CHAPTER III

SOME NEW CONGRUENCES OF THE PARTITION FUNCTION MODULO POWERS OF 7

3.1 INTRODUCTION.

In this chapter we obtain analogous results to (2.1.2) and (2.1.3) for powers of 7. Watson [16] obtained the following result. If $\alpha \geqslant 3$ and λ_α is the reciprocal modulo 7^α of 24 then for $n \geqslant 0$,

$$(3.1.1) \qquad p(7^\alpha n + \lambda_\alpha) \equiv \begin{cases} p(\lambda_\alpha) \dfrac{p(7n+5)}{7} \pmod{7^{[\alpha/2]+2}}, & \alpha \text{ odd}, \\ 24\, p(\lambda_\alpha) \dfrac{p(49n+47)}{49} \pmod{7^{[\alpha/2]+3}}, & \alpha \text{ even}. \end{cases}$$

For $\alpha = 3$ and $\alpha = 4$ (3.1.1) is equivalent to

$$(3.1.2) \qquad p(343n + 243) \equiv 5 \times 7\, p(7n + 5) \mod 7^3$$

and

$$p(2401n + 2301) \equiv 47 \times 7\, p(49n + 47) \mod 7^5 .$$

Using methods completely analogous to those of the previous chapter we are able to obtain the following new congruence relations

$$(3.1.3) \qquad p(16807n + 11905) \equiv 439 \times 7\, p(343n + 243) \mod 7^7,$$

$$p(117649n + 112747) \equiv 5241 \times 7\, p(2401n + 2301) \mod 7^9,$$

$$p(823543n + 583343) \equiv 374995 \times 7\, p(16807n + 11905) \mod 7^{11},$$

$$p(5764801n + 5524601) \equiv 1198538 \times 7\, p(117649n + 112747) \mod 7^{13},$$

$$p(40353607n + 28583805) \equiv 243320180 \times 7\, p(823543n + 583343) \mod 7^{15},$$

and $$p(282475249n + 270705447) \equiv 1655696425 \times 7\, p(5764801n + 5524601) \mod 7^{17}.$$

In fact there exist similar congruence relations for all higher powers of 7. The main result of this chapter is the following.

If $\alpha \geqslant 1$ and if λ_α is the reciprocal modulo 7^α of 24, then there exists an integral constant ℓ_α not divisible by 7 such that for all $n \geqslant 0$

(3.1.4) $$p(7^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \ell_\alpha \times 7\, p(7^\alpha n + \lambda_\alpha) \mod 7^{2\alpha+1}$$

and this is best possible.

In section 3.3 we give the generating functions for

$$p(7^\alpha n + \lambda_\alpha) \mod 7^{17} \qquad (1 \leqslant \alpha \leqslant 10),$$

which enable us to calculate ℓ_α for $1 \leqslant \alpha \leqslant 8$ thus obtaining (3.1.2) and (3.1.3).

3.2 As in 2.2 we define

$$\delta_{\beta,i,j} = x_{2\beta+1,i}\, x_{2\beta-1,j} - x_{2\beta-1,i}\, x_{2\beta+1,j} \tag{3.2.1}$$

and

$$\varepsilon_{\beta,i,j} = x_{2\beta+2,i}\, x_{2\beta,j} - x_{2\beta,i}\, x_{2\beta+2,j} \quad \text{for} \quad \beta,i,j \geqslant 1, \tag{3.2.2}$$

where the $\underset{\sim}{x}_\alpha$ are defined in (1.1.17). Again we have

$$\delta_{\beta,i,j} = -\delta_{\beta,j,i},\ \varepsilon_{\beta,i,j} = -\varepsilon_{\beta,j,i},\ \delta_{\beta,i,i} = \varepsilon_{\beta,i,i} = 0,$$

$$\delta_{\beta+1,i,j} = \sum_{k\geq 1}\sum_{\ell\geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j} \quad \text{and} \tag{3.2.3}$$

$$\varepsilon_{\beta,i,j} = \sum_{k\geq 1}\sum_{\ell\geq 1} \delta_{\beta,i,j}\, a_{k,i}\, a_{\ell,j}\ ,$$

where the $a_{i,j}$ and $b_{i,j}$ are defined in (1.1.18) - (1.1.20).

