z)$, or as a polynomial in G_4, G_6. In fact it can be easily shown that polynomial has the form

$$G_n = \sum_{\substack{a,b\geq 0\\ 4a+6b=n}} c_{a,b}\, G_4^a\, G_6^b$$

We have

Theorem Every entire modular form f of weight k can be written as a polynomial in G_4, G_6 of the form

$$f = \sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c_{a,b}\, G_4^a\, G_6^b$$

where the $c_{a,b}\in\mathbb{C}$.

Suppose $k\geq 0$, k even.

Theorem:

$$\dim M_k = \begin{cases} \left[\frac{k}{12}\right] & \text{if } k\equiv 2 \pmod{12} \\ \left[\frac{k}{12}\right]+1 & \text{if } k\not\equiv 2 \pmod{12}. \end{cases}$$

Proof: $M_2=\{0\}$ so $\dim M_2 = 0$ & result is true for $k=2$.
If $k\geq 0$, even, $k\neq 2$, a basis for M_k is

$$G_{k-12r}\,\Delta^r, \quad 0\leq r\leq \left[\tfrac{k}{12}\right], \quad k-12r\neq 2$$

If $k\not\equiv 2 \pmod{12}$ & $0\leq r\leq \left[\frac{k}{12}\right]$, then $k-12r\neq 2$ & $\dim M_k = \left[\frac{k}{12}\right]+1$.
If $k\equiv 2 \pmod{12}$, then $\exists\, r_0\geq 0$ s.t. $k = 12r_0+2$, then $0\leq 12r_0 < k$, $0\leq r_0 < [k/12]$ & $k-12r_0 = 2$. It follows that $\dim M_k = [k/12]$. $\square$

Recall, S_k is the space of cusp forms of weight k. (8

<u>Theorem</u> Let $k \geq 0$, k even. Then

$$M_k = \mathbb{C}\, G_k \oplus S_k \qquad (k \neq 2).$$

<u>Proof:</u> Let $f \in M_k$. Then

let $g(z) = f(z) - \frac{f(i\infty)}{G_k(i\infty)} G_k(z) \in S_k$

So

$$M_k = \mathbb{C} G_k + S_k.$$

Clearly $\mathbb{C} G_k \cap S_k = \{0\}$ since $G_k(i\infty) \neq 0$. □

<u>Cor.</u> $\dim M_k = 1 + \dim S_k$.

<u>Theorem:</u> ~~[illegible]~~ Let $k \geq 12$, k even.
The map $T: M_{k-12} \longrightarrow S_k$ by $T(h) = h\Delta$ is an isomorphism.

<u>Proof:</u> Suppose $k \geq 12$, k even. If $h \in M_{k-12}$ then clearly $T(h) = h\Delta \in S_k$ since weight of $h\Delta = (k-12)+12 = k$ and $v_{i\infty}(h\Delta) = v_{i\infty}(h) + v_{i\infty}(\Delta) \geq 1$.

Suppose $f \in S_k$, then $v_{i\infty}(f) \geq 1$

so $h = \frac{f}{\Delta} \in M_{k-12}$ (since Δ is nonzero H

& $v_{i\infty}(h) = v_{i\infty}(f) - v_{i\infty}(\Delta)$

$= v_{i\infty}(f) - 1 \geq 0.$)

<u>Cor:</u> $\dim S_k = \dim M_{k-12}$ for $k \geq 12$

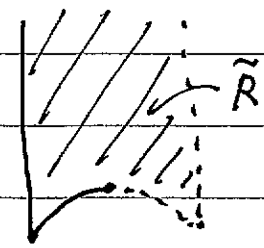
$\dim S_k = 0$ if $k < 12$.

From proof of Thm 2.7 p. 60, we have

$$G_4(\omega) = 0 \quad \& \quad G_6(i) = 0.$$

$$v_{i\infty}(G_4) + \tfrac{1}{3} v_\omega(G_4) + \tfrac{1}{2} v_i(G_4) + \sum_{\substack{p \in \tilde{R} \\ p \neq i, \omega}} v_p(G_4) = \tfrac{1}{3}$$

The only zero of G_4 is a single zero at $\tau = \omega$.



$$v_{i\infty}(G_6) + \tfrac{1}{3} v_\omega(G_6) + \tfrac{1}{2} v_i(G_6) + \sum_{\substack{p \in \tilde{R} \\ p \neq i, \omega}} v_p(G_6) = \tfrac{1}{2}$$

The only zero of G_6 is a single zero at $\tau = i$.

$\tilde{R}$ ✓ & v_i

Theorem Let f be an entire modular form of weight k.

Suppose f has N zeros in $\tilde{R} - \{i, \omega\}$, $z_1, z_2, \dots, z_N$.

Then

$$f(\tau) = c\,(G_4(\tau))^{v_\omega(f)}\,(G_6(\tau))^{v_i(f)}\,(\Delta(\tau))^{v_{i\infty}(f)}\,(\Delta(\tau))^{N}$$

$$\cdot \prod_{n=1}^{N} (J(\tau) - J(z_n)).$$

Proof:

$$g(\tau) = \prod_{n=1}^{N} (J(\tau) - J(z_n))$$

is a modular function with its only zeros in $\tilde{R}$ at $z_1, z_2, \dots, z_n$ (since J is one-to-one on $\tilde{R}$ ~~[illegible]~~ & $\tilde{R}\cup\{i\infty\}$) and a pole at $i\infty$ of order N (only singularity).

& $(J(\tau) - J(p))^{-1}$ for $p \in \tilde{R}$

So $v_{i\infty}(g) = -N$

$$v_{i\infty}(g\,\Delta^N) = -N + N = 0$$

Hence $g\Delta^N$ is an entire modular form of weight $12N$ which only vanishes at $z_1, z_2, \dots, z_N$ in $\tilde{R}$.

~~The weight of f~~

~~$= 4 v_\omega(f)$~~

Let $h = G_4^{v_\omega(f)}\, G_6^{v_i(f)}\, \Delta^{v_{i\infty}(f)}\, \Delta^N g.$

$$v_{i\infty}(h) = v_{i\infty}(f) + 0 = v_{i\infty}(f)$$

$$v_\omega(h) = v_\omega(f) \qquad \text{Note: } J(\omega) = 0 \quad J(i) = 1$$

$$v_i(h) = v_i(f).$$

The weight of h is

$$4 v_\omega(f) + 6 v_i(f) + 12 v_{i\infty}(f) + 12N = k \quad \text{(by Thm)}$$

Hence, $\frac{f}{h}$ is an entire form of weight 0 & so must be a constant. □

The Hecke Operators

Notes: (Automorphy factor)

For $A \in GL_2^+(\mathbb{R})$, define

$$j_A(z) = (c\tau + d), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

j satisfies:

$$j_{AB}(\tau) = j_A(B\tau)\, j_B(\tau) \qquad (\underline{EX})$$

where $A\tau = \dfrac{a\tau + b}{c\tau + d}$.

also

$$(AB)\tau = A(B\tau).$$

SLASH OPERATOR For a function $f : H \to \mathbb{C}$ and a fixed integer k define

$$\left(f\,|_k\,A\right)(\tau) := (\det A)^{k/2} \frac{f(A\tau)}{(c\tau + d)^k} = (\det A)^{k/2} \frac{f(A\tau)}{(j_A(\tau))^k}$$

Note (1) If f is a modular form of weight k on Γ

$$f\,|_k\,A = f \qquad \text{for } A \in \Gamma.$$

(2) $$\left(f\,|_k\,A\right)|_k\,B = f\,|_k\,(AB)$$

Definition: Let k, n be fixed. $n \geq 1$. ~~The Hecke operator~~ Let f be a entire modular form of weight k. We define the Hecke operator T_n by

$$T_n f = n^{\frac{k}{2} - 1} \sum_{A \in \Gamma \backslash \Gamma(n)} f\,|\,A$$

Sum over right cosets reps of Γ in $\Gamma(n)$

Note: Suppose $A_2 \in \Gamma A_1$, $A_2 = VA_1$ some $V \in \Gamma$.

