# CHAPTER I

## A SIMPLE PROOF OF WATSON'S PARTITION CONGRUENCES FOR POWERS OF 7

### 1.1 INTRODUCTION

In 1919 Ramanujan [ 15 ] conjectured that if $\alpha \geq 1$, if $\delta_\alpha$ is the reciprocal modulo $5^\alpha$ of 24, if $\lambda_\alpha$ is the reciprocal modulo $7^\alpha$ of 24 and if $\psi_\alpha$ is the reciprocal modulo $11^\alpha$ of 24, then

(1.1.1) $$p(5^\alpha n + \delta_\alpha) \equiv 0 \mod 5^\alpha ,$$

(1.1.2) $$p(7^\alpha n + \lambda_\alpha) \equiv 0 \mod 7^\alpha ,$$

(1.1.3) $$p(11^\alpha n + \psi_\alpha) \equiv 0 \mod 11^\alpha .$$

In each case he proved his conjecture for $\alpha = 1$ and $\alpha = 2$. In 1938 Watson [ 16 ] proved (1.1.1) for general $\alpha$. Watson's proof has been simplified by many writers of which the most recent are Hirschhorn and Hunt [ 10 ].

Ramanujan proved (1.1.1) for the case $\alpha = 1$ by showing that

(1.1.4) $$\sum_{n \geq 0} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6} ,$$

where

$$E(q) = \prod_{n \geq 1} (1-q^n) .$$

In fact there exist similar formulae for the generating functions

(1.1.5) $$\sum_{n \geq 0} p(5^\alpha n + \delta_\alpha)q^n \qquad \text{for} \quad \alpha \geq 1 .$$

Watson was able to derive such formulae from the modular equation of fifth order and prove (1.1.1) by showing that the coefficients in these formulae are divisible by $5^\alpha$. However Watson's notation is formidable and as a result his proof is complex and almost unreadable. By using elementary methods Hirschhorn and Hunt have been able to derive simpler formulae for (1.1.5). They are stated below.

Theorem (1.1.6). For $\alpha \geq 1$,

$$\sum_{n\geq 0} p(5^{\alpha}n+\delta_{\alpha})q^{n} = \begin{cases} \sum_{i\geq 1} x_{\alpha,i}q^{i-1}\dfrac{E(q^5)^{6i-1}}{E(q)^{6i}}, & \alpha \text{ odd}, \\ \sum_{i\geq 1} x_{\alpha,i}q^{i-1}\dfrac{E(q^5)^{6i}}{E(q)^{6i+1}}, & \alpha \text{ even}, \end{cases} \tag{1.1.7}$$

where

$$E(q) = \prod_{n\geq 1}(1-q^{n}),$$

$$\underset{\sim}{x}_1 = (x_{1,1}, x_{1,2}, \ldots) = (5,0,0,0,\ldots),$$

and for $\alpha \geq 1$,

$$\underset{\sim}{x}_{\alpha+1} = \begin{cases} \underset{\sim}{x}_{\alpha}A, & \alpha \text{ odd}, \\ \underset{\sim}{x}_{\alpha}B, & \alpha \text{ even}. \end{cases} \tag{1.1.8}$$

Here $A = (a_{i,j})_{i,j\geq 1}$ and $B = (b_{i,j})_{i,j\geq 1}$ are defined by

$$a_{i,j} = m_{6i,i+j}, \quad b_{i,j} = m_{6i+1,i+j}, \tag{1.1.9}$$

where $M = (m_{i,j})_{i,j\geq 1}$ is defined as follows:

The first five rows of $M$ are

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2\times 5 & 5^3 & 0 & 0 & 0 & 0 & \ldots \\ 9 & 3\times 5^3 & 5^5 & 0 & 0 & 0 & \ldots \\ 4 & 22\times 5^2 & 4\times 5^5 & 5^7 & 0 & 0 & \ldots \\ 1 & 4\times 5^3 & 8\times 5^5 & 5^8 & 5^9 & 0 & \ldots \end{bmatrix} \tag{1.1.10}$$

and for $i \geq 6$, $m_{i,1} = 0$, and for $j \geq 2$,

$$m_{i,j} = 25m_{i-1,j-1} + 25m_{i-2,j-1} + 15m_{i-3,j-1} + 5m_{i-4,j-1} + m_{i-5,j-1}. \tag{1.1.11}$$

They then proved (1.1.1) by showing that

$$x_{\alpha,i} \equiv 0 \mod 5^{\alpha} \quad \text{for} \quad i \geq 1.$$

Watson also proved that if $\alpha$ is odd and at least 3, then

$$p(5^{\alpha}n + \delta_{\alpha} \pm 5^{\alpha-1}) \equiv 0 \mod 5^{\alpha}. \tag{1.1.12}$$

Chowla [ 7 ] noticed that (1.1.2) fails for $\alpha = 3$. In fact from Gupta [ 9 ] we have

$$p(\lambda_3) = p(243) = 13397\ 82593\ 44888,$$

which is divisible by $7^2$ but not by $7^3$. Watson [16] proved the appropriate modification of (1.1.2), viz. that if $\beta \geq 1$ then

$$(1.1.13) \qquad p(7^{2\beta-1}n + \lambda_{2\beta-1}) \equiv 0 \mod 7^{\beta}$$

$$\text{and} \qquad p(7^{2\beta}n + \lambda_{2\beta}) \equiv 0 \mod 7^{\beta+1}.$$

Watson also proved that if $\beta \geqslant 1$, then

$$(1.1.14) \qquad p(7^{2\beta}n + \lambda_{2\beta} - 4.7^{2\beta-1}) \equiv p(7^{2\beta}n + \lambda_{2\beta} - 2.7^{2\beta-1})$$
$$\equiv p(7^{2\beta}n + \lambda_{2\beta} - 7^{2\beta-1}) \equiv 0 \mod 7^{\beta+1}.$$

In this chapter we establish identities analogous to (1.1.7) for the generating functions

$$(1.1.15) \qquad \sum_{n\geq 0} p(7^{\alpha}n + \lambda_{\alpha})q^n \quad \text{for} \quad \alpha \geq 1,$$

fromwhich (1.1.13) and (1.1.14) follow easily. Watson's proofs rely on the modular equation of seventh order. We also need the modular equation but we derive it using the elementary techniques of O. Kolberg [11]. The remainder of our proof of (1.1.13) is analogous to that of Hirschhorn and Hunt. The main result of this chapter, stated below, contains an algorithm for calculating the coefficients in the formulae for the generating functions in (1.1.15). We carry out these calculations for $\alpha = 1,2$.

