CHAPTER II

ATKIN'S PARTITION CONGRUENCES MODULO POWERS OF 5

2.1 INTRODUCTION.

Watson [16] obtained the following result. If $\alpha \geqslant 3$ and δ_α is the reciprocal modulo 5^α of 24 then for $n \geqslant 0$,

$$(2.1.1) \qquad p(5^\alpha n + \delta_\alpha) \equiv \begin{cases} p(\delta_\alpha)\ \dfrac{p(5n+4)}{5} \pmod{5^{\alpha+2}}, & \alpha \text{ odd}, \\ 2p(\delta_\alpha)\ \dfrac{p(25n+24)}{25} \pmod{5^{\alpha+3}}, & \alpha \text{ even}. \end{cases}$$

In 1981 Hirschhorn and Hunt [10] obtained stronger results for the cases $\alpha = 3,4$ namely

$$(2.1.2) \qquad P(125n + 99) \equiv 89 \times 25\ p(5n + 4) \bmod 5^6$$

and

$$p(625n + 599) \equiv 3339 \times 25\ p(25n + 24) \bmod 5^{10}.$$

In this chapter we obtain the following new congruence relations

$$(2.1.3) \qquad p(3125n + 2474) \equiv 240839 \times 25\ p(125n + 99) \bmod 5^{13},$$

$$p(15625n + 14974) \equiv 2193964 \times 25\ p(625n + 599) \bmod 5^{17},$$

$$p(78125n + 61849) \equiv 256100214 \times 25\ p(3125n + 2474) \bmod 5^{20}.$$

In fact, there exist similar congruence relations for all higher powers of 5. Atkin [2] has stated the following result without proof. If $\alpha \geqslant 1$ and if δ_α is the reciprocal modulo 5^α of 24, then there exists an integral constant k_α not divisible by 5 such that for all $n \geqslant 0$

$$(2.1.4) \qquad p(5^{\alpha+2}n + \delta_{\alpha+2}) \equiv k_\alpha \times 5^2\ p(5^\alpha n + \delta_\alpha) \bmod 5^{[7\alpha/2]+3}.$$

We provide the details of a proof and show that (2.1.4) is best possible. The methods we use are analogous to those of Atkin and O'Brien [3] where it is proved that if $\alpha \geqslant 1$ and if μ_α is the reciprocal modulo 13^α of 24, then there exists an integral constant m_α not divisible by 13 such that for all $n \geqslant 0$

(2.1.5) $$p(13^{\alpha+2}n + \mu_{\alpha+2}) \equiv m_\alpha\, p(13^\alpha n + \mu_\alpha) \mod 13^\alpha ,$$

and this is best possible.

In section 2.3 we give the generating functions for

$$p(5^\alpha n + \delta_\alpha) \mod 5^{20} \quad (1 \leqslant \alpha \leqslant 7) ,$$

which enable us to calculate k_α for $1 \leqslant \alpha \leqslant 5$ thus obtaining (2.1.2) and (2.1.3).

2.2. In this section we prove (2.1.4) and show that it best possible. First we need some results of Hirschhorn and Hunt [10]. Lemmata (2.2.1) and (2.2.2) are respectively Lemmata (4.3) and (4.2) in Hirschhorn and Hunt's paper. Let $\nu(n)$ denote the exact power of 5 dividing n. Then

Lemma (2.2.1). For $\alpha, i \geqslant 1$,

$$\nu(x_{\alpha,i}) \geqslant \alpha + [\tfrac{1}{2}(5i - 5)], \quad \alpha \text{ odd},$$
$$\nu(x_{\alpha,i}) \geqslant \alpha + [\tfrac{1}{2}(5i - 4)], \quad \alpha \text{ even},$$

where the $x_{\alpha,i}$ are defined by (1.1.8).

Lemma (2.2.2).

$$\nu(a_{i,j}) \geqslant [\tfrac{1}{2}(5j - i - 1)], \quad \nu(b_{i,j}) \geqslant [\tfrac{1}{2}(5j - i - 2)],$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (1.1.9) - (1.1.11).