Lemma (3.2.4). For $\beta \geqslant 1$,

$$\nu(\delta_{\beta,1,2}) = \nu(\delta_{\beta,2,1}) = 5\beta - 1$$

$$\nu(\delta_{\beta,i,j}) \geqslant 5\beta - 3 + \left[\frac{7(i+j)-12}{4}\right] \geqslant 5\beta + 1 \quad \text{for} \quad i+j > 3,$$

$$\nu(\varepsilon_{\beta,1,2}) = \nu(\varepsilon_{\beta,2,1}) = 5\beta + 2$$

$$\nu(\varepsilon_{\beta,i,j}) \geqslant 5\beta + 1 + \left[\frac{7(i+j)-16}{4}\right] \geqslant 5\beta + 4 \quad \text{for} \quad i+j > 3.$$

Proof. $\delta_{1,i,j} = x_{3,i}\, x_{1,j} - x_{1,i}\, x_{3,j}$ and $\underset{\sim}{x}_1 = (7, 49, 0, 0, \ldots)$

so

$$\delta_{1,2,1} = -\delta_{1,1,2} = 7x_{3,2} - 49x_{3,1}\ .$$

Now from Theorem (1.1.16) we have

$$\sum_{n\geq 0} p(343n + 243)q^n = x_{3,1}\frac{E(q^7)^3}{E(q)^4} + x_{3,2}\ q\,\frac{E(q^7)^7}{E(q)^8} + \ldots\ .$$

Therefore from Gupta [9] we have

$$x_{3,1} = p(243) = 13397\ 82593\ 44888 = 273\ 42501\ 90712 \times 7^2$$

and

$$4x_{3,1} + x_{3,2} = p(586) = 2242\ 82898\ 59904\ 68310\ 34631$$
$$= 45\ 77202\ 01222\ 54455\ 31319 \times 7^2 .$$

So
$$x_{3,2} = p(586) - 4\ p(243)$$

and

$$\delta_{1,2,1} = 7(x_{3,2} - 7x_{3,1})$$
$$= 7(p(586) - 11\ p(243))$$
$$= 45\ 77201\ 98214\ 86934\ 33487 \times 7^3$$
$$= 6\ 53885\ 99744\ 98133\ 47641 \times 7^4 .$$

$$6\ 53885\ 99744\ 98133\ 47641 = 7 \times 93412\ 28534\ 99733\ 35377 + 2 .$$

So
$$\nu(\delta_{1,2,1}) = \nu(\delta_{1,1,2}) = 4 .$$

For $j > 2$ $\delta_{1,j,1} = -\delta_{1,1,j} = x_{1,1}\ x_{3,j}$. Therefore from Lemma (1.5.3) we have

$$\nu(\delta_{1,j,1}) = \nu(\delta_{1,1,j}) = \nu(x_{1,1}) + \nu(x_{3,j}) \geqslant 3 + [\frac{7j - 4}{4}]$$

For $i,j > 2$ $\delta_{1,i,j} = 0$ hence for $i + j > 3$ we have

$$\nu(\delta_{1,i,j}) \geqslant 3 + [\frac{7(i + j) - 11}{4}] \geqslant 2 + [\frac{7(i + j) - 12}{4}] .$$

Now suppose $\nu(\delta_{\beta,1,2}) = 5\beta - 1$

and

$$\nu(\delta_{\beta,i,j}) \geqslant 5\beta - 3 + [\frac{7(i + j) - 12}{4}] \quad \text{for } i + j > 3.$$

From (3.2.3) we have $\varepsilon_{\beta,i,j} = \sum_{k\geq 1} \sum_{\ell\geq 1} \delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}$ and

$\delta_{\beta,k,\ell} = 0$ for $k = \ell$ so by Lemma (1.5.2)

$$\nu(\varepsilon_{\beta,i,j}) \geqslant \min_{k\neq\ell}\{\nu(\delta_{\beta,k,\ell}) + \nu(a_{k,i}) + \nu(a_{\ell,j})\}$$

$$= \min\Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{\nu(\delta_{\beta,k,\ell}) + \nu(a_{k,i}) + \nu(a_{\ell,j})\},$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{\nu(\delta_{\beta,k,\ell}) + \nu(a_{k,i}) + \nu(a_{\ell,j})\}\Big\}$$

$$\geqslant \min\Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{5\beta - 1 + [\tfrac{7i - k - 1}{4}] + [\tfrac{7j - \ell - 1}{4}]\},$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{5\beta - 3 + [\tfrac{7(k + \ell) - 12}{4}] + [\tfrac{7i - k - 1}{4}] + [\tfrac{7j - \ell - 1}{4}]\}\Big\}$$

$$\geqslant \min\Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{5\beta - 1 + [\tfrac{7(i + j) - (k - \ell) - 5}{4}]\},$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{5\beta - 3 + [\tfrac{7(k + \ell) - 12}{6}] + [\tfrac{7(i + j) - (k + \ell) - 5}{4}]\}\Big\}.$$