The main result of this chapter is

<u>THEOREM (1.1.16)</u> If $\alpha \geq 1$,

$$\sum_{n\geq 0} p(7^{\alpha}n + \lambda_{\alpha})q^n = \begin{cases} \displaystyle\sum_{i\geq 1} x_{\alpha,i}\, q^{i-1} \frac{E(q^7)^{4i-1}}{E(q)^{4i}}, & \alpha \text{ odd}, \\[2ex] \displaystyle\sum_{i\geq 1} x_{\alpha,i}\, q^{i-1} \frac{E(q^7)^{4i}}{E(q)^{4i+1}}, & \alpha \text{ even}, \end{cases}$$

where

$$E(q) = \prod_{n\geq 1} (1 - q^n) ,$$

$$\underset{\sim}{x}_1 = (7,\ 49,\ 0,\ 0, \ldots),$$

and for $\alpha \geq 1$,

(1.1.17)
$$\underset{\sim}{x}_{\alpha+1} = \begin{cases} \underset{\sim}{x}_\alpha A, & \alpha \text{ odd}, \\ \underset{\sim}{x}_\alpha B, & \alpha \text{ even}. \end{cases}$$

Here $A = (a_{i,j})_{i,j\geq 1}$ , $B = (b_{i,j})_{i,j\geq 1}$ are defined by

(1.1.18)
$$a_{i,j} = m_{4i,i+j} , \quad b_{i,j} = m_{4i+1,i+j} ,$$

where $M = (m_{i,j})_{i,j\geq 1}$ is defined as follows:

The first seven rows of $M$ are

$$\begin{pmatrix} 7 & 7^{2} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 10 & 9\times7^{2} & 2\times7^{4} & 7^{5} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 114\times7 & 85\times7^{3} & 24\times7^{5} & 3\times7^{7} & 7^{8} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 82\times7 & 176\times7^{3} & 845\times7^{4} & 272\times7^{6} & 46\times7^{8} & 4\times7^{10} & 7^{11} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 190 & 1265\times7^{2} & 1895\times7^{4} & 1233\times7^{6} & 3025\times7^{7} & 620\times7^{9} & 75\times7^{11} & 5\times7^{13} & 7^{14} & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 27 & 736\times7^{2} & 16782\times7^{3} & 20424\times7^{5} & 12825\times7^{7} & 4770\times7^{9} & 7830\times7^{10} & 1178\times7^{12} & 111\times7^{14} & 6\times7^{16} & 7^{17} & 0 & 0 & \cdot \\ \cdot & 1 & 253\times7^{2} & 1902\times7^{4} & 4246\times7^{6} & 31540\times7^{7} & 19302\times7^{9} & 7501\times7^{11} & 1944\times7^{13} & 2397\times7^{14} & 285\times7^{16} & 22\times7^{18} & 7^{20} & 7^{20} & 0 \end{pmatrix} \qquad (1.1.19)$$

and for $i \geq 4$ $m_{i,1} = 0$, for $i \geq 8$ $m_{i,2} = 0$ and for $i \geq 8,\ j \geq 3$,

$$(1.1.20) \quad m_{i,j} = 7m_{i-3,j-1} + 35m_{i-2,j-1} + 49m_{i-1,j-1} + m_{i-7,j-2} + 7m_{i-6,j-2} + 21m_{i-5,j-2} + 49m_{i-4,j-2} + 147m_{i-3,j-2} + 343m_{i-2,j-2} + 343m_{i-1,j-2}.$$

The case $\alpha = 1$ in Theorem (1.1.16) namely

$$(1.1.21) \quad \sum_{n\geq 0} p(7n+5)q^n = 7\frac{E(q^7)^3}{E(q)^4} + 49q\frac{E(q^7)^7}{E(q)^8}$$

was known to Ramanujan.

In 1966 Atkin [ 1 ] proved (1.1.3) for general $\alpha$. It appears that Watson's method of modular equations is not sufficient. Atkin's method of proof relies on the behaviour and the Fourier series of entire modular functions on

$$\Gamma_0(11) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1,\ c \equiv 0 \pmod{11} \right\},$$

as well as Fine's [ 8 ] modular equation of eleventh order.

<u>1.2.</u> We need some preliminary results.

<u>Lemma (1.2.1)</u>.

$$E(q) = E(q^{49})[Q_0 - qQ_1 - q^2 + q^5 Q_5],$$

where $Q_0$, $Q_1$ and $Q_5$ are power series in $q^7$ which satisfy

$$(1.2.2) \quad \begin{cases} Q_0 Q_1^2 - Q_0^2 + q^7 Q_5 = 0 \\ Q_0 - Q_1^2 - q^7 Q_1 Q_5^2 = 0 \\ Q_0^2 Q_5 - q^7 Q_5^2 - Q_1 = 0, \end{cases}$$

$$(1.2.3) \quad Q_0 Q_1 Q_5 = 1.$$

<u>Proof.</u> Here we assume

$$E(q) = \sum_{-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(3n^2-n)} \qquad \text{(Euler)}$$

and

$$E(q)^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{\frac{1}{2}(n^2+n)} \qquad \text{(Jacobi)} .$$

Write $E(q) = E_0 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ where $E_i$ contain those terms of $E(q)$ in which the power of $q$ is congruent to $i$ mod 7.

Since $\frac{1}{2}(3r^2 - r) \not\equiv 3,4$ and 6 mod 7,

$$E_3 = E_4 = E_6 = 0 \quad \text{and} \quad E(q) = E_0 + E_1 + E_2 + E_5 .$$

Now

$$E_2 = \sum_{\substack{\frac{1}{2}(3r^2-r)\equiv 2 \\ \text{mod } 7}} (-1)^r q^{\frac{1}{2}(3r^2-r)} = \sum_{\substack{r\equiv 6 \\ \text{mod } 7}} (-1)^r q^{\frac{1}{2}(3r^2-r)}$$

$$= \sum_{-\infty}^{\infty} (-1)^{7n-1} q^{\frac{1}{2}[147n^2-49n+4]} = - q^2 \sum_{-\infty}^{\infty} (-1)^n (q^{49})^{\frac{1}{2}(3n^2-n)}$$

$$= - q^2 E(q^{49}) .$$

$$E(q)^3 = (E_0 + E_1 + E_2 + E_5)^3$$

$$= (E_0^3 + 6E_0E_2E_5 + 3E_1^2E_5) + (E_5^3 + 3E_0^2E_1 + 6E_1E_2E_5) + (3E_0E_1^2 + 3E_0^2E_2 + 3E_2^2E_5)$$

$$+ (E_1^3 + 6E_0E_1E_2 + 3E_0E_5^2) + (3E_0E_2^2 + 3E_1^2E_2 + 3E_1E_5^2)$$

$$+ (3E_0^2E_5 + 3E_2E_5^2 + 3E_1E_2^2) + (E_2^3 + 6E_0E_1E_5)$$

$$= \sum_{n\geq 0} (-1)^n (2n + 1) q^{\frac{1}{2}(n^2+n)}$$

and since $\frac{1}{2}(n^2 + n) \not\equiv 2,4$ and 5 mod 7 we have

$$(1.2.4) \qquad \begin{cases} 3E_0E_1^2 + 3E_0^2E_2 + 3E_2^2E_5 = 0 \\ 3E_0E_2^2 + 3E_1^2E_2 + 3E_1E_5^2 = 0 \\ 3E_0^2E_5 + 3E_2E_5^2 + 3E_1E_2^2 = 0 . \end{cases}$$

If we define $Q_0$, $Q_1$ and $Q_5$ by

$$E_0 = E(q^{49})Q_0, \quad E_1 = -q\,E(q^{49})Q_1 \quad \text{and} \quad E_5 = q^5\,E(q^{49})Q_5$$

then $Q_0$, $Q_1$ and $Q_5$ are power series in $q^7$. (1.2.2) follows from (1.2.4) and we have

$$E(q) = E(q^{49})\,[Q_0 - qQ_1 - q^2 + q^5 Q_5].$$

Multiplying the first equation in (1.2.2) by $Q_5$ and substituting $Q_0^2 Q_5 = Q_1 + q^7 Q_5^2$ we obtain

$$Q_0 Q_1^2 Q_5 = Q_1$$

or $$Q_0 Q_1 Q_5 = 1, \quad \text{which is (1.2.3).}$$

In fact, it can be shown that

$$Q_0 = \prod_{n\geq 1} \frac{(1-q^{49n-14})(1-q^{49n-35})}{(1-q^{49n-7})(1-q^{49n-42})},$$

$$Q_1 = \prod_{n\geq 1} \frac{(1-q^{49n-21})(1-q^{49n-28})}{(1-q^{49n-14})(1-q^{49n-35})},$$

$$Q_5 = \prod_{n\geq 1} \frac{(1-q^{49n-7})(1-q^{49n-42})}{(1-q^{49n-21})(1-q^{49n-28})},$$

using Watson's quintuple product identity.