Then
$$f|_k A_2 = f|_k (VA_1) = (f|_k V)|_k A_1 = f|_k A_1$$
since f is a modular ~~fn~~ of weight k and $V \in \Gamma$

So ~~def~~ T_n is well-defined.

Note A complete set of right coset reps is given by
$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \geq 0,\ ad = n,\ 0 \leq b < d \right\}$$

Let $A \in \Delta_n$, $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$$A\tau = \frac{a\tau + b}{d} = \frac{ad\tau + bd}{ad^2} = \frac{n\tau + bd}{d^2}$$

$$j_A(\tau) = d = \frac{n}{a}$$

$$n^{\frac{k}{2}-1} f|_k A = n^{\frac{k}{2}-1} n^{\frac{k}{2}} \frac{f(A\tau)}{(j_A(\tau))^k} = n^{k-1} \frac{f(A\tau)}{(n/a)^k} = \frac{1}{n} a^k f(A\tau)$$

So $$T_n f = \frac{1}{n} \sum_{\substack{ad=n \\ 0 \leq b < d}} a^k f(A\tau)$$

Transformations of order n

Let n be fixed positive integer.

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc = n \right\} / \pm I$$

Let $A_1, A_2 \in \Gamma(n)$

We write $A_1 \sim A_2 \pmod{\Gamma}$

if $\exists\, V \in \Gamma = \Gamma(1)$ s.t.

$$A_1 = VA_2.$$

$\sim$ is an equivalence relation on $\Gamma(n)$.

Theorem In every equivalence class of $\Gamma(n)$ there is a representative in triangular form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad d > 0.$$

Proof: Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(n)$.

If $\gamma = 0$ we are done.

If $\gamma \neq 0$, $-\frac{\alpha}{\gamma} = \frac{s}{r}$ (where $(s,r)=1$).

$\exists$ integers p, q s.t.

$$ps - qr = 1.$$

Let $V = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, so that $V \in \Gamma$ and

$$VA = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} * & * \\ r\alpha + s\gamma & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

$\det(VA) = \det V \det A = 1 \cdot n = n$.

So $VA \in \Gamma(n)$, $VA \sim A$. Hence VA or $-VA$ is the desired representative. $\square$

Theorem A complete system of non-equivalent elements in $\Gamma(n)$ is given by the set of transformations

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

where $d \mid n$, $a = \frac{n}{d}$ and b runs thru a complete residue system mod d.

Proof:

Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} \in \Gamma(n)$

Claim: $A_1 \sim A_2$ iff $a_1 = a_2$, $d_1 = d_2$ and $b_1 \equiv b_2 \pmod{d_1}$

($\Leftarrow$) If $a_1 = a_2$, $d_1 = d_2$ & $b_1 \equiv b_2 \pmod{d}$, then

$$b_2 = b_1 + qd \quad \text{for some integer } q.$$

Let $V = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in \Gamma$

$$VA_1 = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 + qd_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$$

$$= A_2,$$

& $A_1 \sim A_2$.

($\Rightarrow$) Suppose $A_1 \sim A_2$. Then $\exists\, V = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma$

s.th. $A_2 = VA_1$

$$\begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} pa_1 & pb_1 + qd_1 \\ ra_1 & rb_1 + sd_1 \end{pmatrix}$$

$ra_1 = 0$, $r = 0$ (since $a_1 d_1 = n \neq 0$).

So $ps - qr = 1$ & $ps = 1$ & $p = s = \pm 1$.

... $A_1 V_1 \sim A_2$ i.e. $A_1 V_1 = V_2 A_2$. (1

wlog suppose $\rho = \delta = 1$ (since can replace V by $-V$). Therefore $a_1 = a_2$ & $d_1 = d_2$.
Also, $b_2 = b_1 + q d_1$, & $b_2 \equiv b_1 \pmod{d_1}$. □

<u>Note</u>: Let $\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d, b \in \mathbb{N},\ ad = n \text{ and } 0 \le b < d \right\}$

Then $\Gamma(n) = \bigcup_{A \in \Delta_n} \Gamma A$ ("right cosets").

<u>Theorem</u> There is a 1-1 correspondence between $\Delta_n \times \Gamma$ and $\Gamma \times \Delta_n$ such that for any $A_1 \in \Delta_n$ & $V_1 \in \Gamma$ $\exists!$ $V_2 \in \Gamma$ and $A_2 \in \Delta_n$

(1) such that $A_1 V_1 = V_2 A_2$.

(2) Further, for fixed $V_1 \in \Gamma$, as A_1 ranges over Δ_n so does A_2.

Further, if $A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}$, $V_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$, then

(3) $$a_1 (\gamma_2 A_2 \tau + \delta_2) = a_2 (\gamma_1 \tau + \delta_1).$$

<u>NOTE:</u> A_2, V_2 exist because $A_1 V_1 \in \Gamma(n)$ so exist $A_2 \in \Delta_n$ & $V_2 \in \Gamma$ such that $A_1 V_1 = V_2 A_2$

Proof: (1) We want to show uniqueness:

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \alpha_1 + b_1 \gamma_1 & \\ d_1 \gamma_1 & d_1 \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2 a_2 & \\ \gamma_2 a_2 & \gamma_2 b_2 + \delta_2 d_2 \end{pmatrix}$$

<u>Note</u> we require $(\alpha_2, \gamma_2) = 1$ since $\alpha_2 \delta_2 - \gamma_2 \beta_2 = 1$.

$$\alpha_2 a_2 = a_1 \alpha_1 + b_1 \gamma_1$$
$$\gamma_2 a_2 = d_1 \gamma_1$$

It follows that $a_2 = (a_1 \alpha_1 + b_1 \gamma_1, d_1 \gamma_1)$ (EX)

$$\alpha_2 = (a_1 \alpha_1 + b_1 \gamma_1) / (a_1 \alpha_1 + b_1 \gamma_1, d_1 \gamma_1)$$

$$\gamma_2 = d_1\gamma_1 / (a_1\alpha_1 + b_1\gamma_1, d_1\gamma_1)$$
$$d_2 = n / (a_1\alpha_1 + b_1\gamma_1, d_1\gamma_1)$$

since $a_2 d_2 = n$.

(Case 1 $\gamma_1 \neq 0$. so $\gamma_2 \neq 0$.)

$$\alpha_2\delta_2 - \gamma_2\beta_2 = 1$$
$$\text{so} \quad \alpha_2\delta_2 \equiv 1 \pmod{|\gamma_2|}$$

omit?

δ_2 is given mod $|\gamma_2|$

$$d_1\delta_1 = \gamma_2 b_2 + \delta_2 d_2$$

~~So b_2 is given mod~~

$$\gamma_2 b_2 \equiv d_1\delta_1 \pmod{d_2}$$

what if $(\gamma_2, d_2) \neq 1$? *

See next page

so b_2 is given $(\mathrm{mod}\ d_2)$ and uniquely since $0 \leq b_2 < d_2$. Hence δ_2 is unique.

(2) Let $V_1 \in \Gamma$ be fixed. For any $A_2 \in \Delta_n$ we want to show $\exists A_1 \in A_n$, $V_2 \in \Gamma$ s.t.

$$A_1 V_1 = V_2 A_2.$$

$A_2 V_1^{-1} \in \Gamma(n)$. So $\exists W_2 \in \Gamma$ & $A_1 \in \Delta_n$ s.t.

$$A_2 V_1^{-1} = W_2 A_1$$
$$W_2^{-1} A_2 = A_1 V_1$$
$$A_1 V_1 = W_2^{-1} A_2, \quad \text{take } V_2 = W_2^{-1}.$$

(3) $$\gamma_2 = \frac{d_1\gamma_1}{a_2} = \frac{d_1\gamma_1}{a_2}\,\frac{a_2 d_2}{a_1 d_1} = \frac{\gamma_1 d_2}{a_1}$$

and $a_1\gamma_2 = d_2\gamma_1$.