The minimum of the latter term is attained when $k + \ell = 4$. So

$$\nu(\varepsilon_{\beta,i,j}) \geqslant \min\{5\beta - 1 + [\tfrac{7(i + j) - 8}{4}],\ 5\beta + 1 + [\tfrac{7(i + j) - 9}{4}]\}$$

$$\geqslant \min\{5\beta + 1 + [\tfrac{7(i + j) - 16}{4}],\ 5\beta + 1 + [\tfrac{7(i + j) - 9}{4}]\}$$

$$= 5\beta + 1 + [\tfrac{7(i + j) - 16}{4}] .$$

We will now show that $\nu(\varepsilon_{\beta,1,2}) = 5\beta - 2$. From (1.1.18), (1.1.19) and (1.1.20) we obtain

$$a_{1,1} = m_{4,2} = 82 \times 7,\quad a_{1,2} = m_{4,3} = 176 \times 7^3,$$

$$a_{2,1} = m_{8,3} = 352 \times 7,\quad a_{2,2} = m_{8,4} = 48758 \times 7^2 .$$

We also have $\nu(\delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}) \geqslant 5\beta + 1 + [\frac{7(i+j)-9}{4}]$ for $k + \ell > 3$.

So if $k + \ell > 3$ and $i + j = 3$ then

$$\nu(\delta_{\beta,k,\ell}\, a_{k,i}\, a_{\ell,j}) \geqslant 5\beta + 4.$$

It follows that

$$\varepsilon_{\beta,1,2} = -\varepsilon_{\beta,2,1} = \sum_{k\geq 1} \sum_{\ell\geq 1} \delta_{\beta,k,\ell}\, a_{k,1}\, a_{\ell,2}$$

$$\equiv \delta_{\beta,1,2}\, a_{1,1}\, a_{2,2} + \delta_{\beta,2,1}\, a_{2,1}\, a_{1,2} \mod 7^{5\beta+3}$$

$$\equiv \delta_{\beta,1,2}\,(3998156 \times 7^3 - 61952 \times 7^4) \mod 7^{5\beta+3}$$

$$\equiv 3564492 \times 7^3\, \delta_{\beta,1,2} \mod 7^{5\beta+3}$$

$\nu(\delta_{\beta,1,2}) = 5\beta - 1$ so $\varepsilon_{\beta,1,2} \not\equiv 0 \mod 7^{5\beta+3}$ and $\nu(\varepsilon_{\beta,1,2}) = 5\beta + 2$.

Now suppose $\nu(\varepsilon_{\beta,1,2}) = 5\beta + 2$

and

$$\nu(\varepsilon_{\beta,i,j}) \geqslant 5\beta + 1 + [\frac{7(i+j)-16}{4}] \quad \text{for} \quad i + j > 3.$$

From (3.2.3) we have $\delta_{\beta+1,i,j} = \sum_{k\geq 1} \sum_{\ell\geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j}$ and

$\varepsilon_{\beta,k,\ell} = 0$ for $k = \ell$ so by Lemma (1.5.2)

$$\nu(\delta_{\beta+1,i,j}) \geqslant \min_{k\neq\ell} \{\nu(\varepsilon_{\beta,k,\ell}) + \nu(b_{k,i}) + \nu(b_{\ell,j})\}$$

$$= \min \Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{\nu(\varepsilon_{\beta,k,\ell}) + \nu(b_{k,i}) + \nu(b_{\ell,j})\},$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{\nu(\varepsilon_{\beta,k,\ell}) + \nu(b_{k,i}) + \nu(b_{\ell,j})\}\Big\}.$$

$$\geqslant \min \Big\{ \min_{\substack{k\neq\ell \\ k+\ell=3}} \{5\beta + 2 + [\frac{7i-k-3}{4}] + [\frac{7j-\ell-3}{4}]\},$$

$$\min_{\substack{k\neq\ell \\ k+\ell>3}} \{5\beta + 1 + [\frac{7(k+\ell)-16}{4}] + [\frac{7i-k-3}{4}] + [\frac{7j-\ell-3}{4}]\}\Big\}$$

$$\geqslant \min \left\{ \min_{\substack{k \neq \ell \\ k+\ell=3}} \{5\beta + 2 + [\frac{7(i+j) - (k+\ell) - 9}{4}] \right. ,$$

$$\left. \min_{\substack{k \neq \ell \\ k+\ell>3}} \{5\beta + 1 + [\frac{7(k+\ell) - 16}{4}] + [\frac{7(i+j) - (k+\ell) - 9}{4}] \}\right\} .$$