<u>Lemma (1.2.5)</u>. If $\omega^7 = 1$, $\omega \neq 1$ then

$$E(q)E(\omega q)E(\omega^2 q)E(\omega^3 q)E(\omega^4 q)E(\omega^5 q)E(\omega^6 q) = \frac{E(q^7)^8}{E(q^{49})}.$$

Proof.

$$\prod_{k=0}^{6} E(\omega^k q) = \prod_{n\geq 1} (1 - q^n)(1 - \omega^n q^n)(1 - \omega^{2n} q^n) \ \ldots \ (1 - \omega^{6n} q^n)$$

$$= \prod_{\substack{n\geq 1 \\ n\equiv 0 \bmod 7}} (1 - q^n)^7 \prod_{\substack{n\geq 1 \\ n\not\equiv 0 \bmod 7}} (1 - q^{7n})$$

$$= \prod_{n\geq 1} (1 - q^{7n})^7 \cdot \frac{\prod_{n\geq 1} (1 - q^{7n})}{\prod_{\substack{n\geq 1 \\ n\equiv 0 \bmod 7}} (1 - q^{7n})}$$

$$= \frac{\prod_{n\geq 1} (1 - q^{7n})^8}{\prod_{n\geq 1} (1 - q^{49n})} = \frac{E(q^7)^8}{E(q^{49})} .$$

1.3. The main result of this section is lemma (1.3.1). Our proof relies on the modular equation of seventh order (1.3.14), which appears in Watson's paper but which we obtain by an elementary method.

We now introduce the operators $H_i$, $0 \leq i \leq 6$ which act on a series of powers of $q$ and simply pick out those terms in which the power of $q$ is congruent to $i$ modulo 7. Set $H = H_0$.

Lemma (1.3.1). For $i \geq 1$,

(1.3.2) $$H(\xi^{-i}) = \sum_{j\geq 1} m_{i,j} T^{-j}$$

(1.3.3) where $$\xi(q) = \frac{E(q)}{q^2 \, E(q^{49})} , \qquad T(q) = \frac{E(q^7)^4}{q^7 \, E(q^{49})^4}$$

and the $m_{i,j}$ are defined by (1.1.19) and (1.1.20).

We leave the proof of Lemma (1.3.1) till later. As an immediate consequence we have the following Lemma.

Lemma (1.3.4). For $i \geq 1$,

$$H(\xi^{-4i}) = \sum_{j\geq 1} a_{i,j} T^{-i-j}$$

and

$$H(\xi^{-(4i+1)}) = \sum_{j\geq 1} b_{i,j} T^{-i-j}$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (1.1.18) .

Proof. It is easy to check that $H(\xi^{-4i})$ as a polynomial in $T^{-1}$ has no terms of degree $i$ or less. So by Lemma (1.3.1)

$$H(\xi^{-4i}) = \sum_{j\geq 1} a'_{i,j} T^{-i-j} = \sum_{j\geq 1} m_{4i,j} T^{-j} .$$

Therefore

$$a'_{i,j} = m_{4i,i+j} = a_{i,j} ,$$

and

$$H(\xi^{-4i}) = \sum_{j\geq 1} a_{i,j} T^{-i-j} .$$

We can argue similarly to show that

$$H(\xi^{-(4i+1)}) = \sum_{j\geq 1} b_{i,j} T^{-i-j} .$$

In order to derive the modular equation we first need some preliminary results. Following Kolberg [11] we define

(1.3.5) $$\alpha = -q^{-2}Q_0, \quad \beta = q^{-1}Q_1 \quad \text{and} \quad \gamma = -q^3 Q_5 .$$

From Lemma (1.2.1) we have

(1.3.6) $$\xi(q) = \frac{E(q)}{q^2 E(q^{49})} = q^{-2}Q_0 - q^{-1}Q_1 - 1 + q^3 Q_5 = -(\alpha + \beta + \gamma + 1) .$$

From (1.2.2) and (1.2.3) we obtain

(1.3.7) $$\begin{cases} \alpha\beta^2 + \alpha^2 + \gamma = 0 \\ \beta\gamma^2 + \beta^2 + \alpha = 0 \\ \gamma\alpha^2 + \gamma^2 + \beta = 0 \\ \alpha\beta\gamma = 1 . \end{cases}$$

Let

(1.3.8) $$y_1 = \alpha^3\beta, \quad y_2 = \beta^3\gamma \quad \text{and} \quad y_3 = \gamma^3\alpha .$$

Then, by (1.3.7) we easily find $y_1y_2 = -y_1 - 1,\ y_2y_3 = -y_2 - 1,$

$$y_3y_1 = -y_3 - 1, \quad y_1y_2y_3 = 1,$$

$$\alpha^2\beta^3 = -y_1 - 1,\ \beta^2\gamma^3 = -y_2 - 1,\ \gamma^2\alpha^3 = -y_3 - 1$$

$$\alpha\beta^5 = y_1 - y_2 + 1, \quad \beta\gamma^5 = y_2 - y_3 + 1, \quad \gamma\alpha^5 = y_3 - y_1 + 1,$$

$$\alpha^7 = -y_1^2 + y_1 - y_3 - 1,\ \beta^7 = -y_2^2 + y_2 - y_1 - 1,\ \gamma^7 = -y_3^2 + y_3 - y_2 - 1.$$

The following Lemma is 5.20, 5.21 and 5.14 of Kolberg's paper.

<u>Lemma (1.3.9).</u>

$$y_1 + y_2 + y_3 = -T - 8,$$

$$y_1y_2 + y_2y_3 + y_3y_1 = T + 5,$$

$$y_1y_2y_3 = 1,$$

where $T$ is defined in (1.3.3) and the $y_i$ are defined in (1.3.5) and (1.3.8).