Following Atkin and O'Brien [3] we define

$$\delta_{\beta,i,j} = x_{2\beta+1,i}\, x_{2\beta-1,j} - x_{2\beta-1,i}\, x_{2\beta+1,j} \tag{2.2.3}$$

and

$$\varepsilon_{\beta,i,j} = x_{2\beta+2,i}\, x_{2\beta,j} - x_{2\beta,i}\, x_{2\beta+2,j} \quad \text{for} \quad \beta, i, j \geqslant 1, \tag{2.2.4}$$

where the $\underset{\sim}{x}_\alpha$ are defined in (1.1.8). We have

$$\delta_{\beta,i,j} = -\delta_{\beta,j,i} \quad \text{and} \quad \varepsilon_{\beta,i,j} = -\varepsilon_{\beta,j,i}$$

so that

$$\delta_{\beta,i,i} = \varepsilon_{\beta,i,i} = 0.$$

Lemma (2.2.5)

$$\delta_{\beta+1,i,j} = \sum_{k\geq 1} \sum_{\ell\geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j}$$

and

$$\varepsilon_{\beta,i,j} = \sum_{k\geq 1} \sum_{\ell\geq 1} \delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}.$$

Proof. From (1.1.8) and (2.2.3) we have

$$\delta_{\beta+1,i,j} = x_{2\beta+3,i}\, x_{2\beta+1,j} - x_{2\beta+1,i}\, x_{2\beta+3,j}$$

$$= \Big(\sum_{k\geq 1} x_{2\beta+2,k}\, b_{k,i}\Big)\Big(\sum_{\ell\geq 1} x_{2\beta,\ell}\, b_{\ell,j}\Big) - \Big(\sum_{k\geq 1} x_{2\beta,k} b_{k,i}\Big)\Big(\sum_{\ell\geq 1} x_{2\beta+2,\ell} b_{\ell,j}\Big)$$

$$= \sum_{k\geq 1}\sum_{\ell\geq 1} (x_{2\beta+2,k}\, x_{2\beta,\ell} - x_{2\beta,k}\, x_{2\beta+2,\ell}) b_{k,i}\, b_{\ell,j}$$

$$= \sum_{k\geq 1}\sum_{\ell\geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j}.$$

Similarly

$$\varepsilon_{\beta,i,j} = x_{2\beta+2,i}\, x_{2\beta,j} - x_{2\beta,i}\, x_{2\beta+2,j}$$

$$= \Big(\sum_{k\geq 1} x_{2\beta+1,k}\, a_{k,i}\Big)\Big(\sum_{\ell\geq 1} x_{2\beta-1,\ell}\, a_{\ell,j}\Big) - \Big(\sum_{k\geq 1} x_{2\beta-1,k}\, a_{k,i}\Big)\Big(\sum_{\ell\geq 1} x_{2\beta+1,\ell} a_{\ell,j}\Big)$$

$$= \sum_{k\geq 1}\sum_{\ell\geq 1} (x_{2\beta+1,k}\, x_{2\beta-1,\ell} - x_{2\beta-1,k}\, x_{2\beta+1,\ell}) a_{k,i}\, a_{\ell,j}$$

$$= \sum_{k\geq 1}\sum_{\ell\geq 1} \delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}\ .$$

Lemma (2.2.6). For $\beta \geqslant 1$,

$$\nu(\delta_{\beta,1,2}) = \nu(\delta_{\beta,2,1}) = 9\beta - 2,$$

$$\nu(\delta_{\beta,i,j}) \geqslant 9\beta - 6 + \Big[\frac{5(i+j)-8}{2}\Big] \geqslant 9\beta \quad \text{for} \quad i+j > 3,$$

$$\nu(\varepsilon_{\beta,1,2}) = \nu(\varepsilon_{\beta,2,1}) = 9\beta + 3,$$

$$\nu(\varepsilon_{\beta,i,j}) \geqslant 9\beta - 2 + \Big[\frac{5(i+j)-6}{2}\Big] \geqslant 9\beta + 5 \quad \text{for} \quad i+j > 3.$$

Proof. From Hirschhorn and Hunt [10], 5.1, we have

$$\underset{\sim}{x}_1 = (5,\ 0,\ 0,\ 0,\ \ldots) \text{ and}$$

$$\underset{\sim}{x}_3 = (1353839 \times 5^3,\ \ 1885026212 \times 5^6,\ \ldots)$$

Now $\delta_{1,2,1} = -\delta_{1,1,2} = x_{1,1}\, x_{3,2}$ so

$$\nu(\delta_{1,2,1}) = \nu(\delta_{1,1,2}) = \nu(x_{1,1}) + \nu(x_{3,2}) = 1 + 6 = 7.$$