Case 1 $\gamma_1 \neq 0$ so $\gamma_2 \neq 0$

$$\alpha_2 \delta_2 - \gamma_2 \beta_2 = 1$$

so $\alpha_2 \delta_2 \equiv 1 \pmod{\gamma_2}$

and δ_2 is determined mod $|\gamma_2|$.

$$d_1 \delta_1 = \gamma_2 b_2 + \delta_2 d_2.$$

We know b_2, δ_2 exist. So suppose δ_2', b_2' also satisfy this eqn.

$$\delta_2 = \delta_2^* + k_1 \gamma_2$$

$$\delta_2' = \delta_2^* + k_2 \gamma_2 \qquad (\text{since } \delta_2 \equiv \delta_2' \bmod \gamma$$

Hence $\gamma_2 b_2 + (\delta_2^* + k_1 \gamma_2) d_2$

$= \gamma_2 b_2' + (\delta_2^* + k_2 \gamma_2) d_2$

and $\gamma_2 (b_2 - b_2') = \gamma_2 d_2 (k_2 - k_1)$

$\gamma_2 \neq 0$ so $b_2 - b_2' = d_2 (k_2 - k_1)$

& $b_2 \equiv b_2' \pmod{d_2}$ & $b_2 = b_2'$

and hence $\delta_2 = \delta_2'$ & we have uniqueness.

Case 2 $\gamma_1 = 0$ & so $\gamma_2 = 0$. $\qquad \alpha_2 \delta_2 - \beta_2 \gamma_2 = 1$

$$d_1 \gamma_1 = \delta_2 d_2 \qquad \alpha_2 \delta_2 = 1$$

$$\text{and } \delta_2 = \frac{d_1 \gamma_1}{d_2}.$$

$$a_1 \beta_1 + b_1 \delta_1 = \alpha_2 b_2 + \beta_2 d_2$$

wlog suppose $\alpha_2 = \delta_2 = 1$ so $b_2 \equiv \alpha_1 \beta_1 + b_1 \delta_1 \ (d_2)$

and b_2 is determined & so hence is β_2.

Note: For fixed $V_1 \in \Gamma$ as A_1 ranges over Δ_n so does A_2. (1) induces a map fr $\Delta_n \rightarrow \Delta_n$.
We show it is 1-1.

Suppose $A_1 V_1 = V_2 A_2$

and $A_1' V_1 = V_2' A_2$.

Then $V_1 = A_1^{-1} V_2 A_2$

&

$$A_1' A_1^{-1} V_2 A_2 = V_2' A_2$$

&

$$A_1' A_1^{-1} V_2 = V_2'$$

$$A_1^{-1} V_2 = (A_1')^{-1} V_2'$$

&

$$(V_2^{-1}) A_1 = (V_2')^{-1} A_1'$$

& $A_1 \sim A_1'$ & $A_1 = A_1'$. $\square$

$$d_1 \delta_1 = \gamma_2 b_2 + \delta_2 d_2$$

$$\delta_1 a_2 = d_1 \delta_1 \left(\frac{a_2}{d_1}\right)$$

$$= \frac{\gamma_2 b_2 a_2}{d_1} + \frac{\delta_2 d_2 a_2}{d_1}$$

$$= \frac{d_2 \gamma_1 b_2 a_2}{a_1 d_1} + \frac{\delta_2 a_1 d_1}{d_1} \qquad \left(\gamma_2 = \frac{d_2 \gamma_1}{a_1}\right)$$

$$(a_1 d_1 = a_2 d_2)$$

$$= \frac{d_2 \gamma_1 b_2 a_2}{a_2 d_2} + \delta_2 a_1$$

$$\delta_1 a_2 = \gamma_1 b_2 + \delta_2 a_1 .$$

$\cancel{\text{So } a_1 \delta_2 = a_2 \delta_1 - b_2 \gamma_1}$

Now

$$a_1 (\gamma_2 A_2 \tau + \delta_2) = a_1 \left(\gamma_2 \left(\frac{a_2 \tau + b_2}{d_2} \right) + \delta_2 \right)$$

$$= d_2 \gamma_1 \left(\frac{a_2 \tau + b_2}{d_2} \right) + a_1 \delta_2$$

$$= \gamma_1 a_2 \tau + \gamma_1 b_2 + a_1 \delta_2$$

$$= \gamma_1 a_2 \tau + \delta_1 a_2$$

$$= a_2 (\gamma_1 \tau + \delta_1) . \ \square$$

See (15A)

Recall,

$$T_n f = n^{\frac{k}{2}-1} \sum_{A \in \Gamma \backslash \Gamma(n)} f|_k A$$

for $A \in M_k$.

Note

Let $A \in \Delta_n$, $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.

Then

$$n^{k/2-1} f|_k A = n^{\frac{k}{2}-1} n^{k/2} \frac{f(A\tau)}{(j_A(\tau))^k}$$

$$= n^{k-1} \frac{f(A\tau)}{d^k} \qquad \left(\begin{array}{l} ad = n \\ \frac{1}{d} = \frac{a}{n} \end{array} \right.$$

$$= n^{k-1} \frac{f(A\tau)}{(n/a)^k}$$

$$= \frac{1}{n} a^k f(A\tau).$$

So $$T_n f = \frac{1}{n} \sum_{A \in \Delta_n} a^k f(A\tau).$$

<u>Theorem</u>: For $f \in M_k$, $V_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \in \Gamma$ then

$$(T_n f)(V_1\tau) = (\gamma_1\tau + \delta_1)^k T_n f(\tau).$$

<u>Proof:</u>

$$(T_n f)(V_1\tau) = \frac{1}{n} \sum_{A_1 \in \Delta_n} a_1^k f(A_1 V_1 \tau)$$

For each $A_1 \in \Delta_n$ $\exists!\, V_2 \in \Gamma$, $A_2 \in \Delta_n$ s.th

$A_1 V_1 = V_2 A_2$

$$\boxed{\gamma_n(1) = c(n)}$$

$$a_1^k f(A_1 V_1 \tau) = a_1^k f(V_2 A_2 \tau)$$
$$= a_1^k (\gamma_2 A_2\tau + \delta_2)^k f(A_2 \tau), \quad \text{since } V_2 \in \Gamma \ \& \ f \in M_k$$
$$= a_2^k (\gamma_1 \tau + \delta_1)^k f(A_2 \tau) \quad \text{(by Thm (2))}$$

Hence

$$(T_n f)(V_1 \tau) = \frac{1}{n} \sum_{A_2 \in \Delta_n} a_2^k (\gamma_1 \tau + \delta_1)^k f(A_2 \tau) \quad \text{(by Thm)}$$

$$= (\gamma_1 \tau_1 + \delta_1)^k (T_n f)(\tau). \quad \square$$

~~Theorem~~ $\cancel{T_n : M_k \longrightarrow M_k}$ ~~and~~ $\cancel{T_n : S_k \longrightarrow S_k}$.