<u>Proof.</u> Let $\omega^7 = 1,\ \omega \neq 1$ .

$$\prod_{i=0}^{6} \xi(\omega^i q) = -\prod_{i=0}^{6} [\alpha(\omega^i q) + \beta(\omega^i q) + \gamma(\omega^i q) + 1]$$

$$= -\prod_{i=0}^{6} [\omega^{-2i}\alpha + \omega^{-i}\beta + \omega^{3i}\gamma + 1]$$

$$= -\prod_{i=0}^{6} [1 + \omega^{3i}\gamma + \omega^{5i}\alpha + \omega^{6i}\beta]$$

$$= -\det\begin{pmatrix} 1 & 0 & 0 & \gamma & 0 & \alpha & \beta \\ \beta & 1 & 0 & 0 & \gamma & 0 & \alpha \\ \alpha & \beta & 1 & 0 & 0 & \gamma & 0 \\ 0 & \alpha & \beta & 1 & 0 & 0 & \gamma \\ \gamma & 0 & \alpha & \beta & 1 & 0 & 0 \\ 0 & \gamma & 0 & \alpha & \beta & 1 & 0 \\ 0 & 0 & \gamma & 0 & \alpha & \beta & 1 \end{pmatrix}$$

On the other hand, from Lemma (1.2.5) we have

(1.3.10) $$\prod_{i=0}^{6} \xi(\omega^i q) = \frac{\prod_{i=0}^{6} E(\omega^i q)}{q^{14} E(q^{49})^7} = \frac{E(q^7)^8}{q^{14}\ E(q^{49})^8} = T^2 .$$

Using (1.3.7), we find by evaluating the determinant that

$$T^2 = -(\alpha^7 + \beta^7 + \gamma^7) + 7(\alpha\beta^5 + \beta\gamma^5 + \gamma\alpha^5) - 14(\alpha^2\beta^3 + \beta^2\gamma^3 + \gamma^2\alpha^3) - 8$$

$$= -(-y_1^2 - y_2^2 - y_3^2 - 3) + 21 - 14(-y_1 - y_2 - y_3 - 3) - 8$$

$$= (y_1^2 + y_3^2 + y_3^2) + 14(y_1 + y_2 + y_3) + 58$$

$$= (y_1 + y_2 + y_3)^2 - 2(y_1y_2 + y_2y_3 + y_3y_1) + 14(y_1 + y_2 + y_3) + 58$$

$$= (y_1 + y_2 + y_3)^2 + 16(y_1 + y_2 + y_3) + 64$$

$$= (y_1 + y_2 + y_3 + 8)^2.$$

So,

(1.3.11) $$y_1 + y_2 + y_3 + 8 = \pm T .$$

We now calculate the first term in the expansion of each $y_i$.

$$y_1 = \alpha^3\beta = -q^{-7}Q_3^3Q_1 = -q^{-7} + \dots , \quad y_2 = \beta^2\gamma = -Q_1^3Q_5 = -1 + \dots ,$$

$$y_3 = \gamma^3\alpha = q^7Q_5^3Q_0 = q^7 + \dots .$$

We therefore have to take the $-$ sign in (1.3.11) so,

$$y_1 + y_2 + y_3 = -T - 8$$

$$y_1y_2 + y_2y_3 + y_3y_1 = -y_1 - y_2 - y_3 - 3 = T + 8 - 3 = T + 5$$

$$y_1y_2y_3 = (\alpha\beta\gamma)^4 = 1$$

and the Lemma is proved.

Lemma (1.3.12).

| | |
|---|---|
| $H(\xi) = -1,$ | $H(\xi^4) = -4T - 7,$ |
| $H(\xi^2) = 1,$ | $H(\xi^5) = 10T + 49,$ |
| $H(\xi^3) = -7,$ | $H(\xi^6) = 49.$ |

Proof. From (1.3.7) and Lemma (1.3.9) we have

$$H(\alpha + \beta + \gamma) = 0,$$

$$H((\alpha + \beta + \gamma)^2) = 0,$$

$$H((\alpha + \beta + \gamma)^3) = 6\alpha\beta\gamma = 6,$$

$$\begin{aligned} H((\alpha + \beta + \gamma)^4) &= 4(\alpha^3\beta + \beta^3\gamma + \gamma^3\alpha) \\ &= 4(y_1 + y_2 + y_3) \\ &= -4T - 32, \end{aligned}$$

$$\begin{aligned} H((\alpha + \beta + \gamma)^5) &= 10(\alpha^3\gamma^2 + \beta^3\alpha^2 + \gamma^3\beta^2) \\ &= 10(-y_1 - 1 - y_2 - 1 - y_3 - 1) \\ &= 10T + 50, \end{aligned}$$

$$\begin{aligned} H((\alpha + \beta + \gamma)^6) &= 6(\alpha^5\gamma + \beta^5\alpha + \gamma^5\beta) + 90\alpha^2\beta^2\gamma^2 \\ &= 6(y_3 - y_1 + 1 + y_1 - y_2 + 1 + y_2 - y_3 + 1) + 90 \\ &= 108, \end{aligned}$$

so

$$H(\xi) = H(-(\alpha + \beta + \gamma) - 1) = -1,$$

$$H(\xi^2) = H((\alpha + \beta + \gamma)^2 + 2(\alpha + \beta + \gamma) + 1) = 1,$$

$$H(\xi^3) = H(-(\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)^2 - 3(\alpha + \beta + \gamma) - 1) = -6 - 1 = -7,$$

$$H(\xi^4) = (-4T - 32) + 4 \times 6 + 1 = -4T - 7,$$

$$H(\xi^5) = -(10T + 50) - 5(-4T - 32) - 10 \times 6 - 1 = 10T + 49,$$

$$H(\xi^6) = 108 + 6(10T + 50) + 15(-4T - 32) + 20 \times 6 + 1 = 49.$$

Lemma (1.3.13). $H_3(\xi^3) = 0, \quad H_5(\xi^3) = 0, \quad H_6(\xi^3) = 0.$

Proof. From (1.3.7) we have

$$H_3(\alpha + \beta + \gamma) = \gamma.$$

$$H_3((\alpha + \beta + \gamma)^2) = \alpha^2,$$

$$H_3((\alpha + \beta + \gamma)^3) = 3\alpha\beta^2,$$

so

$$\begin{aligned} H_3(\xi^3) &= H_3(-(\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)^2 - 3(\alpha + \beta + \gamma) - 1) \\ &= -3\alpha\beta^2 - 3\alpha^2 - 3\gamma = 0. \end{aligned}$$

Similarly,

$$H_5(\xi^3) = -3\beta\gamma^2 - 3\beta^2 - 3\alpha = 0,$$

$$H_6(\xi^3) = -3\gamma\alpha^2 - 3\gamma^2 - 3\beta = 0.$$

We note here that some of the formulae in Lemma (1.3.12) were known to Morris Newman [12]. If we define $p_r$ by

$$\sum_{n\geq 0} p_r(n)q^n = \prod_{n\geq 1} (1-q^n)^r,$$

then from (1.3.3) we find that

$$H(\xi^5) = 10T + 49$$

is equivalent to

$$H(q^{-10}E(q)^5) = 10q^{-7} E(q^7)^4 E(q^{49}) + 49 E(q^{49})^5,$$

$$H_3(E(q)^5) = 10q^3 E(q^7)^4 E(q^{49}) + 49q^{10} E(q^{49})^5,$$

$$\sum_{n\geq 0} p_5(7n+3)q^{7n+3} = 10 q^3 E(q^7)^4 E(q^{49}) + 49q^{10} E(q^{49})^5$$

or

$$\sum_{n\geq 0} p_5(7n+3)q^n = 10E(q)^4 E(q^7) + 49q E(q^7)^5,$$

which is the second identity at the end of Newman's paper. Similarly we also find that

$$\sum_{n\geq 0} p_1(7n+2)q^n = -E(q^7),$$

$$\sum_{n\geq 0} p_2(7n+4)q^n = E(q^7)^2,$$

$$\sum_{n\geq 0} p_3(7n+6)q^n = -7E(q^7)^3,$$

$$\sum_{n\geq 0} p_4(7n+1)q^n = -4E(q)^4 - 7q E(q^7)^4$$

and

$$\sum_{n\geq 0} p_6(7n+5)q^n = 49q E(q^7)^6.$$

Lemma (1.3.14). (The Modular Equation of Seventh Order)

$$T^2 = (7\xi^3 + 35\xi^2 + 49\xi)T + \xi^7 + 7\xi^6 + 21\xi^5 + 49\xi^4 + 147\xi^3 + 343\xi^2 + 343\xi$$

where $T$ and $\xi$ are defined in (1.3.3).