The minimum of the latter term is obtained when $k + \ell = 4$. So

$$\nu(\delta_{\beta+1,i,j}) \geqslant \min\{5\beta + 2 + [\frac{7(i+j) - 12}{4}],\ 5\beta + 4 + [\frac{7(i+j) - 13}{4}]\}$$

$$= 5\beta + 2 + [\frac{7(i+j) - 12}{4}] = 5(\beta + 1) - 3 + [\frac{7(i+j) - 12}{4}] .$$

We will now show that $\nu(\delta_{\beta+1,1,2}) = 5\beta + 4$. From (1.1.18), (1.1.19) and (1.1.20) we obtain

$$b_{1,1} = m_{5,2} = 190, \quad b_{1,2} = m_{5,3} = 1265 \times 7^2 ,$$

$$b_{2,1} = m_{9,3} = 255, \quad b_{2,2} = m_{9,4} = 17118 \times 7^2 .$$

We also have $\nu(\varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j}) \geqslant 5\beta + 4 + [\frac{7(i+j) - 13}{4}]$ for $k + \ell > 3$.

So if $k + \ell > 3$ and $i + j = 3$ then

$$\nu(\varepsilon_{\beta,k,\ell}\, b_{k,i}\, b_{\ell,j}) \geqslant 5\beta + 6.$$

It follows that

$$\delta_{\beta+1,1,2} = - \delta_{\beta+1,2,1} = \sum_{k\geq 1} \sum_{\ell\geq 1} \varepsilon_{\beta,k,\ell}\, b_{k,1}\, b_{\ell,2}$$

$$\equiv \varepsilon_{\beta,1,2}\, b_{1,1}\, b_{2,2} + \varepsilon_{\beta,2,1}\, b_{2,1}\, b_{1,2} \bmod 7^{5\beta+5}$$

$$\equiv \varepsilon_{\beta,1,2}\, (3252420 \times 7^2 - 322575 \times 7^2) \bmod 7^{5\beta+5}$$

$$\equiv 2929845 \times 7^2\, \varepsilon_{\beta,1,2} \bmod 7^{5\beta+5}$$

$\nu(\varepsilon_{\beta,1,2}) = 5\beta + 2$ so $\delta_{\beta+1,1,2} \not\equiv 0 \bmod 7^{5\beta+5}$ and

$$\nu(\delta_{\beta+1,1,2}) = 5\beta + 4 = 5(\beta + 1) - 1.$$

Lemma (3.2.4) follows by induction on β .

The following result was known to Watson [16].

Lemma (3.2.5). For $\beta \geqslant 1$,

$$p(\lambda_{2\beta-1}) = x_{2\beta-1,1} \equiv 5^{\beta-1} \times 7^{\beta} \bmod 7^{\beta+1}$$

and

$$p(\lambda_{2\beta}) = x_{2\beta,1} \equiv 5^{\beta} \times 7^{\beta+1} \bmod 7^{\beta+2} .$$

Proof.

$$p(\lambda_1) = x_{1,1} = 7 \equiv 5^0 \times 7^1 \bmod 7^2.$$

Now suppose

$$p(\lambda_{2\beta-1}) = x_{2\beta-1,1} \equiv 5^{\beta-1} \times 7^{\beta} \bmod 7^{\beta+1}.$$

From (1.1.17) we have

$$x_{2\beta,1} = \sum_{i\geq 1} x_{2\beta-1,i}\, a_{i,1} .$$

So if $\beta = 1$ then

$$\begin{aligned} x_{2,1} &= 7\, a_{1,1} + 7^2\, a_{2,1} \\ &= 82 \times 7^2 + 352 \times 7^3 \\ &\equiv 5 \times 7^2 \bmod 7^3 . \end{aligned}$$

If $\beta > 1$ then from Lemma (1.5.3) we have

$$\nu(x_{2\beta-1,i}) \geqslant \beta + [\frac{7i - 4}{4}] \geqslant \beta + 2 \quad \text{for} \quad i \geqslant 2 .$$