For $j > 2$ $\delta_{1,j,1} = -\delta_{1,1,j} = x_{1,1}\, x_{3,j}$. Therefore from Lemma (2.2.1) we have

$$\nu(\delta_{1,j,1}) = \nu(\delta_{1,1,j}) \geqslant 4 + [\frac{5j-5}{2}] .$$

If $i > 1$ and $j > 1$ $\delta_{1,i,j} = 0$ hence for $i + j > 3$

$$\nu(\delta_{1,i,j}) \geqslant 4 + [\frac{5(i+j)-10}{2}] = 3 + [\frac{5(i+j)-8}{2}] .$$

Now suppose

$$\nu(\delta_{\beta,1,2}) = 9\beta - 2$$

and $$\nu(\delta_{\beta,i,j}) \geqslant 9\beta - 6 + [\frac{5(i+j)-8}{2}] \quad \text{for } i + j > 3.$$

From Lemma (2.2.5) we have

$$\varepsilon_{\beta,i,j} = \sum_{k\geq 1} \sum_{\ell\geq 1} \delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j} \quad \text{and} \quad \delta_{\beta,k,\ell} = 0 \quad \text{for} \quad k = \ell$$

so by Lemma (2.2.2)

$$\nu(\varepsilon_{\beta,i,j}) \geqslant \min_{k\neq\ell}\{\nu(\delta_{\beta,k,\ell}) + \nu(a_{k,i}) + \nu(a_{\ell,j})\}$$

$$= \min\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{\nu(\delta_{\beta,k,\ell})+\nu(a_{k,i})+ \nu(a_{\ell,j})\}, \min_{\substack{k\neq\ell \\ k+\ell>3}} \{\nu(\delta_{\beta,k,\ell})+\nu(a_{k,i})+\nu(a_{\ell,j})\}\}$$

$$\geqslant \min\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{9\beta - 2 + [\frac{5i-k-1}{2}] + [\frac{5j-\ell-1}{2}] ,$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{9\beta - 6 + [\frac{5(k+\ell)-8}{2}] + [\frac{5i-k-1}{2}] + [\frac{5j-\ell-1}{2}]\}\}$$

$$\geqslant \min \{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{9\beta - 2 + [\frac{5(i+j)-(k+\ell)-3}{2}],$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{9\beta - 6 + [\frac{5(k+\ell)-8}{2}] + [\frac{5(i+j)-(k+\ell)-3}{2}]\}\}$$

The minimum of the latter term is attained when $k + \ell = 4$ so

$$\nu(\varepsilon_{\beta,i,j}) \geqslant \min\{ 9\beta - 2 + [\frac{5(i+j)-6}{2}] , 9\beta + [\frac{5(i+j)-7}{2}]\}$$

$$= 9\beta - 2 + [\frac{5(i+j)-6}{2}] .$$

We will now show that $\nu(\varepsilon_{\beta,1,2}) = 9\beta + 3$. From (1.1.9), (1.1.10) and (1.1.11) we obtain

$$a_{1,1} = m_{6,2} = 63 \times 5, \quad a_{1,2} = m_{6,3} = 52 \times 5^4 ,$$

$$a_{2,1} = m_{12,3} = 104 \times 5, \quad a_{2,2} = m_{12,4} - 819 \times 5^4 .$$

We also have

$$\nu(\delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}) \geqslant 9\beta + [\frac{5(i+j)-7}{2}] \quad \text{for} \quad k + \ell > 3 .$$

So if $k + \ell > 3$ and $i + j = 3$ then

$$\nu(\delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}) \geqslant 9\beta + 4.$$

It follows that

$$\varepsilon_{\beta,1,2} = \sum_{k\geq 1} \sum_{\ell\geq 1} \delta_{\beta,k,\ell}\, a_{k,1}\, a_{\ell,2}$$

$$\equiv \delta_{\beta,1,2}\, a_{1,1}\, a_{2,2} + \delta_{\beta,2,1}\, a_{2,1}\, a_{1,2} \bmod 5^{9\beta+4}$$

$$\equiv \delta_{\beta,1,2}(51597 \times 5^5 - 5408 \times 5^5) \bmod 5^{9\beta+4}$$

$$\equiv 46189 \times 5^5\, \delta_{\beta,1,2} \bmod 5^{9\beta+4} .$$

$\nu(\delta_{\beta,1,2}) = 9\beta - 2$ so $\varepsilon_{\beta,1,2} \not\equiv 0 \bmod 5^{9\beta+4}$ and $\nu(\varepsilon_{\beta,1,2}) = 9\beta + 3$.