Proof. For $0 \le i \le 6$ define $\xi_i(q) = \xi(\omega^i q)$ where $\omega^7 = 1$, $\omega \ne 1$, then

$$H(\xi^j) = \frac{1}{7}\sum_{i=0}^{6} \xi^j(\omega^i q) = \frac{1}{7}\sum_{i=0}^{6} \xi_i^j .$$

Therefore from Lemma (1.3.12) we have

$$P_1 = \xi_0 + \xi_1 + \dots + \xi_6 = -7,$$

$$P_2 = \xi_0^2 + \xi_1^2 + \dots + \xi_6^2 = 7,$$

$$P_3 = \xi_0^3 + \xi_1^3 + \dots + \xi_6^3 = -49,$$

$$P_4 = \xi_0^4 + \xi_1^4 + \dots + \xi_6^4 = -28T - 49,$$

$$P_5 = \xi_0^5 + \xi_1^5 + \dots + \xi_6^5 = 70T + 343,$$

$$P_6 = \xi_0^6 + \xi_1^6 + \dots + \xi_6^6 = 343.$$

It follows from standard formulae that

$$S_1 = \sum \xi_i = -7$$

$$S_2 = \sum_{i<j} \xi_i\xi_j = 21,$$

$$S_3 = \sum_{i<j<k} \xi_i\xi_j\xi_k = -49,$$

$$S_4 = \sum_{i<j<k<\ell} \xi_i\xi_j\xi_k\xi_\ell = 7T + 147,$$

$$S_5 = -35T - 343,$$

$$S_6 = 49T + 343.$$

(1.3.10) is $$S_7 = \prod_{i=0}^{6} \xi_i = \prod_{i=0}^{6} \xi(\omega^i q) = T^2.$$

Hence the $\xi_i$ are the roots of

$$X^7 + 7X^6 + 21X^5 + 49X^4 + (7T+147)X^3 + (35T+343)X^2 + (49T+343)X - T^2 = 0.$$

But $\xi_0 = \xi$ and the Lemma is proved.

We are now in a position to prove Lemma (1.3.1). From Lemma (1.3.14) it follows that

$$\xi^{-i} = [7\xi^{-(i-3)} + 35\xi^{-(i-2)} + 49\xi^{-(i-1)}]T^{-1} + [\xi^{-(i-7)} + 7\xi^{-(i-6)} + 21\xi^{-(i-5)} + 49\xi^{-(i-4)} + 147\xi^{-(i-3)} + 343\xi^{-(i-2)} + 343\xi^{-(i-1)}]T^{-2}.$$

Picking out those terms in which the power of $q$ is congruent to 0 mod 7 we obtain:

$$(1.3.15)\quad H(\xi^{-i}) = [7H(\xi^{-(i-3)}) + 35H(\xi^{-(i-2)}) + 49H(\xi^{-(i-1)})]T^{-1} + [H(\xi^{-(i-7)}) + 7H(\xi^{-(i-6)}) + 21H(\xi^{-(i-5)}) + 49H(\xi^{-(i-4)}) + 147H(\xi^{-(i-3)}) + 343H(\xi^{-(i-2)}) + 343H(\xi^{-(i-1)})]T^{-2}.$$

From (1.3.15) and Lemma (1.3.12) we have

$$H(\xi^{-1}) = (7-35+49)T^{-1} + [49+7(10T+49) + 21(-4T-7) + 49(-7)+147-343+343]T^{-2}$$

or

$$(1.3.16)\qquad H(\xi^{-1}) = 7T^{-1} + 49T^{-2}.$$

So (1.3.2) is true for $i = 1$. Similarly it is easily verified that (1.3.2) holds for $i = 2, 3, 4, 5, 6, 7$ from (1.3.15) together with Lemma (1.3.12). Further it is clear from (1.3.15) that we can write

$$H(\xi^{-i}) = \sum_{j\geq 1} m'_{i,j}\, T^{-j}.$$

Already we have $m'_{i,j} = m_{i,j}$ for $1 \leq i \leq 7$. It follows from (1.3.15) that $m'_{i,1} = 0$ for $i \geq 4$ and $m'_{i,2} = 0$ for $i \geq 8$.

From (1.3.15), we have for $i \geq 8$

$$\sum_{j\geq 3} m'_{i,j}\, T^{-j} = \sum_{j\geq 1} (7m'_{i-3,j} + 35m'_{i-2,j} + 49m'_{i-1,j})T^{-j-1} + \sum_{j\geq 1} (m'_{i-7,j} + 7m'_{i-6,j} + 21m'_{i-5,j} + 49m'_{i-4,j} + 147m'_{i-3,j} + 343m'_{i-2,j} + 343m'_{i-1,j})T^{-j-2}$$

$$= \sum_{j\geq 3} (7m'_{i-3,j-1} + 35m'_{i-2,j-1} + 49m'_{i-1,j-1} + m'_{i-1,j-2} + \dots + 343m'_{i-1,j-2})T^{-j}.$$

Hence, for $i \geq 8$, $j \geq 3$

$$m'_{i,j} = 7m'_{i-3,j-1} + 35m'_{i-2,j-1} + 49m'_{i-1,j-1} + m'_{i-7,j-2} + 7m'_{i-6,j-2} + 21m'_{i-5,j-2}$$
$$+ 49m'_{i-4,j-2} + 147m'_{i-3,j-2} + 343m'_{i-2,j-2} + 343m'_{i-1,j-2} .$$

Therefore $m'_{i,j} = m_{i,j}$ for every $i,j \geq 1$.

Before proving our main Theorem we need one more lemma.

Lemma (1.3.17). $\lambda_\alpha$, the reciprocal modulo $7^\alpha$ of 24, satisfies $\lambda_1 = 5$ and for $\alpha \geq 1$,

$$\lambda_{\alpha+1} = \begin{cases} 6 \times 7^\alpha + \lambda_\alpha, & \alpha \text{ odd,} \\ 4 \times 7^\alpha + \lambda_\alpha, & \alpha \text{ even.} \end{cases}$$

Proof.

$$\lambda_\alpha = \begin{cases} \frac{1}{24}(17 \times 7^\alpha + 1), & \alpha \text{ odd,} \\ \frac{1}{24}(23 \times 7^\alpha + 1), & \alpha \text{ even,} \end{cases}$$

since this is an integer which satisfies $0 < \lambda_\alpha < 7^\alpha$ and $24\lambda_\alpha \equiv 1 \bmod 7^\alpha$. It is easily shown that this $\lambda_\alpha$ satisfies recurrence.

1.4 We are now in a position to prove Theorem (1.1.16). For convenience we write the theorem in the following equivalent form

(1.4.1)
$$\sum_{n\geq 0} p(7^\alpha n + \lambda_\alpha) q^n = \begin{cases} [\sum_{i\geq 1} x_{\alpha,i} T^i \xi^{-4i}]/q\, E(q^7), & \alpha \text{ odd}, \\ [\sum_{i\geq 1} x_{\alpha,i} T^i \xi^{-4i-1}]/q^3\, E(q^{49}), & \alpha \text{ even}, \end{cases}$$

where $T$ and $\xi$ are define in (1.3.3).

