SUPPLEMENT TO CH 1

PARTIAL FRACTION DECOMPOSITION of $\cot \pi z$

Let $\tau \notin \mathbb{Z}$. ~~Choose~~ Let $n \in \mathbb{Z}$, $n \geq 2|\tau|$.
Let

$$g(z) = \frac{\pi \cot \pi z}{\tau^2 - z^2} = \frac{-\pi}{2\tau}(\cot \pi z)\left(\frac{1}{z-\tau} - \frac{1}{z+\tau}\right)$$

$$\operatorname*{Res}_{z=\tau} g(z) = \operatorname*{Res}_{z=-\tau} g(z) = -\frac{\pi}{2\tau} \cot \pi\tau$$

Let $n \geq 1$. $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$

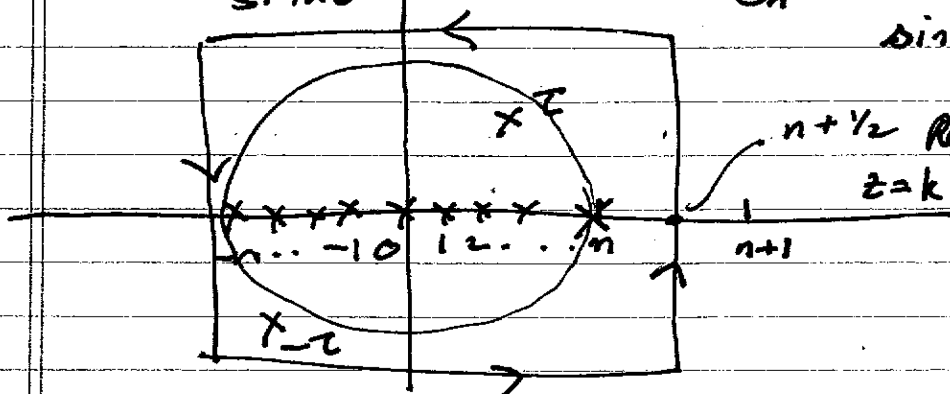
near $z = k$ (integer)

C_n $k = 0, \pm 1, \pm 2, \ldots, \pm n$

$\sin \pi z = \pi (\cos \pi k)(z-k) + \cdots$

&

$$\operatorname*{Res}_{z=k} g(z) = \frac{1}{\tau^2 - k^2}$$



$$|\cos \pi z|^2 = \cos^2 \pi x + \sinh^2 \pi y$$
$$|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y$$

On the vertical sides $z = \pm(n + 1/2) + iy$

So

$$|\sin \pi z|^2 = 1 + \sinh^2 \pi y = \cosh^2 \pi y$$
$$|\cos \pi z|^2 = \sinh^2 \pi y$$

Hence

$$|\cot \pi z|^2 = \frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y} \leq 1$$

On the horizontal sides $z = x \pm i(n + 1/2)$
where $-(n+\tfrac{1}{2}) \leq x \leq (n + 1/2)$.

$$|\cos \pi z|^2 \leq 1 + \sinh^2 \pi y = \cosh^2 \pi y$$

and $|\cos \pi z| \leq \cosh \pi y$

$$|\sin \pi z|^2 \geqslant \sinh^2 \pi y$$

So $\quad |\cot \pi z| \leqslant \dfrac{\cosh \pi y}{\sinh |\pi y|}$ $\quad$ (for $y \neq 0$)

$$= \frac{e^{\pi y} + e^{-y\pi}}{e^{y\pi} - e^{-y\pi}} \quad (y > 0)$$

$$= \frac{e^{\pi y} - e^{-y\pi} + 2e^{-y\pi}}{e^{\pi y} - e^{-\pi y}}$$

$$= 1 + \frac{e^{-y\pi}}{\sinh \pi y}$$

Let $f(x) = e^{-x} / \sinh \pi x$, $f'(x) = \dfrac{-1}{(\sinh x)^2} < 0$
so f is decreasing &

$$|\cot \pi z| \leqslant \frac{\cosh \pi/2}{\sinh \pi/2} \quad \text{for } z \text{ on horizontal edge}$$

$$\leqslant 2$$

Hence for $z \in C_n$, $\quad |\cot \pi z| \leqslant 2$.