Now suppose

$$\nu(\varepsilon_{\beta,1,2}) = 9\beta + 3$$

and $\nu(\varepsilon_{\beta,i,j}) \geqslant 9\beta - 2 + [\frac{5(i+j)-6}{2}]$ for $i + j > 3$.

From Lemma (2.2.5) we have

$$\delta_{\beta+1,i,j} = \sum_{k\geq 1} \sum_{\ell\geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j} \quad \text{and} \quad \varepsilon_{\beta,k,\ell} = 0 \quad \text{for} \quad k = \ell$$

so by Lemma (2.2.2)

$$\nu(\delta_{\beta+1,i,j}) \geqslant \min_{k\neq\ell} \{\nu(\varepsilon_{\beta,k,\ell}) + \nu(b_{k,i}) + \nu(b_{\ell,j})\}$$

$$= \min\Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{ \nu(\varepsilon_{\beta,k,\ell}) + \nu(b_{k,i}) + \nu(b_{\ell,j})\} , \min_{\substack{k\neq\ell \\ k+\ell>3}} \{\nu(\varepsilon_{\beta,k,\ell}) + \nu(b_{k,i}) + \nu(b_{\ell,j})\}\Big\}$$

$$\geqslant \min\Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{9\beta + 3 + [\frac{5i - k - 2}{2}] + [\frac{5j - \ell - 2}{2}]\} , \min_{\substack{k\neq\ell \\ k+\ell>3}} \{9\beta - 2 + [\frac{5(k + \ell) - 6}{2}] + [\frac{5i - k - 2}{2}] + [\frac{5j - \ell - 2}{2}] \}\Big\}$$

$$\geqslant \min\Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{9\beta + 3 + [\frac{5(i + j) - (k + \ell) - 5}{2}] \} , \min_{\substack{k\neq\ell \\ k+\ell>3}} \{9\beta - 2 + [\frac{5(k + \ell) - 6}{2}] + [\frac{5(i + j) - (k +\ell) - 5}{2}] \}\Big\}$$

The minimum of the latter term is attained when $k + \ell = 4$ so

$$\nu(\delta_{\beta+1,i,j}) \geqslant \min\{9\beta + 3 + [\frac{5(i + j) - 8}{2}] , 9\beta + 5 + [\frac{5(i + j) - 9}{2}]\}$$

$$= 9\beta + 3 + [\frac{5(i + j) - 8}{2}] = 9(\beta + 1) - 6 + [\frac{5(i + j) - 8}{2}] .$$

We will now show that $\nu(\delta_{\beta+1,1,2}) = 9\beta + 7$. From (1.1.9), (1.1.10) and (1.1.11) we obtain

$$b_{1,1} = m_{7,2} = 28 \times 5, \quad b_{1,2} = m_{7,3} = 49 \times 5^4 ,$$

$$b_{2,1} = m_{13,3} = 104, \quad b_{2,2} = m_{13,4} = 364 \times 5^4 .$$

We also have

$$\nu(\varepsilon_{\beta,k,\ell} b_{k,i}\, b_{\ell,j}) \geqslant 9\beta + 5 + [\frac{5(i + j) - 9}{2}] \quad \text{for} \quad k + \ell > 3.$$

So if $k + \ell > 3$ and $i + j = 3$ then

$$\nu(\varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j}) \geqslant 9\beta + 8.$$

It follows that

$$\delta_{\beta+1,1,2} = \sum_{k\geq 1} \sum_{\ell \geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,1}\, b_{\ell,2}$$

$$\equiv \varepsilon_{\beta,1,2}\, b_{1,1}\, b_{2,2} + \varepsilon_{\beta,2,1}\, b_{2,1}\, b_{1,2} \bmod 5^{9\beta+8}$$

$$\equiv \varepsilon_{\beta,1,2}(10192 \times 5^5 - 5096 \times 5^4) \bmod 5^{9\beta+8}$$

$$\equiv 45864 \times 5^4\, \varepsilon_{\beta,1,2} \bmod 5^{9\beta+8}$$

$\nu(\varepsilon_{\beta,1,2}) = 9\beta + 3$ so $\delta_{\beta+1,1,2} \not\equiv 0 \bmod 5^{9\beta+8}$ and $\nu(\delta_{\beta+1,1,2}) = 9\beta + 7 = 9(\beta + 1) - 2$. Lemma (2.2.6) follows by induction on β .