We have

$$\sum_{n\geq 0} p(n) q^n = \frac{1}{E(q)} = \frac{\xi^{-1}}{q^2\, E(q^{49})}$$

Picking out those terms in which the power of $q$ is congruent to 5 mod 7, we have by (1.3.16)

$$\sum_{n\geq 0} p(7n+5) q^{7n+5} = \frac{H(\xi^{-1})}{q^2\, E(q^{49})} = \frac{7T^{-1} + 49T^{-2}}{q^2\, E(q^{49})}$$

or

$$\sum_{n\geq 0} p(7n+5) q^{7n} = \frac{7T^{-1} + 49T^{-2}}{q^7\, E(q^{49})} .$$

Now
$$T(q^{1/7}) = \frac{E(q)^4}{q\, E(q^7)^4} = \frac{q^8\, \xi^4\, E(q^{49})^4}{q\, E(q^7)^4} = \xi^4\, T^{-1},$$

so

(1.4.2)
$$T^{-1}(q^{1/7}) = T\, \xi^{-4}(q).$$

So we have

$$\sum_{n\geq 0} p(7n+5) q^n = \frac{7T\, \xi^{-4} + 49T^2\, \xi^{-8}}{q\, E(q^7)} .$$

Substituting

$$T = q^{-7} \frac{E(q^7)^4}{E(q^{49})^4} \quad \text{and} \quad \xi^{-1} = q^2 \frac{E(q^{49})}{E(q)}$$

we obtain Ramanujan's result:

$$\sum_{n\geq 0} p(7n+5)q^n = 7\,\frac{E(q^7)^3}{E(q)^4} + 49q\,\frac{E(q^7)^7}{E(q)^8} ,$$

which is the case $\alpha = 1$ of Theorem (1.1.16).

We now proceed by induction on $\alpha$. Suppose $\alpha$ is odd and

$$\sum_{n\geq 0} p(7^\alpha n + \lambda_\alpha)q^n = [\sum_{i\geq 1} x_{\alpha,i}\, T^i\, \xi^{-4i}]/q\, E(q^7).$$

Picking out those terms in which the power of $q$ is congruent to 6 mod 7 we have by Lemma (1.3.4),

$$\sum_{n\geq 0} p(7^\alpha(7n+6) + \lambda_\alpha)q^{7n+6} = [\sum_{i\geq 1} x_{\alpha\,i}\, T^i\, H(\xi^{-4i})]/q\, E(q^7)$$

$$= [\sum_{i\geq 1} x_{\alpha,i}\, T^i \sum_{j\geq 1} a_{i,j}\, T^{-i-j}]/q\, E(q^7)$$

$$= [\sum_{j\geq 1} (\sum_{i\geq 1} x_{\alpha,i}\, a_{i,j})T^{-j}]/q\, E(q^7).$$

It follows from (1.1.17) and Lemma (1.3.17) that

$$\sum_{n\geq 0} p(7^{\alpha+1}n + \lambda_{\alpha+1})q^{7n} = [\sum_{i\geq 1} x_{\alpha+1,i}\, T^{-i}]/q^7\, E(q^7).$$

From (1.3.3) and (1.4.2) we have

$$\sum_{n\geq 0} p(7^{\alpha+1}n + \lambda_{\alpha+1})q^n = [\sum_{i\geq 1} x_{\alpha+1,i}\, T^i\, \xi^{-4i}]/q\, E(q)$$

$$= [\sum_{i\geq 1} x_{\alpha+1,i}\, T^i\, \xi^{-4i-1}]/q^3\, E(q^{49}) .$$

Now suppose $\alpha$ is even and

$$\sum_{n\geq 0} p(7^\alpha n + \lambda_\alpha)q^n = [\sum_{i\geq 1} x_{\alpha,i}\, T^i\, \xi^{-4i-1}]/q^3\, E(q^{49}).$$

Picking out those terms in which the power of $q$ is congruent to 4 mod 7, we have by Lemma (1.3.4),

$$\sum_{n\geq 0} p(7^{\alpha}(7n+4)+\lambda_{\alpha})q^{7n+4} = [\sum_{i\geq 1} x_{\alpha,i}\, T^{i}\, H(\xi^{-4i-1})]/q^{3}\, E(q^{49})$$

$$= [\sum_{i\geq 1} x_{\alpha,i}\, T^{i} \sum_{j\geq 1} b_{i,j}\, T^{-i-j}]/q^{3}\, E(q^{49})$$

$$= [\sum_{j\geq 1} \Big(\sum_{i\geq 1} x_{\alpha,i}\, b_{i,j}\Big) T^{-j}]/q^{3}\, E(q^{49}).$$

It follows from (1.1.17) and Lemma (1.3.17) that

$$\sum_{n\geq 0} p\big(7^{\alpha+1}n+\lambda_{\alpha+1}\big)q^{7n} = [\sum_{i\geq 1} x_{\alpha+1,i}\, T^{-i}]/q^{7}\, E(q^{49}).$$

From (1.4.2) we have

$$\sum_{n\geq 0} p\big(7^{\alpha+1}n+\lambda_{\alpha+1}\big)q^{n} = [\sum_{i\geq 1} x_{\alpha+1,i}\, T^{i}\, \xi^{-4i}]/q\, E(q^{7}).$$

This completes the proof of Theorem (1.1.16).

1.5. We now turn to Watson's results. Let $\nu(n)$ denote the exact power of 7 dividing $n$. Then

**Lemma (1.5.1).** $\nu(m_{i,j}) \geqslant [\frac{1}{4}(7j - 2i - 1)]$.

**Proof.** Consider the matrix $V = (\nu(m_{i,j}))_{i,j\geq 1}$. The first seven rows of $V$ are:

$$\begin{pmatrix}
1 & 2 & \infty & \infty & & & & & & & & & & & & & & \\
0 & 2 & 4 & 5 & \infty & \infty & & & & & & & & & & & & \\
0 & 1 & 3 & 5 & 7 & 8 & \infty & \infty & & & & & & & & & & \\
\infty & 1 & 3 & 4 & 6 & 8 & 10 & 11 & \infty & \infty & & & & & & & & \\
\infty & 0 & 2 & 4 & 6 & 7 & 9 & 11 & 13 & 14 & \infty & \infty & & & & & & \\
\infty & 0 & 2 & 3 & 5 & 7 & 9 & 10 & 12 & 14 & 16 & 17 & \infty & \infty & & & & \\
\infty & 0 & 2 & 4 & 6 & 7 & 9 & 11 & 13 & 14 & 16 & 18 & 20 & 20 & \infty & \infty & \cdot & \cdot \\
\infty & \infty & & & & & & & & & & & & & & & & \\
\infty & \infty & & & & & & & & & & & & & & & & \\
\cdot & \cdot & & & & & & & & & & & & & & & & \\
\cdot & \cdot & & & & & & & & & & & & & & & &
\end{pmatrix}$$

Define the matrix $N = (\nu_{i,j})_{i,j\geq 1}$ by $\nu_{i,j} = [\frac{1}{4}(7j - 2i - 1)]$.