It follows that

$$\begin{aligned} p(\lambda_{2\beta}) = x_{2\beta,1} &\equiv x_{2\beta-1,1}\, a_{1,1} \bmod 7^{\beta+2} \\ &\equiv 82 \times 7\, x_{2\beta-1,1} \bmod 7^{\beta+2} \\ &\equiv 5^{\beta} \times 7^{\beta+1} \bmod 7^{\beta+2} . \end{aligned}$$

Now suppose

$$p(\lambda_{2\beta}) = x_{2\beta,1} \equiv 5^{\beta} \times 7^{\beta+1} \bmod 7^{\beta+2} .$$

From Lemma (1.5.3) we have

$$\nu(x_{2\beta,i}) \geqslant (\beta + 1) + [\frac{7i - 6}{4}] \geqslant \beta + 3 \quad \text{for} \quad i \geqslant 2 .$$

Therefore from (1.1.17) it follows that

$$
\begin{aligned}
x_{2\beta+1,1} &= \sum_{i\geq 1} x_{2\beta,i}\, b_{i,1} \\
&\equiv x_{2\beta,1}\, b_{1,1} \mod 7^{\beta+2} \\
&\equiv 190\, x_{2\beta,1} \mod 7^{\beta+2} \\
&\equiv 5^{\beta} \times 7^{\beta+1} \mod 7^{\beta+2} .
\end{aligned}
$$

Lemma (3.2.5) follows by induction on β.

We are in a position to prove (3.1.4).

<u>Theorem (3.2.6)</u>. For every $\alpha \geqslant 1$ there exists an integral constant ℓ_{α} not divisible by 7 such that for all $n \geqslant 0$

(3.2.7) $$p(7^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \ell_{\alpha} \times 7\, p(7^{\alpha}n + \lambda_{\alpha}) \mod 7^{2\alpha+1} ,$$

where λ_{α} is the reciprocal modulo 7^{α} of 24, and this is best possible in the sense that the congruence does not hold for a higher power of 7.

<u>Proof</u>. Suppose α is odd with $\alpha = 2\beta - 1$, say. From Lemma (3.2.4) we have

$$\nu(\delta_{\beta,i,j}) \geqslant 5\beta - 1$$

or

(3.2.8) $$x_{2\beta+1,i}\, x_{2\beta-1,j} \equiv x_{2\beta-1,i}\, x_{2\beta+1,j} \mod 7^{5\beta-1} .$$

From Lemma (3.2.5) it follows that $\nu(x_{2\beta-1,1}) = \beta$ and $\nu(x_{2\beta+1,1}) = \beta + 1$. Therefore there is an integer $\ell_{2\beta-1}$ with $(\ell_{2\beta-1}, 7) = 1$ such that

(3.2.9) $$\frac{x_{2\beta+1,1}}{7^{\beta+1}} \equiv \ell_{2\beta-1} \frac{x_{2\beta-1,1}}{7^{\beta}} \mod 7^{3\beta-2} .$$

$$x_{2\beta-1,j} \equiv 0 \mod 7^{\beta}$$

so putting $i = 1$ in (3.2.8) and dividing both sides by 7^{β} we obtain

$$x_{2\beta+1,j}\frac{x_{2\beta-1,1}}{7^{\beta}} \equiv 7\frac{x_{2\beta+1,1}}{7^{\beta+1}}\, x_{2\beta-1,j} \mod 7^{4\beta-1}$$

$$\equiv \ell_{2\beta-1} \times 7\frac{x_{2\beta-1,1}}{7^{\beta}}\, x_{2\beta-1,j} \mod 7^{4\beta-1} .$$

Hence

$$x_{2\beta+1,j} \equiv \ell_{2\beta-1} \times 7\, x_{2\beta-1,j} \mod 7^{4\beta-1}$$

or

$$x_{\alpha+2,j} \equiv \ell_{\alpha} \times 7\, x_{\alpha,j} \mod 7^{2\alpha+1} .$$

From Theorem (1.1.16) it follows that

$$p(7^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \ell_{\alpha} \times 7\, p(7^{\alpha}n + \lambda_{\alpha}) \mod 7^{2\alpha+1} ,$$

for all $n \geqslant 0$.

We will now show that (3.2.7) is best possible when α is odd. Suppose

$$p(7^{2\beta+1}n + \lambda_{2\beta+1}) \equiv \ell^{*}_{2\beta-1} \times 7\, p(7^{2\beta-1}n + \lambda_{2\beta-1}) \mod 7^{4\beta} ,$$

for all $n \geqslant 0$.