The following result was known to Watson [16].

<u>Lemma (2.2.7)</u> For $\alpha \geqslant 1$,

$$p(\delta_\alpha) = x_{\alpha,1} \equiv 3^{\alpha-1} \times 5^\alpha \bmod 5^{\alpha+1} .$$

<u>Proof</u>. $p(\delta_1) = x_{1,1} = 5$ so the statement is true for $\alpha = 1$. Suppose α is odd and $p(\delta_\alpha) = x_{\alpha,1} \equiv 3^{\alpha-1} \times 5^\alpha \bmod 5^{\alpha+1}$. From Lemma (2.2.1) we have

$$\nu(x_{\alpha,i}) \geqslant \alpha + \left[\frac{5i - 5}{2}\right] \geqslant \alpha + 2 \quad \text{for} \quad i \geqslant 2.$$

Hence from (1.1.8) it follows that

$$x_{\alpha+1,1} = \sum_{i\geq 1} x_{\alpha,i}\, a_{i,1}$$

$$\equiv x_{\alpha,1}\, a_{1,1} \bmod 5^{\alpha+2}$$

$$\equiv 63 \times 5 \times x_{\alpha,1} \bmod 5^{\alpha+2}$$

$$\equiv 3^\alpha \times 5^{\alpha+1} \bmod 5^{\alpha+2} .$$

Now suppose α is even and $p(\delta_\alpha) = x_{\alpha,1} \equiv 3^{\alpha-1} \times 5^\alpha \bmod 5^{\alpha+1}$.

From Lemma (2.2.1) we have

$$\nu(x_{\alpha,i}) \geqslant \alpha + [\frac{5i-4}{2}] \geqslant \alpha + 3 \quad \text{for} \quad i \geqslant 2.$$

Hence from (1.1.8) it follows that

$$\begin{aligned} x_{\alpha+1,1} &= \sum_{i\geq 1} x_{\alpha,i}\, b_{i,1} \\ &\equiv x_{\alpha,1}\, b_{1,1} \bmod 5^{\alpha+2} \\ &\equiv 28 \times 5 \times x_{\alpha,1} \bmod 5^{\alpha+2} \\ &\equiv 3^{\alpha} \times 5^{\alpha+1} \bmod 5^{\alpha+2} . \end{aligned}$$

Lemma (2.2.7) follows by induction on α .

We are now in a position to prove (2.1.4).

Theorem (2.2.8). For every $\alpha \geqslant 1$ there exists an integral constant k_{α} not divisible by 5 such that for all $n \geqslant 0$

(2.2.9) $$p(5^{\alpha+2}n + \delta_{\alpha+2}) \equiv k_{\alpha} \times 5^2 p(5^{\alpha}n + \delta_{\alpha}) \bmod 5^{[7\alpha/2]+3} ,$$

where δ_{α} is the reciprocal modulo 5^{α} of 24, and this best possible in the sense that the congruence does not hold for a higher power of 5.

Proof. Suppose α is odd with $\alpha = 2\beta - 1$, say. From Lemma (2.2.6) we have

$$\nu(\delta_{\beta,i,j}) \geqslant 9\beta - 2$$

or

(2.2.10) $$x_{2\beta+1,i}\, x_{2\beta-1,j} \equiv x_{2\beta-1,i}\, x_{2\beta+1,j} \bmod 5^{9\beta-2} .$$

From Lemma (2.2.7) it follows that $\nu(x_{\alpha,1}) = \alpha$ for $\alpha \geqslant 1$. Therefore there is an integer $k_{2\beta-1}$ with $(k_{2\beta-1}, 5) = 1$ such that

(2.2.11) $$\frac{x_{2\beta+1,1}}{5^{2\beta+1}} \equiv k_{2\beta-1} \frac{x_{2\beta-1,1}}{5^{2\beta-1}} \bmod 5^{5\beta-2} .$$

$$x_{2\beta-1,j} \equiv 0 \bmod 5^{2\beta-1}$$

so putting $i = 1$ into (2.2.10) and dividing both sides by $5^{2\beta-1}$ we obtain

$$\begin{aligned} x_{2\beta+1,j} \frac{x_{2\beta-1,1}}{5^{2\beta-1}} &\equiv 5^2 \frac{x_{2\beta+1,1}}{5^{2\beta+1}} x_{2\beta-1,j} \bmod 5^{7\beta-1} \\ &\equiv k_{2\beta-1} \times 5^2 \frac{x_{2\beta-1,1}}{5^{2\beta-1}} x_{2\beta-1,j} \bmod 5^{7\beta-1} . \end{aligned}$$