<u>Theorem</u>: If $f \in M_k$, ~~then~~ and

$$f(\tau) = \sum_{m=0}^{\infty} c(m) q^m \qquad (q = e^{2\pi i \tau})$$

Then

$$(T_n f)(\tau) = \sum_{m=0}^{\infty} \gamma_n(m) q^m$$

where
$$\gamma_n(m) = \sum_{d \mid (n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

<u>Proof</u>:
$$T_n f = \frac{1}{n} \sum_{A \in \Delta_n} a^k f(A\tau)$$
$$= \frac{1}{n} \sum_{ad=n} \sum_{a<b<d} a^k f\left(\frac{a\tau + b}{d}\right)$$

$1+x+x^2+\cdots = \frac{1}{1-x}$

(18)

$$e^{\left(\frac{a\tau+b}{d}\right)2\pi i} = e^{2\pi i\left(\frac{b}{d}\right)} e^{\frac{2\pi i a\tau}{d}}$$

$$= e^{2\pi i\left(\frac{b}{d}\right)} e^{\frac{2\pi i n\tau}{d^2}}$$

$$e^{\left(\frac{a\tau+b}{d}\right)2\pi i m} = e^{2\pi i b m/d} e^{\frac{2\pi i m n\tau}{d^2}}$$

$$\sum_{0\le b<d} e^{2\pi i b m/d}$$

$$= \begin{cases} d & \text{if } d \mid m \\ \dfrac{e^{2\pi i m}-1}{e^{2\pi i m/d}-1} = 0 & d \nmid m \end{cases}$$

$$\text{So } (T_n f)(\tau) = \frac{1}{n}\sum_{d\mid n}\sum_{0\le b<d} a^k \sum_{m=0}^{\infty} c(m) e^{\left(\frac{a\tau+b}{d}\right)2\pi i m}$$

$$= \frac{1}{n}\sum_{d\mid n}\sum_{0\le b<d}\sum_{m=0}^{\infty} a^k c(m) e^{2\pi i b m/d} e^{2\pi i\frac{m n\tau}{d^2}}$$

$$\cancel{= \frac{1}{n}\sum_{d\mid n}\sum_{m=0}^{\infty} a^k \cdot d\, c}$$

$$= \sum_{m=0}^{\infty}\sum_{d\mid n}\frac{1}{n}\left(\frac{n}{d}\right)^k c(md)\cdot d\, e^{2\pi i m n\tau/d}$$

$$= \sum_{m=0}^{\infty}\sum_{d\mid n}\left(\frac{n}{d}\right)^{k-1} c(md)\, e^{2\pi i m n\tau/d}$$

(replace d by n/d)

$$= \sum_{m=0}^{\infty} \sum_{d \mid n} d^{k-1} c\left(\frac{mn}{d}\right) e^{2\pi i m d \tau}$$

$$\cancel{= \sum_{m=0}^{\infty} \sum_{\substack{d \mid n \\ d \mid m}} d^{k-1} c\left(mn\right)}$$

$$= \sum_{l=0}^{\infty} \left(\sum_{\substack{md = l \\ d \mid n}} d^{k-1} c\left(\frac{mn}{d}\right) \right) q^{l}$$

$$= \sum_{l=0}^{\infty} \left(\sum_{\substack{d \mid l \\ d \mid n}} d^{k-1} c\left(\frac{ln}{d^2}\right) \right) q^{l}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right) \right) q^{m} \quad . \square$$

<u>Theorem</u>: $T_n : M_k \longrightarrow M_k$

$T_n : S_k \longrightarrow S_k$

and T_n is linear.

<u>Note</u> If $c(0)=0$ then $\gamma_n(0)=0$.

<u>Theorem</u> If $(m,n)=1$ then

$$T_m T_n = T_{mn}$$

Proof: If $f \in M_k$ then

$$(T_n f) = \frac{1}{n} \sum_{d|n} \sum_{b=0}^{d-1} a^k f(A\tau) \qquad A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$(T_m T_n f)(\tau) = \frac{1}{m} \sum_{d'|m} \sum_{b'=0}^{d'-1} (a')^k \frac{1}{n} \sum_{d|n} \sum_{b=0}^{d-1} a^k f(AB$$

$$B = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

$$= \frac{1}{mn} \sum_{d'|m} \sum_{b'=0}^{d'-1} \sum_{d|n} \sum_{b=0}^{d-1} (aa')^k f(C\tau)$$

$$C = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad ad = n$$

$$a'd' = m$$

$$= \begin{pmatrix} aa' & a'b + b'd \\ 0 & d'd \end{pmatrix}$$

Since $(m,n)=1$ every divisor of mn can be written uniquely as $d' d$ where $d'|m$ & $d|n$

<u>Claim</u> $a'b + b'd$ for $0 \le b < d$, $0 \le b' < d'$ are distinct mod $d'd$

For fixed b', $a'b + b'd$ $0 \le b < d$ gives d distinct residue classes (mod d), since $(a',d)=1$.

For fixed b, $a'b + b'd$, $0 \le b' < d'$ gives d' distinct residue classes (mod d') since $(d,d')=1$.

It follows that

$$T_m T_n f = \frac{1}{mn} \sum_{C \in \Delta_{mn}} (c_1)^k f(C\tau)$$

$$= T_{mn} f. \quad \square$$

<u>Notation</u> $T_n = T(n)$

<u>Theorem</u>: The Hecke operators commute and

$$T(m)\,T(n) = \sum_{d \mid (m,n)} d^{k-1}\, T\!\left(\frac{mn}{d^2}\right)$$

<u>Proof</u>: Case 1 The result holds for $(m,n)=1$.

<u>Case 2</u> $m = p$, $n = p^r$

$$T(p^r) f(\tau) = \cancel{\frac{1}{p^r} \sum_{d \mid p^r} \left(\frac{p^r}{d}\right)^k f(}$$

$$= \frac{1}{p^r} \sum_{d \mid p^r} \sum_{0 \le b < d} \left(\frac{p^r}{d}\right)^k f\left(\frac{\frac{p^r}{d}\tau + b}{d}\right)$$

$$= \frac{1}{p^r} \sum_{t=0}^{r} \sum_{0 \le b < p^t} p^{(r-t)k} f\left(\frac{p^{r-t}\tau + b}{p^t}\right)$$

$$T(p) g = \frac{1}{p} \sum_{t=0}^{1} \sum_{0 \le b < p^t} p^{(1-t)k} g\left(\frac{p^{1-t}\tau + b}{p^t}\right)$$

$$= p^{k-1} g(p\tau) + p^{-1} \sum_{b=0}^{p-1} g\left(\frac{\tau + b}{p}\right)$$

(2

$$T(p)\,T(p^r) f = p^{k-1-r} \sum_{t=0}^{r} \sum_{0\le b_t<p^t} p^{(r-t)k} f\left(\frac{p(p^{r-t}z+b_t)}{p^t}\right)$$

$$+\ p^{-1-r} \sum_{b=0}^{p-1} \sum_{t=0}^{r} \sum_{0\le b_t<p^t} p^{(r-t)k} f\left(\frac{\left(\frac{p^{r-t}z+b_t}{p^t}\right)+b}{p}\right)$$

$$= p^{k-1-r} \sum_{t=0}^{r} \sum_{0\le b_t<p^t} p^{(r-t)k} f\left(\frac{p^{r+1-t}z+p\,b_t}{p^t}\right) \quad (*)$$

$$+\ p^{-1-r} \sum_{t=0}^{r} \sum_{0\le b_t<p^t} p^{(r-t)k} \sum_{b=0}^{p-1} f\left(\frac{p^{r-t}z+b_t+b\,p^t}{p^{t+1}}\right)$$

Note (1) $b_t + bp^t$ runs thro a complete residue system mod p^{t+1}

(2) $(r+1)-(t+1) = r-t$

$$\{T(p^{r+1}) f\}(z) = p^{-r-1} \sum_{t=0}^{r+1} \sum_{0\le b_t'<p^t} p^{((r+1)-t)k} f\left(\frac{p^{r+1-t}z+b_t'}{p^t}\right)$$

$$= p^{-r-1} \sum_{t=0}^{r} \sum_{0\le b_{t+1}'<p^{t+1}} p^{(r-t)k} f\left(\frac{p^{r-t}z+b_{t+1}'}{p^{t+1}}\right)$$

$$+\ p^{-r-1} \sum_{0\le b_0'<1} p^{(r+1)k} f\left(\frac{p^{r+1}z+b_0'}{1}\right)$$

This is the $t=0$ term of $(*)$

Hence,

$$\{T(p)\,T(p^r) f\}(z) = \{T(p^{r+1}) f\}(z) + p^{-r-1} \sum_{t=1}^{r} \sum_{0\le b_t<p^t} p^{(r+1-t)k} f\left(\frac{p^{r-t}z+b_t}{p^{t-1}}\right)$$

For $1 \leq t \leq r$, $\exists\ q_t, r_t$

$$b_t = q_t p^{t-1} + r_t,$$

where $0 \leq q_t < p$, $0 \leq r_t < p^{t-1}$.