The first seven rows of $N$ are:

$$\begin{pmatrix}
1 & 2 & 4 & 6 & 8 & 9 & 11 & 13 & 15 & 16 & 18 & 20 & 22 & 23 & \cdot & \cdot \\
0 & 2 & 4 & 5 & 7 & 9 & 11 & 12 & 14 & 16 & 18 & 19 & 21 & 23 & \cdot & \cdot \\
0 & 1 & 3 & 5 & 7 & 8 & 10 & 12 & 14 & 15 & 17 & 19 & 21 & 22 & \cdot & \cdot \\
-1 & 1 & 3 & 4 & 6 & 8 & 10 & 11 & 13 & 15 & 17 & 18 & 20 & 22 & \cdot & \cdot \\
-1 & 0 & 2 & 4 & 6 & 7 & 9 & 11 & 13 & 14 & 16 & 18 & 20 & 21 & \cdot & \cdot \\
-2 & 0 & 2 & 3 & 5 & 7 & 9 & 10 & 12 & 14 & 16 & 17 & 19 & 21 & \cdot & \cdot \\
-2 & -1 & 1 & 3 & 5 & 6 & 8 & 10 & 12 & 13 & 15 & 17 & 19 & 20 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

Observe that for $i \leqslant 7$ and for $i > 7$, $j \leqslant 2$, $\nu(m_{i,j}) \geqslant \nu_{i,j}$.

From (1.1.20) it follows that for $i > 7$, $j > 2$,

$$\begin{aligned}
\nu(m_{i,j}) \geqslant \min\{&\nu(m_{i-3,j-1}) + 1,\ \nu(m_{i-2,j-1}) + 1,\ \nu(m_{i-1,j-1}) + 2,\ \nu(m_{i-7,j-2}),\\
&\nu(m_{i-6,j-2}) + 1,\ \nu(m_{i-5,j-2}) + 1,\ \nu(m_{i-4,j-2}) + 2,\ \nu(m_{i-3,j-2}) + 2,\\
&\nu(m_{i-2,j-2}) + 3,\ \nu(m_{i-2,j-2}) + 3\ \}
\end{aligned}$$

while, as is easily checked

$$\nu_{i,j} = \min\{\nu_{i-3,j-1}+1,\ \nu_{i-2,j-1}+1,\ \nu_{i-1,j-1}+2,\ \nu_{i-7,j-2},\ \nu_{i-6,j-2}+1,$$
$$\nu_{i-5,j-2}+1,\ \nu_{i-4,j-2}+2,\ \nu_{i-3,j-2}+2,\ \nu_{i-2,j-2}+3,\ \nu_{i-1,j-2}+3\}.$$

Lemma (1.5.1) follows by induction.

Lemma (1.5.2).

$$\nu(a_{i,j}) \geqslant [\frac{7j - i - 1}{4}],\quad \nu(b_{i,j}) \geqslant [\frac{7j - i - 3}{4}] .$$

Proof. From (1.5.1) we have

$$\nu(a_{i,j}) = \nu(m_{4i,i+j}) \geqslant [\frac{7(i + j) - 8i - 1}{4}] = [\frac{7j - i - 1}{4}] .$$

$$\nu(b_{i,j}) = \nu(m_{4i+1,i+j}) \geqslant [\frac{7(i + j) - 2(4i + 1) - 1}{4}] = [\frac{7j - i - 3}{4}] .$$

Lemma (1.5.3).

$$\nu(x_{1,1}) = 1, \quad \nu(x_{1,2}) = 2$$

and for $\beta \geqslant 1$,

$$\nu(x_{2\beta,j}) \geqslant (\beta + 1) + [\frac{7j - 6}{4}] ,$$

$$\nu(x_{2\beta+1,j}) \geqslant (\beta + 1) + [\frac{7j - 4}{4}] .$$

Proof.

$$\underset{\sim}{x}_1 = (7,\ 49,\ 0,\ 0,\ \ldots), \quad \text{so} \quad \nu(x_{1,1}) = 1, \quad \nu(x_{1,2}) = 2 .$$

We have

$$x_{2,j} = \sum_{i\geq 1} x_{1,i}\, a_{i,j} = 7a_{1,j} + 49a_{2,j}$$

so

$$\begin{aligned} \nu(x_{2,j}) &\geqslant \min\{1 + \nu(a_{1,j}),\ 2 + \nu(a_{2,j})\} \\ &\geqslant \min\{1 + [\frac{7j - 2}{4}],\ 2 + [\frac{7j - 3}{4}]\} \\ &= \min\{2 + [\frac{7j - 6}{4}],\ 2 + [\frac{7j - 3}{4}]\} \\ &= 2 + [\frac{7j - 6}{4}], \quad \text{as required.} \end{aligned}$$

Now suppose $\beta \geqslant 1$ and

$$\nu(x_{2\beta,i}) \geqslant (\beta + 1) + [\frac{7i - 6}{4}] .$$

We have
$$x_{2\beta+1,j} = \sum_{i\geq 1} x_{2\beta,i}\, b_{i,j}$$

so

$$\nu(x_{2\beta+1,j}) \geqslant \min_{i\geq 1} \{\nu(x_{2\beta,i}) + \nu(b_{i,j})\}$$

$$\geqslant \min_{i\geq 1} \{(\beta + 1) + [\frac{7i - 6}{4}] + [\frac{7j - i - 3}{4}]\}$$

$$= (\beta + 1) + [\frac{7j - 4}{4}] , \quad \text{as required.}$$

Finally suppose $\beta \geqslant 1$ and

$$\nu(x_{2\beta+1,i}) \geqslant (\beta + 1) + [\frac{7i - 4}{4}] .$$

We have
$$x_{2\beta+2,j} = \sum_{i\geq 1} x_{2\beta+1,i}\, a_{i,j}$$

so
$$\nu(x_{2\beta+2,j}) \geqslant \min_{i\geq 1} \{\nu(x_{2\beta+1,i}) + \nu(a_{i,j})\}$$

$$\geqslant \min_{i\geq 1} \{(\beta + 1) + [\frac{7i - 4}{4}] + [\frac{7j - i - 1}{4}]\}$$

$$= (\beta + 1) + [\frac{7j - 2}{4}]$$

$$= (\beta + 2) + [\frac{7j - 6}{4}], \quad \text{as required.}$$

Lemma (1.5.3) follows by induction.

<u>Theorem (1.5.4).</u>

For $\beta \geqslant 1$,

$$p(7^{2\beta-1}n + \lambda_{2\beta-1}) \equiv 0 \bmod 7^{\beta} ,$$

$$p(7^{2\beta}n + \lambda_{2\beta}) \equiv 0 \bmod 7^{\beta+1} .$$

Proof. From Theorem (1.1.16) we have

$$\sum_{n\geq 0} p(7^{2\beta-1}n + \lambda_{2\beta-1})q^n = \sum_{i\geq 1} x_{2\beta-1,i}\, q^{i-1} \frac{E(q^7)^{4i-1}}{E(q)^{4i}} .$$

By Lemma (1.5.3)

$$\nu(x_{1,i}) \geqslant 1 \quad \text{and for} \quad \beta \geqslant 2,$$

$$\nu(x_{2\beta-1,i}) \geqslant \beta + [\frac{7i-4}{4}] \geqslant \beta$$

or, $$x_{2\beta-1,i} \equiv 0 \mod 7^{\beta}, \quad \text{for} \quad \beta \geqslant 1.$$

It follows that $p(7^{2\beta-1}n + \lambda_{2\beta-1}) \equiv 0 \mod 7^{\beta}$ .