Hence

$$x_{2\beta+1,j} \equiv k_{2\beta-1} \times 5^2 x_{2\beta-1,j} \bmod 5^{7\beta-1}$$

or

$$x_{\alpha+2,j} \equiv k_\alpha \times 5^2 x_{\alpha,j} \bmod 5^{[7\alpha/2]+3} .$$

From Theorem (1.1.6) it follows that

$$p(5^{\alpha+2}n + \delta_{\alpha+2}) \equiv k_\alpha \times 5^2 p(5^\alpha n + \delta_\alpha) \bmod 5^{[7\alpha/2]+3} ,$$

for all $n \geqslant 0$. We will now show that (2.2.9) is best possible when α is odd.

Suppose $p(5^{2\beta+1}n + \delta_{2\beta+1}) \equiv k^*_{2\beta-1} \times 5^2 p(5^{2\beta-1}n + \delta_{2\beta-1}) \bmod 5^{7\beta}$,

for all $n \geqslant 0$.

Then from Theorem (1.1.6) we have

$$\sum_{n\geq 0} p(5^{2\beta+1}n + \delta_{2\beta+1})q^n = x_{2\beta+1,1}\frac{E(q^5)^5}{E(q)^6} + x_{2\beta+1,2}\, q\frac{E(q^5)^{11}}{E(q)^{12}} + \ldots$$

$$= x_{2\beta+1,1} + (x_{2\beta+1,2} + 6x_{2\beta+1,1})q + \ldots$$

and

$$\sum_{n\geq 0} p(5^{2\beta-1}n + \delta_{2\beta-1})q^n = x_{2\beta-1,1}\frac{E(q^5)^5}{E(q)^6} + x_{2\beta-1,2}\, q\frac{E(q^5)^{11}}{E(q)^{12}} + \ldots$$

$$= x_{2\beta-1,1} + (x_{2\beta-1,2} + 6x_{2\beta-1,1})\, q + \ldots .$$

Therefore

$$x_{2\beta+1,1} \equiv k^*_{2\beta-1} \times 5^2 x_{2\beta-1,1} \bmod 5^{7\beta} ,$$

$$x_{2\beta+1,2} + 6x_{2\beta+1,1} \equiv k^*_{2\beta-1} \times 5^2 (x_{2\beta-1,2} + 6x_{2\beta-1,1}) \bmod 5^{7\beta}$$

and

$$x_{2\beta+1,2} \equiv k^*_{2\beta-1} \times 5^2 x_{2\beta-1,2} \bmod 5^{7\beta}$$

or

$$\frac{x_{2\beta+1,1}}{5^{2\beta+1}} \equiv k^*_{2\beta-1} \frac{x_{2\beta-1,1}}{5^{2\beta-1}} \bmod 5^{5\beta-1}$$

and

$$\frac{x_{2\beta+1,2}}{5^{2\beta+1}} \equiv k^{*}_{2\beta-1} \frac{x_{2\beta-1,2}}{5^{2\beta-1}} \mod 5^{5\beta-1} \quad .$$

Therefore

$$\frac{x_{2\beta+1,2}}{5^{2\beta+1}} \frac{x_{2\beta-1,1}}{5^{2\beta-1}} \equiv \frac{x_{2\beta-1,2}}{5^{2\beta-1}} \frac{x_{2\beta+1,2}}{5^{2\beta+1}} \mod 5^{5\beta-1}$$

$$x_{2\beta+1,2}\ x_{2\beta-1,1} \equiv x_{2\beta-1,2}\ x_{2\beta+1,1} \mod 5^{9\beta-1}$$

or

$$\delta_{\beta,1,2} \equiv 0 \mod 5^{9\beta-1} \quad .$$

But from Lemma (2.2.6) we have $\nu(\delta_{\beta,1,2}) = 9\beta - 2$, a contradiction. Hence (2.2.9) is best possible when α is odd.