$$f\left(\frac{p^{r-t}z + b_t}{p^{t-1}}\right) = f\left(\frac{p^{r-t}z + q_t p^{t-1} + r_t}{p^{t-1}}\right)$$

$$= f\left(\left(\frac{p^{r-t}z + r_t}{p^{t-1}}\right) + q_t\right)$$

$$= f\left(\frac{p^{r-t}z + r_t}{p^{t-1}}\right) \qquad 0 \leq q_t < p$$

So

$$\{T(p)\, T(p^r)\, f\}(z) = \{T(p^{r+1}) f\}(z)$$

$$+ \; p^{-r-1} \sum_{t=1}^{r} \sum_{0 \leq r_t < p^{t-1}} p^{(r+1-t)k+1} f\left(\frac{p^{r-t}z + r_t}{p^{t-1}}\right)$$

(repl. t_1 by $t+1$; $r-1-t$ ~~$r-t$~~ $t = t' +$)

$$= \{T(p^{r+1}) f\}(z) + p^{-r} \sum_{t=0}^{r-1} \sum_{0 \leq r_t < p^t} p^{(r-t)k} f\left(\frac{p^{r-1-t}z + r_t}{p^t}\right)$$

$$= \{T(p^{r+1}) f\}(z) + p\, p^{k-1} p^{-(r-1)} \sum_{t=0}^{r-1} \sum_{0 \leq r_t < p^t} p^{(r-1-t)k} f\left(\frac{p^{r-1-t}z + r_t}{p^t}\right)$$

$$= \{T(p^{r+1}) f\}(z) + p^{k-1} \{T(p^{r-1}) f\}(z)$$

Hence

$$(*) \quad T(p)\, T(p^r) = T(p^{r+1}) + p^{k-1}\, T(p^{r-1})$$

We prove

$$(**)\quad T(p^s)\,T(p^r) = \sum_{t=0}^{r} p^{t(k-1)}\,T(p^{r+s-2t})$$

$$= \sum_{d\mid(p^r,p^s)} d^{k-1}\,T\left(\frac{p^{r+s}}{d^2}\right)$$

for all r, and all $s \geq r$.
We proceed by induction on r. The case $r=1$ follows from $(*)$.
Assume it holds for r and smaller powers and all $s \geq r$.

$$T(p^s)\,T(p^r)\,T(p) = T(p^s)\,T(p^{r+1}) + p^{k-1}T(p^s)\,T(p^{r-1})$$

By induction hypoth.

$$T(p^s)\,T(p^r)\,T(p) = \sum_{t=0}^{r} p^{t(k-1)}\,T(p^{r+s-2t})\,T(p)$$

$$= \sum_{t=0}^{r} p^{t(k-1)}\left(T(p^{r+s+1-2t}) + p^{k-1}\,T(p^{r+s-1-2t})\right)$$

so

$$T(p^s)\,T(p^{r+1}) = \sum_{t=0}^{r} p^{t(k-1)}\,T(p^{r+s+1-2t})$$

$$+ \sum_{t=0}^{r} p^{(t+1)(k-1)}\,T(p^{r+s-1-2t})$$

$$- p^{k-1}\,T(p^s)\,T(p^{r-1})$$

note by ind. hypoth

$$p^{k-1}\,T(p^s)\,T(p^{r-1}) = \sum_{t=0}^{r-1} p^{(t+1)(k-1)}\,T(p^{r+s-1-2t})$$

and

$$T(p^s)\,T(p^{r+1}) = \sum_{t=0}^{r} p^{t(k-1)}\,T(p^{r+s+1-2t})$$

$$+ p^{(r+1)(k-1)}\,T(p^{s-1-r})$$

$$= \sum_{t=0}^{r+1} p^{t(k-1)}\,T(p^{r+s+1-2t})$$ which gives $(**)$ by induction.

<u>Completion of proof:</u>

Let $m = \prod_{i=1}^{j} p_i^{m_i}$ $n = \prod_{i=1}^{j} p_i^{n_i}$, be prime factorizations of m, n.

Then $T(m)\,T(n) = T(p_1^{m_1})\,T(p_2^{m_2}) \cdots T(p_j^{m_j})\,T(p_1^{n_1}) \cdots T(p_j^{n_j})$

(since $T(mn) = T(m)\,T(n)$ for $(m,n)=1$),

$$= T(p_1^{m_1})\,T(p_1^{n_1})\,T(p_2^{m_2})\,T(p_2^{n_2}) \cdots T(p_j^{m_j})\,T(p_j^{n_j})$$

$$= \sum_{t_1=0}^{\min(m_1,n_1)} p_1^{t_1(k-1)}\,T(p_1^{m_1+n_1-2t_1}) \cdots \sum_{t_j=0}^{\min(m_j,n_j)} p_j^{t_j(k-1)}\,T(p_j^{m_j+n_j-2t_j})$$

$$= \sum_{\substack{0 \le t_i \le \min(m_i,n_i) \\ 1 \le i \le j}} \left(p_1^{t_1} \cdots p_j^{t_j}\right)^{(k-1)} T\left(\frac{mn}{\left(p_1^{t_1} \cdots p_j^{t_j}\right)^2}\right)$$

$$= \sum_{d \mid (m,n)} d^{k-1}\,T\left(\frac{mn}{d^2}\right). \quad \square$$

Eigenfunctions of Hecke operators

Recall, if $f \in M_k$ &

$$f(z) = \sum_{m=0}^{\infty} c(m) q^m \qquad (q = e^{2\pi i z})$$

Then

$$(T_n f)(z) = \sum_{m=0}^{\infty} \gamma_n(m) q^m$$

where $\displaystyle \gamma_n(m) = \sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right)$.

NOTE:

$$\gamma_n(0) = \sum_{d \mid n} d^{k-1} c(0) = c(0)\, \sigma_{k-1}(n).$$

$$\gamma_n(1) = \sum_{d \mid (1,n)} d^{k-1} c\left(\frac{n}{d^2}\right) = c(n).$$

Example: $\dim S_{12} = 1$, & $\Delta \in S_{12}$

so $S_{12} = \mathbb{C}\Delta$.

Let $\displaystyle f(z) = \frac{1}{(2\pi)^{12}} \Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$; $\tau(1) = 1$.

$T_n : S_{12} \longrightarrow S_{12}$ so

$$T_n f = c_n f \qquad \text{some constant } c_n$$

Coeff of q^1 in $T_n f = \tau(n)$

Coeff of q^1 in $c_n f = c_n \tau(1) = c_n$

so $c_n = \tau(n)$ and

$$T_n f = \tau(n) f$$

Hence $\tau(n)$ is eigenvalue of T_n.

$$(T_m T_n) f = T_m(\tau(n) f) = \tau(n)\tau(m) f$$

$$T_m T_n = \sum_{d \mid (m,n)} d^{k-1} T_{\frac{mn}{d^2}}$$

$$T_m T_n f = \sum_{d \mid (m,n)} d^{k-1} T_{\frac{mn}{d^2}} f$$

$$= \sum_{d \mid (m,n)} d^{k-1} \tau\left(\frac{mn}{d^2}\right) f$$

Hence $\tau(m)\tau(n) f = \sum_{d \mid (m,n)} d^{k-1} \tau\left(\frac{mn}{d^2}\right) f$

and $\tau(m)\tau(n) = \sum_{d \mid (m,n)} d^{k-1} \tau\left(\frac{mn}{d^2}\right)$ since $f \neq 0$.

Cor If $(m,n)=1$ $\tau(m)\tau(n) = \tau(mn)$.

Definition: A function $f \in M_k$, $f \neq 0$ is an eigenfunction (or eigenform) of the Hecke operator T_n if $\exists\ \lambda(n) \in \mathbb{C}$ s.th.

$$T_n f = \lambda(n) f.$$

If f is an eigenform for every Hecke operator T_n, $n \geq 1$ then f is called a simultaneous eigenform.