Then from Theorem (1.1.16) we have

$$\sum_{n\geq 0} p(7^{2\beta+1}n + \delta_{2\beta+1})q^{n} = x_{2\beta+1,1}\frac{E(q^{7})^{3}}{E(q)^{4}} + x_{2\beta+1,2}\, q\frac{E(q^{7})^{7}}{E(q)^{8}} + \dots$$

$$= x_{2\beta+1,1} + (x_{2\beta+1,2} + 4\, x_{2\beta+1,1})q + \dots$$

and

$$\sum_{n\geq 0} p(7^{2\beta-1}n + \delta_{2\beta-1})q^{n} = x_{2\beta-1,1}\frac{E(q^{7})^{3}}{E(q)^{4}} + x_{2\beta-1,2}\, q\frac{E(q^{7})^{7}}{E(q)^{8}} + \dots$$

$$= x_{2\beta-1,1} + (x_{2\beta-1,2} + 4x_{2\beta-1,1})\, q + \dots .$$

Therefore,

$$x_{2\beta+1,1} \equiv \ell^*_{2\beta-1} \times 7\, x_{2\beta-1,1} \bmod 7^{4\beta}$$

$$x_{2\beta+1,2} + 4\, x_{2\beta+1,1} \equiv \ell^*_{2\beta-1} \times 7\, (x_{2\beta-1,2} + 4\, x_{2\beta-1,1}) \bmod 7^{4\beta}$$

and

$$x_{2\beta+1,2} \equiv \ell^*_{2\beta-1} \times 7\, x_{2\beta-1,2} \bmod 7^{4\beta}\ ,$$

or

$$\frac{x_{2\beta+1,1}}{7^{\beta+1}} \equiv \ell^*_{2\beta-1} \frac{x_{2\beta-1,1}}{7^{\beta}} \bmod 7^{3\beta-1}$$

and

$$\frac{x_{2\beta+1,2}}{7^{\beta+1}} \equiv \ell^*_{2\beta-1} \frac{x_{2\beta-1,1}}{7^{\beta}} \bmod 7^{3\beta-1}\ .$$

Therefore

$$\frac{x_{2\beta+1,2}}{7^{\beta+1}} \frac{x_{2\beta-1,1}}{7^{\beta}} \equiv \frac{x_{2\beta-1,2}}{7^{\beta}} \frac{x_{2\beta+1,1}}{7^{\beta+1}} \bmod 7^{3\beta-1}\ ,$$

$$x_{2\beta+1,2}\, x_{2\beta-1,1} \equiv x_{2\beta-1,2}\, x_{2\beta+1,1} \bmod 7^{5\beta}$$

or

$$\delta_{\beta,1,2} \equiv 0 \bmod 7^{5\beta}\ .$$

But from Lemma (3.2.4) we have $\nu(\delta_{\beta,1,2}) = 5\beta - 1$, a contradiction. Hence (3.2.7) is best possible when α is odd.

Now suppose α is even with $\alpha = 2\beta$, say. From Lemma (3.2.4) we have

$$\nu(\varepsilon_{\beta,i,j}) \geqslant 5\beta + 2$$

or

(3.2.10) $$x_{2\beta+2,i}\, x_{2\beta,j} \equiv x_{2\beta,i}\, x_{2\beta+2,j} \bmod 7^{5\beta+2}\ .$$

From Lemma (3.2.5) it follows that $\nu(x_{2\beta+2,1}) = \beta + 2$ and $\nu(x_{2\beta,1}) = \beta + 1$. Therefore there is an integer $\ell_{2\beta}$ with $(\ell_{2\beta}, 7) = 1$ such that

(3.2.11) $$\frac{x_{2\beta+2,1}}{7^{\beta+2}} \equiv \ell_{2\beta} \frac{x_{2\beta,1}}{7^{\beta+1}} \bmod 7^{3\beta-1}\ .$$

$$x_{2\beta,j} \equiv 0 \bmod 7^{\beta+1}\ .$$

so putting $i = 1$ in (3.2.10) and dividing both sides by $7^{\beta+1}$ we obtain

$$x_{2\beta+2,j} \frac{x_{2\beta,1}}{7^{\beta+1}} \equiv 7 \frac{x_{2\beta+2,1}}{7^{\beta+1}} x_{2\beta,j} \mod 7^{4\beta+1}$$

$$\equiv \ell_{2\beta} \times 7 \frac{x_{2\beta,1}}{7^{\beta+1}} x_{2\beta,j} \mod 7^{4\beta+1} .$$

Hence
$$x_{2\beta+2,j} \equiv \ell_{2\beta} \times 7 x_{2\beta,j} \mod 7^{4\beta+1}$$

or
$$x_{\alpha+2,j} \equiv \ell_{\alpha} \times 7 x_{\alpha,j} \mod 7^{2\alpha+1} .$$

From Theorem (1.1.16) it follows that

$$p(7^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \ell_{\alpha} \times 7 p(7^{\alpha}n + \lambda_{\alpha}) \mod 7^{2\alpha+1} ,$$

for all $n \geqslant 0$.