Now suppose α is even with $\alpha = 2\beta$, say. From Lemma (2.2.6) we have

$$\nu(\varepsilon_{\beta,i,j}) \geqslant 9\beta + 3$$

or

(2.2.12) $$x_{2\beta+2,i}\ x_{2\beta,j} \equiv x_{2\beta,i}\ x_{2\beta+2,j} \mod 5^{9\beta+3} \quad .$$

From Lemma (2.2.7) it follows that $\nu(x_{\alpha,1}) = \alpha$ for $\alpha \geqslant 1$. Therefore there is an integer $k_{2\beta}$ with $(k_{2\beta}, 5) = 1$ such that

(2.2.13) $$\frac{x_{2\beta+2,1}}{5^{2\beta+2}} \equiv k_{2\beta} \frac{x_{2\beta,1}}{5^{2\beta}} \mod 5^{5\beta+1} \quad .$$

$$x_{2\beta,j} \equiv 0 \mod 5^{2\beta}$$

so putting $i = 1$ in (2.2.12) and dividing both sides by $5^{2\beta}$ we obtain

$$x_{2\beta+2,j} \frac{x_{2\beta,1}}{5^{2\beta}} \equiv 5^2 \frac{x_{2\beta+2,1}}{5^{2\beta+2}} x_{2\beta,j} \mod 5^{7\beta+3}$$

$$\equiv k_{2\beta} \times 5^2 \frac{x_{2\beta,1}}{5^{2\beta}} x_{2\beta,j} \mod 5^{7\beta+3} \quad .$$

Hence

$$x_{2\beta+2,j} \equiv k_{2\beta} \times 5^2\, x_{2\beta,j} \mod 5^{7\beta+3}$$

or

$$x_{\alpha+2,j} \equiv k_{\alpha} \times 5^2\, x_{\alpha,j} \mod 5^{[7\alpha/2]+3} \quad .$$

From Theorem (1.1.6) it follows that

$$p(5^{\alpha+2}n + \delta_{\alpha+2}) \equiv k_{\alpha} \times 5^2 p(5^{\alpha}n + \delta_{\alpha}) \mod 5^{[7\alpha/2]+3} \quad ,$$

for all $n \geqslant 0$. We will now show that (2.2.9) is best possible when α is even.

Suppose $p(5^{2\beta+2}n + \delta_{2\beta+2}) \equiv k^*_{2\beta} \times 5^2(5^{2\beta}n + \delta_{2\beta}) \mod 5^{7\beta+4}$, for all $n \geqslant 0$.

Then from Theorem (1.1.6) we have

$$\sum_{n\geq 0} p(5^{2\beta+2}n + \delta_{2\beta+2})q^n = x_{2\beta+2,1}\frac{E(q^5)^6}{E(q)^7} + x_{2\beta+2,2}\, q\, \frac{E(q^5)^{12}}{E(q)^{13}} + \ldots$$

$$= x_{2\beta+2,1} + (x_{2\beta+2,2} + 7\, x_{2\beta+2,1})q + \ldots$$

and

$$\sum_{n\geq 0} p(5^{2\beta}n + \delta_{2\beta})q^n = x_{2\beta,1}\frac{E(q^5)^6}{E(q)^7} + x_{2\beta,2}\, q\, \frac{E(q^5)^{12}}{E(q)^{13}} + \ldots$$

$$= x_{2\beta,1} + (x_{2\beta,2} + 7x_{2\beta,1})q + \ldots \quad .$$

Therefore

$$x_{2\beta+2,1} \equiv k^*_{2\beta} \times 5^2\, x_{2\beta,1} \mod 5^{7\beta+4}$$

$$x_{2\beta+2,2} + 7\, x_{2\beta+2,1} \equiv k^*_{2\beta} \times 5^2(x_{2\beta,2} + 7\, x_{2\beta,1}) \mod 5^{7\beta+4}$$

and

$$x_{2\beta+2,2} \equiv k^*_{2\beta} \times 5^2\, x_{2\beta,2} \mod 5^{7\beta+4} \quad ,$$

or

$$\frac{x_{2\beta+2,1}}{5^{2\beta+2}} \equiv k^*_{2\beta}\frac{x_{2\beta,1}}{5^{2\beta}} \mod 5^{5\beta+2}$$

and

$$\frac{x_{2\beta+2,2}}{5^{2\beta+2}} \equiv k^{*}_{2\beta} \frac{x_{2\beta,2}}{5^{2\beta}} \mod 5^{5\beta+2} .$$