Similarly,

$$\sum_{n\geq 0} p(7^{2\beta}n + \lambda_{2\beta})q^n = \sum_{i\geq 1} x_{2\beta,i}\, q^{i-1} \frac{E(q^7)^{4i}}{E(q)^{4i+1}}$$

and $$\nu(x_{2\beta,i}) \geqslant (\beta+1) + [\frac{7i-6}{4}] \geqslant \beta + 1$$

or, $$x_{2\beta,i} \equiv 0 \mod 7^{\beta+1}$$

so $$p(7^{2\beta}n + \lambda_{2\beta}) \equiv 0 \mod 7^{\beta+1} .$$

Theorem (1.5.5). For $\beta \geqslant 1$,

$$p(7^{2\beta}n + \lambda_{2\beta} - 4.7^{2\beta-1}) \equiv p(^{2\beta-1}n + \lambda_{2\beta} - 2.7^{2\beta-1}) \equiv p(7^{2\beta}n + \lambda_{2\beta} - 7^{2\beta-1}) \equiv 0 \mod 7^{\beta+1}.$$

Proof. From (1.4.1) we have

$$\sum_{n\geq 0} p(7^{2\beta-1}n + \lambda_{2\beta-1})q^n = [\sum_{i\geq 1} x_{2\beta-1,i}\, T^i \xi^{-4i}]/q\, E(q^7) .$$

If we pick out those terms in which the power of $q$ is congruent to $k$ mod 7 we have

$$\sum_{n\geq 0} p(7^{2\beta-1}(7n+k) + \lambda_{2\beta-1})q^{7n+k} = [\sum_{i\geq 1} x_{2\beta-1,i}\, T^i H_{k+1}(\xi^{-4i})]/q\, E(q^7).$$

From Lemma (1.3.17) it follows that

$$\sum_{n\geq 0} p(7^{2\beta}n + \lambda_{2\beta} + (k-6)7^{2\beta-1})q^{7n+k} = [\sum_{i\geq 1} x_{2\beta-1,i}\, T^i H_{k+1}(\xi^{-4i})]/q\, E(q^7).$$

From Lemma (1.5.3)

$$\nu(x_{1,1}) = 1, \quad \nu(x_{1,2}) = 2 \quad \text{and for} \quad \beta \geqslant 2,$$

$$\nu(x_{2\beta-1,i}) \geqslant \beta + \left[\frac{7i - 4}{4}\right].$$

So modulo $7^{\beta+1}$ we have

$$\sum_{n\geq 0} p(7^{2\beta}n + \lambda_{2\beta} + (k - 6)7^{2\beta-1})q^{7n+k} \equiv [x_{2\beta-1,1}\ T\ H_{k+1}(\xi^{-4})]/q\ E(q^7).$$

From the modular equation (1.3.14) we have

$$\xi^{-4} = \xi^3 T^{-2} + 7(\xi^{-1}T^{-1} + 5\xi^{-2}T^{-1} + 7\xi^{-3}T^{-1} + \xi^2 T^{-2} + 3\xi T^{-2} + 7T^{-2} + 21\xi^{-1}T^{-2} + 49\xi^{-2}T^{-2} + 49\xi^{-3}T^{-2}).$$

Picking out those terms in which the power of $q$ is congruent to $(k + 1)$ mod 7 we have

$$H_{k+1}(\xi^{-4}) \equiv T^{-2}\ H_{k+1}\ (\xi^3) \bmod 7.$$

But from (1.3.13) we have $H_{k+1}(\xi^3) = 0$ for $k = 2$, 4 and 5, so

$$H_{k+1}(\xi^{-4}) \equiv 0 \bmod 7 \quad \text{for} \quad k = 2,\ 4 \quad \text{and} \quad 5.$$

Since $x_{2\beta-1,1} \equiv 0 \bmod 7^{\beta}$ we obtain

$$\sum_{n\geq 0} p(7^{2\beta}n + \lambda_{2\beta} + (k - 6)7^{2\beta-1})q^{7n+k} \equiv 0 \bmod 7^{\beta+1} \quad \text{for} \quad k = 2,\ 4 \text{ and } 5,$$

which is the required result.

1.6. We have calculated $\underset{\sim}{x}_\alpha$ for $\alpha = 1, 2$. They are

$$\underset{\sim}{x}_1 = (7,\ 7^2,\ 0,\ 0,\ 0,\ \ldots\ ),$$

$$\underset{\sim}{x}_2 = (2546 \times 7^2,\ 48934 \times 7^4,\ 1418989 \times 7^5,\ 2488800 \times 7^7,\ 2394438 \times 7^9,\ 1437047 \times 7^{11},\ 4043313 \times 7^{12},\ 161744 \times 7^{15},\ 32136 \times 7^{17},\ 31734 \times 7^{18},\ 3120 \times 7^{20},\ 204 \times 7^{22},\ 8 \times 7^{24},\ 7^{25},\ 0,\ 0,\ 0,\ \ldots)$$

Therefore from Theorem (1.1.16) we have

$$\sum_{n\geq 0} p(7n+5)q^n = 7\,\frac{E(q^7)^3}{E(q)^4} + 49\,q\,\frac{E(q^7)^7}{E(q)^8}$$

which is Ramanujan's result and

$$\begin{aligned}\sum_{n\geq 0} p(49n+47)q^n = {} & 2546 \times 7^2\,\frac{E(q^7)^4}{E(q)^5} + 48934 \times 7^4\,q\,\frac{E(q^7)^8}{E(q)^9} \\ & + 1418989 \times 7^5\,q^2\,\frac{E(q^7)^{12}}{E(q)^{13}} + 2488800 \times 7^7\,q^3\,\frac{E(q^7)^{16}}{E(q)^{17}} \\ & + 2394438 \times 7^9\,q^4\,\frac{E(q^7)^{20}}{E(q)^{21}} + 1437047 \times 7^{11}\,q^5\,\frac{E(q^7)^{24}}{E(q)^{25}} \\ & + 4043313 \times 7^{12}\,q^6\,\frac{E(q^7)^{28}}{E(q)^{29}} + 161744 \times 7^{15}\,q^7\,\frac{E(q^7)^{32}}{E(q)^{33}} \\ & + 32136 \times 7^{17}\,q^8\,\frac{E(q^7)^{36}}{E(q)^{37}} + 31734 \times 7^{18}\,q^9\,\frac{E(q^7)^{40}}{E(q)^{41}} \\ & + 3120 \times 7^{20}\,q^{10}\,\frac{E(q^7)^{44}}{E(q)^{45}} + 204 \times 7^{22}\,q^{11}\,\frac{E(q^7)^{48}}{E(q)^{49}} \\ & + 8 \times 7^{24}\,q^{12}\,\frac{E(q^7)^{52}}{E(q)^{53}} + 7^{25}\,q^{13}\,\frac{E(q^7)^{56}}{E(q)^{57}}\ .\end{aligned}$$

This confirms a result of H. Zuckerman [18]. We have omitted the calculation of $\underset{\sim}{x}_\alpha$ for $\alpha \geqslant 3$, since there are computational difficulties. For an idea of the size of these numbers see Hirschhorn and Hunt [10], who have calculated the first four coefficient vectors corresponding to the generating functions for $p(5^\alpha n + \delta_\alpha)$.