For example, Δ is a simultaneous eigenform for the Hecke operators T_n on M_{12}, $n \geq 1$.

Theorem Let $k \geq 4$ be even. If the space M_k contains a simultaneous eigenform f

$$f = \sum_{n=0}^{\infty} c(n)\, q^n \qquad (q = e^{2\pi i z})$$

then

$$c(1) \neq 0.$$

Proof. Suppose $T_n f = \lambda(n) f$

Then let

$$T_n f = \sum_{m \geq 0} \gamma_n(m)\, q^m,$$

and

$$\gamma_n(1) = \sum_{d \mid (n,1)} d^k\, c\left(\frac{n}{d^2}\right) = c(n)$$

and

$$c(n) = \lambda(n)\, c(1) \qquad \text{for all } n \geq 1.$$

If $c(1) = 0$, then $c(n) = 0$ for all $n \geq 1$ and ~~for any~~ $f = c(0)$ a constant so $f = 0$ ※.

Hence $c(1) \neq 0$. □

An eigenform is called normalized if $c(1) = 1$.

Example $f(z) = \dfrac{\Delta}{(2\pi)^{12}} = \sum_{n=1}^{\infty} \tau(n)\, q^n$

is a simultaneous normalized eigenform for M_{12}.

Theorem Let $k \geq 12$ be even. Let $f \in S_k$. Assume $c(1) \neq 0$. f is a simultaneous normalized eigenform iff the Fourier coefficients $c(n)$ of f (ie $f(z) = \sum_{n \geq 1} c(n) q^n$) satisfy

$$c(m)\, c(n) = \sum_{d \mid (m,n)} d^{k-1}\, c\left(\frac{mn}{d^2}\right)$$

~~and $c(1) \neq 0$~~ for all $m, n \geq 1$. Further the $c(n)$ is eigenvalue of T_n.

Proof: ($\Rightarrow$) Suppose f is a simult. normalized eigenform.

Then $\qquad T_n f = \lambda(n) f \qquad$ for all n.

$$\gamma_n(1) = c(n) = \lambda(n)\, c(1) = \lambda(n).$$

$$T_m T_n = \sum_{d \mid (m,n)} d^{k-1} T_{\frac{mn}{d^2}}$$

$$T_m T_n f = \sum_{d \mid (m,n)} d^{k-1} T_{\frac{mn}{d^2}} f$$

$$\left(\lambda(m)\lambda(n)\right) f = \left(\sum_{d \mid (m,n)} d^{k-1} \lambda\left(\frac{mn}{d^2}\right)\right) f$$

$$\lambda(m)\lambda(n) = \sum_{d \mid (m,n)} d^{k-1} \lambda\left(\frac{mn}{d^2}\right) \qquad (\text{since } f \neq 0)$$

$$c(m)c(n) = \sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

($\Leftarrow$) Suppose $c(m)\,c(n) = \sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right)$

$$T_n f = \sum_{m \geq 1} \gamma_n(m)\, q^m$$

where $\gamma_n(m) = \sum_{d \mid (m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right)$

$c(1)c(1) = c(1)$ & since $c(1) \neq 0$

$$= c(m)\,c(n)$$

$c(1) = 1$

$$T_n f = \sum_{m \geq 1} c(m)\,c(n)\,q^m = c(n) \sum_{m \geq 1} c(m)\, q^m$$

$$T_n f = c(n)\, f$$

$\cancel{\gamma_n(1) = c(n) = c(1)\,c(n)}$

$c(1) \neq 0$ so $f \neq 0$ & f is a simult. eigenform.

$$\gamma_n(1) = c(n) = c(1)\,c(n)$$

~~$\exists\, c(n) \neq 0 \Rightarrow$~~ $c(1) = 1$ & f is a simult. normalized eigenform. $\square$

Theorem Let $f \in M_{2k}$, $k \geq 2$, and suppose f is not a cusp form (ie $c(0) \neq 0$). Then f is a normalized simultaneous eigenform if & only if

$$f(\tau) = \frac{(2k-1)!}{2(2\pi i)^{2k}} G_{2k}(\tau).$$

Proof: Let $f(\tau) = \sum_{n=0}^{\infty} c(m) q^m$. ~~and suppose~~ ~~f satisfies~~ the relation

$$T_n f = \lambda(n) f$$

$$\Leftrightarrow \quad \gamma_n(m) = \lambda(n)\, c(m) \ (\text{for all } m \geq 0)$$

$$\gamma_n(0) = \left(\sum_{d \mid n} d^{2k-1}\right) c(0) = \sigma_{2k-1}(n)\, c(0)$$

$c(0) \neq 0$.

($\Rightarrow$) If f is a normalized sim. eigenform.

$$T_n f = \lambda(n) f \qquad \text{for all } n \geq 1.$$

$$\gamma_n(0) = \sigma_{2k-1}(n)\, c(0) = \lambda(n)\, c(0)$$

and $\lambda(n) = \sigma_{2k-1}(n)$ for $n \geq 1$

$$\gamma_n(1) = \sum_{d \mid 1} d^{k-1} \left(c\left(\frac{n}{d^2}\right)\right) = c(n)$$

$$c(n) = \lambda(n)\, c(1) \qquad (\text{coeff of } q^1)$$

$$c(n) = \lambda(n)$$

and $c(n) = \sigma_{2k-1}(n)$ for all $n \geq 1$.

$$G_{2k}(\tau) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m$$

$$\frac{(2k-1)!}{2(2\pi i)^{2k}} G_{2k}(\tau) = \frac{2\zeta(2k)}{2(2\pi i)^{2k}} + \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m$$

So $f - \dfrac{(2k-1)!\, G_{2k}}{2(2\pi i)^{2k}} = $ constant & in M_{2k} so cont $= c$

Hence,

$$f(z) = \frac{(2k-1)!}{2(2\pi i)^{2k}} G_{2k}(z).$$

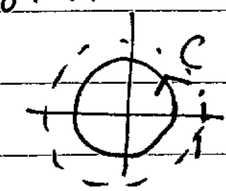
($\Leftarrow$) Suppose $f(z) = \frac{(2k-1)!}{2(2\pi i)^{2k}} G_{2k}(z)$.

<u>Estimates for the Fourier coefficients</u>

$$f(\tau) = \sum_{n=0}^{\infty} c(n)\, q^n \qquad (q = e^{2\pi i \tau})$$

$$|q| < 1$$

$$c(n) = \frac{1}{2\pi i} \int_C \frac{f(\tau)}{q^{n+1}}\, dq$$



$$|q| = e^{-2\pi y_0} \qquad C(y_0) = C : q = e^{2\pi i (x + i y_0)},$$

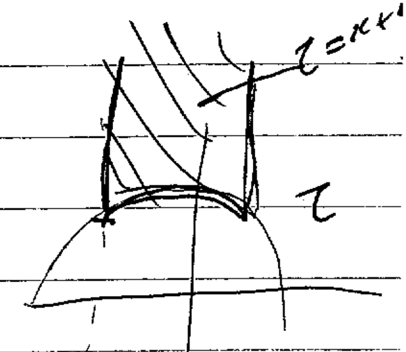
$$0 \le x \le 1$$

$$dq = 2\pi i\, q\, dx$$

$$\frac{dq}{q} = 2\pi i\, dx$$

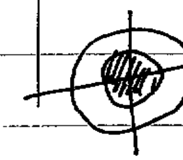
$$c(n) = \frac{1}{2\pi i} \int_0^1 f(x + i y_0) \frac{dx}{q^n}$$

$$= \int_0^1 \frac{f(x+iy_0)\, dx}{e^{2\pi i n (x + i y_0)}}$$



<u>Theorem</u> If $f \in S_k$ then

$$c(n) = O(n^k).$$



<u>Proof:</u> The series $f(\tau) = \sum_{n=0}^{\infty} c(n) q^n$ is absolutely convergent for $|q| < 1$ (since as a function of q it analytic). Let $\tau \in R_\Gamma$ (fundamental domain of Γ),