Therefore

$$\frac{x_{2\beta+2,2}}{5^{2\beta+2}} \frac{x_{2\beta,1}}{5^{2\beta}} \equiv \frac{x_{2\beta,2}}{5^{2\beta}} \frac{x_{2\beta+2,2}}{5^{2\beta+2}} \mod 5^{5\beta+2}$$

$$x_{2\beta+2,2}\, x_{2\beta,1} \equiv x_{2\beta,2}\, x_{2\beta+2,1} \mod 5^{9\beta+4}$$

or

$$\varepsilon_{\beta,1,2} \equiv 0 \mod 5^{9\beta+4} .$$

But from Lemma (2.2.6) we have $\nu(\varepsilon_{\beta,1,2}) = 9\beta + 3$, a contradiction. Hence (2.2.9) is best possible when α is even. This completes the proof of Theorem (2.2.8).

2.3. In this section we provide some details of the calculations for (2.1.2) and (2.1.3).

Theorem (2.3.1).

(2.3.2) $$p(5^{\alpha+2}n+\delta_{\alpha+2}) \equiv k_\alpha \times 5^2\, p(5^\alpha n+\delta_\alpha) \bmod 5^{[7\alpha/2]+3}$$

holds with $k_1 = 89$, $k_2 = 3339$, $k_3 = 240839$, $k_4 = 2193964$ and $k_5 = 256100214$.

Proof. From (2.2.11) and (2.2.13) it follows that (2.3.2) holds if and only if

(2.3.3) $$k_\alpha \equiv \left(\frac{x_{\alpha+2,1}}{5^{\alpha+2}}\right)\left(\frac{x_{\alpha,1}}{5^\alpha}\right)^{-1} \bmod 5^{[5\alpha/2]+1}$$

where the $x_{\alpha,i}$ are defined in (1.1.8). Hence we need only calculate $x_{\alpha+2,1}$ and $x_{\alpha,1}$ mod $5^{[7\alpha/2]+3}$ for $1 \leqslant \alpha \leqslant 5$. We have obtained the following congruences mod 5^{20}

(2.3.4)
$$\underset{\sim}{x}_1 \equiv (5,\ 0,\ 0,\ \ldots),$$
$$\underset{\sim}{x}_2 \equiv (63\times5^2,\ 52\times5^5,\ 63\times5^7,\ 6\times5^{10},\ 5^{12},\ 0,\ 0,\ \ldots),$$
$$\underset{\sim}{x}_3 \equiv (1353839\times5^3,\ 1885026212\times5^6,\ 7201333\times5^9,\ 7307608\times5^{10},\ 72766\times5^{13},\ 313\times5^{15},\ 0,\ 0,\ \ldots),$$
$$\underset{\sim}{x}_4 \equiv (55494085357\times5^4,\ 1015815628\times5^7,\ 8772232\times5^9,\ 179784\times5^{12},\ 3339\times5^{14},\ 0,\ 0,\ \ldots),$$
$$\underset{\sim}{x}_5 \equiv (10192387171\times5^5,\ 143162493\times5^8,\ 867137\times5^{11},\ 149987\times5^{12},\ 49\times5^{15},\ 107\times5^{17},\ 0,0,\ \ldots),$$
$$\underset{\sim}{x}_6 \equiv (1658060148\times5^6,\ 29406892\times5^9,\ 270148\times5^{11},\ 14401\times5^{14},\ 171\times5^{16},\ 0,0,\ \ldots),$$
and
$$\underset{\sim}{x}_7 \equiv (263563969\times5^7,\ 6182877\times5^{10},\ 17318\times5^{13},\ 343\times5^{14},\ 111\times5^{17},\ 3\times5^{19},\ 0,\ 0,\ \ldots).$$

From (2.3.3) and (2.3.4) we obtain $k_1 = 89$, $k_2 = 3339$, $k_3 = 250839$, $k_4 = 2193964$ and $k_5 = 256100214$ if $0 < k_\alpha < 5^{[5\alpha/2]+1}$. All these calculations were done on the Cyber 171 computer at U.N.S.W in about 2 seconds execution time, and these have been checked.

The author has in his possession a table of $p(n)$ for $n < 10700$. From this table we have been able to verify (2.3.2) for $\alpha = 1, 2,$ and 3 and small values of n.