For $z \in C_n \quad |z| \geqslant n \geqslant 2|\tau|$ &

$$\left|\frac{z^2}{\tau^2 - z^2}\right| = \frac{1}{\left|1 - \left(\frac{\tau}{z}\right)^2\right|} \leqslant \frac{1}{1 - \left|\frac{\tau}{z}\right|^2} \leqslant \frac{4}{3}$$

& $\quad \dfrac{1}{|\tau^2 - z^2|} \leqslant \dfrac{4}{3} \dfrac{1}{|z|^2} \leqslant \dfrac{4}{3n^2}$

Hence $\quad \left|\displaystyle\int_{C_n} g(z)dz\right| \leqslant \left(\dfrac{8}{3n^2}\right) \pi \; 4(2n+1) \longrightarrow 0$

as $n \to \infty$

(3)

also $\displaystyle \frac{1}{2\pi i}\int_{C_n} g(t)\,dt = \frac{-\pi}{\tau}\cot \pi\tau + \sum_{k=-n}^{n} \frac{1}{\tau^2-k^2}$

Letting $n \to \infty$ we have

~~Hence~~

$$\frac{\pi \cot \pi\tau}{\tau} = \sum_{-\infty}^{\infty} \frac{1}{\tau^2-k^2}$$

$$\pi \cot \pi\tau = \frac{1}{\tau} + \cancel{\Sigma} \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{\tau-k} + \frac{1}{\tau+k}\right)$$

$$= \frac{1}{\tau} + \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\frac{1}{\tau+k} - \frac{1}{k}\right) + \left(\frac{1}{\tau-k} + \frac{1}{k}\right)\right)$$

$$= \frac{1}{\tau} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left(\frac{1}{\tau+k} - \frac{1}{k}\right).$$

NOTE: $\displaystyle \frac{2\tau}{\tau^2-k^2} = \frac{1}{\tau-k} + \frac{1}{\tau+k}.$

Bernoulli numbers

Let $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2}$

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2}$$

$$= \frac{-x e^x}{1 - e^x} - \frac{x}{2}$$

$$= x \frac{e^x}{e^x - 1} - \frac{x}{2}$$

$$= x\left(\frac{e^x - 1 + 1}{e^x - 1}\right) - \frac{x}{2}$$

$$= \frac{x}{e^x - 1} + \frac{x}{2} = f(x).$$

So $f(x)$ is an even function, &

$$\frac{x}{e^x-1} + \frac{x}{2} = 1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_k x^{2k}}{(2k)!}$$

for some rational $B_k \in \mathbb{Q}$.

$B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$
$B_5 = \frac{5}{66}$, ...

For $\mathrm{Re}\, s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$ (Riemann Zeta function)

Theorem For $k \geq 1$ $(k \in \mathbb{Z})$,

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} \pi^{2k} B_k$$

Proof: We know

$$\frac{\pi \cot \pi z}{z} = \sum_{k=-\infty}^{\infty} \frac{1}{z^2 - k^2} \quad (\text{for } z \notin \mathbb{Z})$$

$$= \frac{1}{z^2} + 2 \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2}$$

$$\pi z \cot \pi z = 1 + 2 \sum_{k=1}^{\infty} \frac{z^2}{z^2 - k^2}$$

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n \quad |x| < 1$$

$$\pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \frac{(z/k)^2}{1 - (z/k)^2}$$

$$= 1 - 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{z^{2n}}{k^{2n}} \quad (\text{for } |z| < 1)$$

$$= 1 - 2 \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \right) z^{2n} \quad (\text{by abs. converge})$$

$$= 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) z^{2n}$$

$$\cot \pi z = i\left(\frac{e^{i\pi z}+e^{-\pi z i}}{e^{i\pi z}-e^{-\pi z i}}\right)$$

$$= i\left(\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}\right)$$

$$= i\left(\frac{e^{2\pi i z}-1+2}{e^{2\pi i z}-1}\right)$$

$$= i + \frac{2i}{e^{2\pi i z}-1}$$

Hence $\pi z \cot \pi z = \pi z i + \dfrac{2\pi z i}{e^{2\pi i z}-1}$

$$\cancel{= \pi z - i\left(\frac{2\pi i z}{e^{2\pi i z}-1}\right)}$$

$$\cancel{= \pi z - i\left(1 - \frac{2\pi i z}{2} + \sum_{k=1}^{\infty}(-1)^{k+1} B_k \frac{(2\pi i z)^{2k}}{(2k)!}\right)}$$

$$= \pi z i + 1 - \frac{2\pi i z}{2} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2\pi i z)^{2k}}{(2k)!}B_k$$

$$= 1 - \sum_{k=1}^{\infty}\frac{\pi^{2k}2^{2k}}{(2k)!}B_k z^{2k}$$

Hence $\zeta(2k) = \dfrac{\pi^{2k}2^{2k-1}}{(2k)!}B_k$

Examples

$$\zeta(2) = \frac{\pi^2 \cdot 2}{2} \cdot \frac{1}{6} = \pi^2/6$$

$$\zeta(4) = \frac{\pi^4 \cdot 8}{4 \cdot 3 \cdot 2} \, \frac{1}{30} = \frac{\pi^4}{90}.$$

Cor $\zeta(2n)$ is irrational (& transcendental).

Theorem $\zeta(3)$ is irrational. [Apery, 1979].

Griess (1982) constructed the monster group F_1 ($|F_1| \approx 8 \times 10^{53}$) as a automorphism group of a commutative nonassoc. algebra B_0 of dim 196883.