$$\tau = x + iy, \qquad y \geq \frac{\sqrt{3}}{2} > \frac{1}{2}.$$

$$|q| = |e^{2\pi i \tau}| = |e^{2\pi i (x + iy)}| = e^{-2\pi y} < e^{-\pi}$$

and

$$|f(\tau)| = \left|\sum_{n=1}^{\infty} c(n) q^n\right| = |q| \sum_{n=1}^{\infty} |c(n)| |q|^{n-1}$$

$$|f(\tau)| \leqslant A|q| = A e^{-2\pi y}$$

where

$$A = \sum_{n=1}^{\infty} |c(n)| e^{-(n-1)\pi} < \infty.$$

Therefore,

$$(*) \quad |f(\tau)|\, y^k \leqslant A\, y^k e^{-2\pi y}.$$

Let $\tau = x + iy$, $\bar{\tau} = x - iy$, $\tau - \bar{\tau} = 2iy$

so $y(\tau) := \frac{1}{2}|\tau - \bar{\tau}| = y$, for $\tau \in H$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$\cancel{y(A\tau)}$

$$\operatorname{Im}(A\tau) = \frac{\operatorname{Im}\tau}{|c\tau+d|^2} = \frac{y}{|c\tau+d|^2}$$

$$\left(\operatorname{Im}(A\tau)\right)^k = \frac{y^k}{|c\tau+d|^{2k}}$$

$$|f(\tau)|\, y^k \xrightarrow{A} |f(A\tau)| \frac{y^k}{|c\tau+d|^{2k}} = \frac{|f(\tau)|\,|c\tau+d|^{2k} y}{|c\tau+d|^{2k}}$$

$$= |f(\tau)|\, y^k$$

Hence $\varphi(\tau) = |f(\tau)|\, y^k = |f(\tau)|(\operatorname{Im}\tau)^k$ is invariant under transformation in Γ.

Therefore $\varphi(H) = \varphi(R_\Gamma)$

(*) implies $|\varphi(\tau)| \longrightarrow 0$ as $y \longrightarrow \infty$ $(\tau \in R_\Gamma)$.

So φ is bounded on R_Γ since φ is continuous and φ is bounded on H, say

$$|\varphi(\tau)| \leqslant M$$

for all $\tau \in H$, and

$$|f(\tau)| \leqslant M y^{-k} \quad \text{for } \tau \in H.$$

Therefore,

$$|c(n)| \leqslant \int_0^1 |f(x+iy_0)|\, e^{+2\pi y_0 n}\, dx$$

$$\leqslant \int_0^1 M y_0^{-k} e^{2\pi y_0 n}\, dx$$

$$= M y_0^{-k} e^{2\pi y_0 n}$$

This holds for all fixed $0 < y_0$. Take $y_0 = 1/n$

$$|c(n)| \leqslant M n^{k} e^{2\pi}$$

and $$c(n) = O(n^{k}). \quad \square$$

<u>Theorem</u> If $f \in M_{2k}$ then

$$c(n) = O(n^{2k-1})$$

<u>Proof</u>:

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^{m}$$

So for $f(z) = G_{2k}(z)$,

$$|c(n)| \leqslant \alpha\, \sigma_{2k-1}(n) \quad \text{some } \alpha = \alpha(k)$$

and

$$\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1} = \sum_{d|n} \left(\frac{n}{d}\right)^{2k-1}$$

$$= n^{2k-1} \sum_{d|n} \frac{1}{d^{2k-1}} \leqslant n^{2k-1} \sum_{d=1}^{\infty} \frac{1}{d^{2k-1}}$$

$$= O(n^{2k-1}) \qquad (\text{for } k \geqslant 2).$$

Let $f \in M_{2k}$. Then

$$f = \lambda G_{2k} + g \qquad \text{where } \lambda = f(i\infty)/G_{2k}(i\infty)$$

and $g \in S_{2k}$ so $c(n) = O(n^{2k-1}) + O(n^{k}) = O(n^{2k-1})$. $\square$

The Petersson Scalar-product

Let $f, g \in M_k$ with at least one of f, g a cusp form then

$$\langle f, g\rangle := \int_F f(z)\,\overline{g(z)}\; y^k \,\frac{dx\,dy}{y^2}$$

where F is a fundamental region for Γ.

NOTE: Let $h(z)$ be analytic.

$$h(z) = u(x,y) + i\,v(x,y).$$

Jacobian determinant $= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \quad \left(\text{since } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \&\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$$

$$= |h'(z)|^2 \quad \left(\text{since } h'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

So $\quad dx\,dy \xrightarrow{h} |h'(z)|^2\,dx\,dy$

Let $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2^+(\mathbb{Q})$.

Then $\quad \operatorname{Im} Az = \dfrac{\det A}{|cz+d|^2}\operatorname{Im} z$

$$\frac{d}{dz}Az = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{\det A}{(cz+d)^2}$$

$$\left|\frac{d}{dz}Az\right|^2 = \frac{(\det A)^2}{|cz+d|^4}$$

$$z \xrightarrow{A} Az$$

$$\frac{dx\,dy}{y^2} \longrightarrow \frac{(\det A)^2}{|cz+d|^4} \frac{1}{\left(\frac{\det A \operatorname{Im} z}{|cz+d|^2}\right)^2} dx\,dy = \frac{dx\,dy}{y^2}$$

Recall

$$f(z)|_k A = (\det A)^{k/2} \frac{f(Az)}{(cz+d)^k}$$

Let $A \in GL_2^+(\mathbb{Q})$

$$f(z)\,\overline{g(z)}\,y^k \xrightarrow{A} f(Az)\,\overline{g(Az)}\,\frac{(\det A)^k}{|cz+d|^{2k}}\,y^k$$

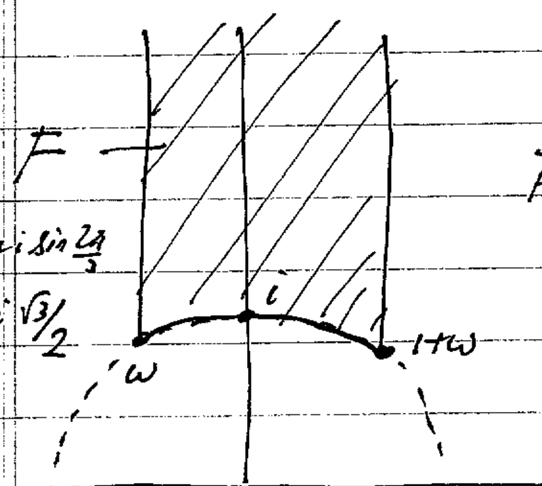
$$= [f(z)|_k A]\,\overline{[g(z)|_k A]}\,y^k$$

Recall for M_k the Hecke operator

$$T_n f(z) = n^{(k/2)-1} \sum_A f|_k A$$

where sum is over representatives A mod $\sim$

$(A_1 \sim A_2$ iff $A_1 = V A_2$, $V \in \Gamma$; $A_1, A_2 \in T(n) = \{A : \det A = n\})$



$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$
$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\int_F \frac{dx\,dy}{y^2} < \int_{-1/2}^{1/2} \int_{\frac{\sqrt{3}}{2}}^{\infty} y^{-2}\,dy\,dx$$

$$= \int_{-1/2}^{1/2} -y^{-1}\Big|_{\sqrt{3}/2}^{\infty} dx$$

$$= \int_{-1/2}^{1/2} 2/\sqrt{3}\,dx = 2/\sqrt{3}$$

NOTE: When at least one $f, g \in M_k$ is a cusp form the integral

$$\int_F f(z)\,\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2}$$

converges.

Suppose wlog $f \in S_k$. Then as on pp 32-33

$$|f(z)| \leq A\,e^{-2\pi y} \quad \text{for } z \in F \supseteq R_r$$

and $A = \sum_{n=1}^{\infty} |a(n)|\,e^{-(n-1)\pi} < \infty$.