We will now show that (3.2.7) is best possible when α is even. Suppose

$$p(7^{2\beta+2}n + \lambda_{2\beta+2}) \equiv \ell^{*}_{2\beta} \times 7 p(7^{2\beta}n + \lambda_{2\beta}) \mod 7^{4\beta+2} ,$$

for all $n \geqslant 0$. Then from Theorem (1.1.16) we have

$$\sum_{n\geq 0} p(7^{2\beta+2}n + \lambda_{2\beta+2})q^n = x_{2\beta+2,1} \frac{E(q^7)^4}{E(q)^5} + x_{2\beta+2,2}\, q \frac{E(q^7)^8}{E(q)^9} + \ldots$$

$$= x_{2\beta+2,1} + (x_{2\beta+2,2} + 5x_{2\beta+2,1})q + \ldots$$

and

$$\sum_{n\geq 0} p(7^{2\beta}n + \lambda_{2\beta})q^n = x_{2\beta,1} \frac{E(q^7)^4}{E(q)^5} + x_{2\beta,2} \frac{E(q^7)^8}{E(q)^9} + \ldots$$

$$= x_{2\beta,1} + (x_{2\beta,2} + 5 x_{2\beta,1})q + \ldots$$

Therefore

$$x_{2\beta+2,1} \equiv \ell^{*}_{2\beta} \times 7 x_{2\beta,1} \mod 7^{4\beta+2} ,$$

$$x_{2\beta+2,2} + 5 x_{2\beta+2,1} \equiv \ell^{*}_{2\beta} \times 7(x_{2\beta,2} + 5 x_{2\beta,1}) \mod 7^{4\beta+2}$$

and $$x_{2\beta+2,2} \equiv \ell^{*}_{2\beta} \times 7\, x_{2\beta,2} \mod 7^{4\beta+2} ,$$

or $$\frac{x_{2\beta+2,1}}{7^{\beta+2}} \equiv \ell^{*}_{2\beta} \frac{x_{2\beta,1}}{7^{\beta+1}} \mod 7^{3\beta}$$

and $$\frac{x_{2\beta+2,2}}{7^{\beta+2}} \equiv \ell^{*}_{2\beta} \frac{x_{2\beta,2}}{7^{\beta+1}} \mod 7^{3\beta} .$$

Therefore $$\frac{x_{2\beta+2,2}}{7^{\beta+2}} \frac{x_{2\beta,1}}{7^{\beta+1}} \equiv \frac{x_{2\beta,2}}{7^{\beta+1}} \frac{x_{2\beta+2,2}}{7^{\beta+2}} \mod 7^{3\beta} ,$$

$$x_{2\beta+2}\, x_{2\beta,1} \equiv x_{2\beta,2}\, x_{2\beta+2,1} \mod 7^{5\beta+3}$$

or $$\varepsilon_{\beta,1,2} \equiv 0 \mod 7^{5\beta+3} .$$

But from Lemma (3.2.4) we have $\nu(\varepsilon_{\beta,1,2}) = 5\beta + 2$, a contradiction. Hence (3.2.7) is best possible when α is even. This completes the proof of Theorem (3.2.6).

3.3. In this section we provide some details of the calculations for (3.1.2) and (3.1.3).

Theorem (3.3.1).

(3.3.2) $$p(7^{\alpha+2}n + \lambda_{\alpha+2}) \equiv \ell_\alpha \times 7\, p(7^{\alpha}n + \lambda_\alpha) \mod 7^{2\alpha+1}$$

holds with $\ell_1 = 5$, $\ell_2 = 47$, $\ell_3 = 439$, $\ell_4 = 5241$, $\ell_5 = 374995$, $\ell_6 = 1198538$, $\ell_7 = 243320180$ and $\ell_8 = 1655696425$.