Similarly $|g(z)| \leq B$ for $z \in F \supseteq R_r$

where $B = \sum_{n=0}^{\infty} |d(n)|\,e^{-n\pi}$

where $g(z) = \sum_{n=0}^{\infty} d(n)\,q^n$. So

$$|f(z)\,\overline{g(z)}|\,y^k \leq AB\,y^k\,e^{-2\pi y} \leq C$$

some constant C (since $\lim_{y\to\infty} y^k e^{-2\pi y} = 0$)

Hence

$$\left|\int_F f(z)\,\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2}\right| \leq \int_F C\,\frac{dx\,dy}{y^2} < \infty.$$

NOTE: The integral $\displaystyle\int_F f(z)\,\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2}$

is indept. of fundamental region F.

Suppose F' is another fundamental region. Divide F' into subregions R' s.t.

$$AR' \subset F$$

for some $A \in \Gamma$. Replace z by $Az \Rightarrow$

$$\cancel{\int_{AR'} f(z)\,\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2} = \int_{R'}}$$

$$\int_{AR'} \left(f\big|_k A^{-1}\right)(z)\, \overline{\left(g\big|_k A^{-1}\right)(z)}\; y^k \frac{dx\,dy}{y^2}$$

$$= \int_{R'} f(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2}$$

& $\displaystyle\int_{R'} f(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2} = \int_{AR'} f(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2}$

since $f, g \in M_k$. It follows that

$$\int_F f(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2} = \int_{F'} f(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2}$$

$\#$

<u>Theorem</u> Let $A \in T(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \;:\; ad - bc = n \right\}$.
Let $A' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ so that $A' = nA^{-1}$.
Then

(a) $\langle f|_k A, g \rangle = \langle f, g|_k A' \rangle$
for $f, g \in M_k$ (at least one in S_k).

(b) If $A_1 \sim A_2$ then $\langle f|_k A_1, g \rangle = \langle f|_k A_2, g \rangle$.

<u>Proof</u>: Let $A \in T(n)$

$$\cancel{\langle f, g \rangle = \int_F f(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2}}$$

$$\cancel{= \int_{A^{-1}(F)} \left(f|_k A\right)(z) \left(g|_k A\right)}$$

$$\langle f|_k A, g \rangle = \int_F \left(f|_k A\right)(z)\,\overline{g(z)}\; y^k \frac{dx\,dy}{y^2}$$

(replace z by $A^{-1}z$)

$$= \int_{AF} \left(f\big|_k A\big|_k A^{-1}\right)(z)\, \overline{\left(g\big|_k A^{-1}\right)(z)}\; y^k \frac{dx\,dy}{y^2}$$

$$= \int_{AF} f(z)\, \overline{\left(g\big|_k A'\right)(z)}\; y^k \frac{dx\,dy}{y^2}.$$

<u>Claim</u>: AF is a fundamental region for Γ.

~~Let $z \in H$. Then $A^{-1}z \in H$. $\exists V_1 \in \Gamma$ s.t. $V_1 A^{-1} z \in F$. $V_1 A^{-1} \in \Gamma(n)$ so $\exists V_2 \in \Gamma$ s.t. $V_1 A^{-1} = A V_2$ so $A V_2 z \in F$~~

<u>NOT TRUE</u> (<u>Eg</u>: $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$)

<u>Claim</u>: AF is a fundamental region for $A\Gamma' A^{-1}$
where $\Gamma' = \Gamma \cap \left(A^{-1}\Gamma A\right)$

<u>Theorem</u> Let $\Gamma' \subset \Gamma$ be a congruence subgroup
~~Define~~ Let $F' \subset H$ be any fundamental region for Γ'.
Define

$$\mu(\Gamma') := \int_{F'} \frac{dx\,dy}{y^2}.$$

(a) The integral converges and is independent of the choice of F'.

(b) $[\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma)$

(c) If $A \in GL_2^+(\mathbb{Q})$ & $A^{-1}\Gamma' A \subset \Gamma$ $\sim$
$[\Gamma : \Gamma'] = [\Gamma : A^{-1}\Gamma' A]$.

<u>Here:</u>

$$\bar{\Gamma}' = \begin{cases} \Gamma'/\{\pm I\} & \text{if } \quad -I \in \Gamma' \\ \Gamma' & \text{otherwise.} \end{cases}$$

<u>Proof:</u> We have seen that the integral for $\mu(\Gamma)$ converges.

Suppose $[\bar{\Gamma} : \bar{\Gamma}'] = d$, $\quad \bar{\Gamma} = \bigcup_{j=1}^{d} A_j \bar{\Gamma}'$ (cosets)

Then

$F' = \bigcup A_j^{-1} F$ is a fundamental region for Γ'

$$\int_{F'} \frac{dx\,dy}{y^2} = \sum_j \int_{A_j^{-1}F} \frac{dx\,dy}{y^2}$$

$$= \sum_j \int_F \frac{dx\,dy}{y^2} \qquad \text{(since each } A_j^{-1}F \text{ is a f. region for } \Gamma)$$

$$= d \int_F \frac{dx\,dy}{y^2}$$

so $\mu(\Gamma') = [\bar{\Gamma} : \bar{\Gamma}']\, \mu(\Gamma)$

<u>Claim:</u> $A^{-1}F'$ is a fundamental region for $A^{-1}\Gamma' A$.

<u>Pf:</u> Let $z \in H$, $Az \in H$. $\exists\, G \in \Gamma'$: $GAz \in F'$

& $(A^{-1}GA)(z) \in A^{-1}F'$, $\quad A^{-1}GA \in A^{-1}\Gamma' A$. <u>etc</u>

$$\int_{A^{-1}F'} \frac{dx\,dy}{y^2} = \int_{F'} \frac{dx\,dy}{y^2}$$

So $\mu(A^{-1}\Gamma' A) = \mu(\Gamma') \Rightarrow [\bar{\Gamma} : \bar{\Gamma}'] = [\bar{\Gamma} : A^{-1}\bar{\Gamma}' A]$.

Definition: Let $\Gamma' \subset \Gamma$ be a cong. subgp and F' a fundamental domain for Γ'. Let $f, g \in M_k(\Gamma')$ with at least two of the functions cusp forms

$$\langle f, g \rangle := \frac{1}{[\Gamma : \Gamma']} \int_{F'} f(z)\, \overline{g(z)}\, y^k \frac{dx\,dy}{y^2}$$

Note $f \in M_k(\Gamma')$ if f is analytic on H,
$(f|_k A)(z) = f(z)$ for all $A \in \Gamma'$,
and for any $V \in \Gamma = SL_2(\mathbb{Z})$ there is an expansion

(*) $$(f|_k V)(z) = \sum_{n \geq 0} a_n \exp\left(\frac{2\pi i z n}{N}\right)$$

Here (1) $\Gamma(N) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : A \equiv I \pmod N\}$
& $\Gamma(N) \subset \Gamma' \subset \Gamma$.

(2) Here $a_n = a_{n,V}$.
If $a_{0,V} = 0$ for all V then f is called a cusp form & $f \in S_k(\Gamma')$.

Lemma 1. Let $A \in GL_2^+(\mathbb{Q})$ have integer entries & let $d = \operatorname{Det} A$. If $\Gamma(N) \subset \Gamma'$ then
$$\Gamma(Nd) \subset A^{-1}\Gamma' A.$$

Proof: Suppose $G \in \Gamma(ND)$.
$$G = I + NdB, \qquad B \in M_2(\mathbb{Z}),$$
$$\det G = 1.$$

$$AGA^{-1} = \underline{I} + Nd\,ABA^{-1}$$
$$= \underline{I} + N\,(AB\,(dA^{-1}))$$
$$\equiv \underline{I} \pmod{N}$$ (since dA^{-1} has integer entries)

so

$$AGA^{-1} \in \Gamma(N) \subset \Gamma'$$

& $G \in A^{-1}\Gamma' A$.

Therefore $\Gamma(Nd) \subset A^{-1}\Gamma' A$. □