Proof. From (3.2.9) and (3.2.11) it follows that (3.3.2) holds if and only if

(3.3.3) $$\ell_\alpha \equiv \left(\frac{x_{\alpha+2,1}}{7^{[\frac{\alpha}{2}]+2}}\right)\left(\frac{x_{\alpha,1}}{7^{[\frac{\alpha}{2}]+1}}\right)^{-1} \mod 7^{[\frac{3\alpha+1}{2}]-1}$$

where the $x_{\alpha,i}$ are defined in (1.1.17). Hence we need only calculate $x_{\alpha+2,1}$ and $x_{\alpha,1}$ mod $7^{[\frac{\alpha}{2}]+[\frac{3\alpha+1}{2}]+1}$ or $x_{\alpha+2,1}$ and $x_{\alpha,1}$ mod $7^{2\alpha+1}$ for $1 \leqslant \alpha \leqslant 8$. We have obtained the following congruences mod 7^{17}

(3.3.4) $$\underset{\sim}{x}_1 \equiv (7,\ 7^2,\ 0,\ 0,\ \ldots),$$

$$\underset{\sim}{x}_2 \equiv (2546\times7^2,\ 48934\times7^4,\ 1418989\times7^5,\ 2488800\times7^7,\ 2394438\times7^9,\ 25259\times7^{11},\ 9633\times7^{12},\ 44\times7^{15},\ 0,\ 0,\ \ldots),$$

$$\underset{\sim}{x}_3 \equiv (27342501907l2\times7^2,\ 83418943353\times7^4,\ 97454417\times7^6,\ 12931672\times7^8,\ 2642302\times7^9,\ 36000\times7^{11},\ 2153\times7^{13},\ 39\times7^{15},\ 5\times7^{16},\ 0,0,\ \ldots),$$

$$\underset{\sim}{x}_4 \equiv (192116669791\times7^3,\ 9806367002\times7^5,\ 1582389141\times7^6,\ 38618596\times7^8,\ 588650\times7^{10},\ 9428\times7^{12},\ 1825\times7^{13},\ 0,0,\ \ldots),$$

$$\underset{\sim}{x}_5 \equiv (472047273940\times7^3,\ 5319341719\times7^5,\ 212629409\times7^7,\ 3551650\times7^9,\ 334569\times7^{10},\ 2676\times7^{12},\ 202\times7^{14},\ 6\times7^{16},0,0,\ \ldots),$$

$$\underset{\sim}{x}_6 \equiv (76637058735\times7^4,\ 1972802199\times7^6,\ 202908614\times7^7,\ 986212\times7^9\ 48141\times7^{11},\ 254\times7^{13},\ 270\times7^{14},\ 0,\ 0,\ \ldots),$$

$$\underset{\sim}{x}_7 \equiv (70283388580\times 7^4,\ 1924477963\times 7^6,\ 7152266\times 7^8,\ 478764\times 7^{10},\ 102661\times 7^{11},\ 675\times 7^{13},\ 37\times 7^{15},\ 0,\ 0,\ \ldots),$$

$$\underset{\sim}{x}_8 \equiv (7381386917\times 7^5,\ 173679190\times 7^7,\ 1894426\times 7^8,\ 231645\times 7^{10},\ 297\times 7^{12},\ 31\times 7^{14},\ 48\times 7^{15},\ 0,\ 0,\ \ldots),$$

$$\underset{\sim}{x}_9 \equiv (10969244911\times 7^5,\ 39089557\times 7^7,\ 3500108\times 7^9,\ 85094\times 7^{11},\ 3810\times 7^{12},\ 316\times 7^{14},\ 3\times 7^{16},\ 0,\ 0,\ \ldots),$$

and

$$\underset{\sim}{x}_{10} \equiv (1213638485\times 7^6,\ 13481638\times 7^8,\ 5497726\times 7^9,\ 48221\times 7^{11},\ 729\times 7^{13},\ 36\times 7^{15},\ 2\times 7^{16},\ 0,\ 0,\ \ldots),$$

From (3.3.3) and (3.3.4) we obtain $\ell_1 = 5$, $\ell_2 = 47$, $\ell_3 = 439$, $\ell_4 = 5241$, $\ell_5 = 374995$, $\ell_6 = 1198538$, $\ell_7 = 243320180$ and $\ell_8 = 1655696425$. All these calculations wer done on the Cyber 171 computer at U.N.S.W. in about 4 seconds execution time, and these have been checked. Also, from our partition table we have been able to verify (3.3.2) for $\alpha = 1, 2$ and small values of n.