CHAPTER IV

CONGRUENCES FOR $p_{-k}(n)$

4.1 INTRODUCTION

Suppose δ_α is the reciprocal modulo 5^α of 24. In Chapter 1 we stated the Ramanujan-type identities that Hirschhorn and Hunt had found for the generating functions $\sum_{n\geq 0} p(5^\alpha n + \delta_\alpha)q^n$. In this chapter we generalise their methods to the functions $p_{-k}(n)$, where $p_{-k}(n)$ is defined by

$$\sum_{n\geq 0} p_{-k}(n)\, q^n = \prod_{n\geq 1} (1-q^n)^{-k},$$

so that $p_{-1}(n) = p(n)$. We obtain Ramanujan-type identities for the generating functions

$$\sum_{\substack{n\equiv\delta_\alpha k \\ \bmod 5^\alpha}} p_{-k}(n)\, q^{(n+(5^\alpha-\delta_\alpha)k)/5^\alpha}$$

and by finding a lower bound on the power of 5 that divides the coefficients in these identities we are able to prove the result, stated below, due to Atkin [2] for the case $p = 5$. We give a detailed discussion of Atkin's proof. Unfortunately we have changed Atkin's notation in order to be consistent with the notation of Hirschhorn and Hunt, and earlier chapters.

Theorem (4.1.1).

Let $k > 0$ and p be one of the primes 2, 3, 5, 7 or 13. Then if $24m \equiv k \bmod p^\alpha$ we have

(4.1.2) $$p_{-k}(m) \equiv 0 \bmod p^{B\alpha/2+\varepsilon},$$

where $\varepsilon = \varepsilon(k) = O(\log k)$ and $B = B(k,p)$ depending on p and the residue of k modulo 24 according to the following table:

	k=1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
p=2	-	-	-	-	-	-	-	3	-	-	-	-	-	-	-	3	-	-	-	-	-	-	-	0
p=3	-	-	3	-	-	2	-	-	3	-	-	2	-	-	1	-	-	2	-	-	1	-	-	0
p=5	2	1	1	1	2	2	1	1	1	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0
p=7	1	1	1	2	1	1	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0
p=13	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Atkin has proved this Theorem in detail for the case $p = 5$ but has only sketched briefly the basic formulae required for the other primes. The proof of the following is completely analogous to that of Theorem (1.1.6) which is the main result of Hirschhorn and Hunt's paper.

Theorem (4.1.3). For $\alpha \geqslant 1$,

(4.1.4)
$$\sum_{\substack{n \equiv \delta_\alpha k \\ \bmod 5^\alpha}} p_{-k}(n)\, q^{(n+(5^\alpha-\delta_\alpha)k)/5^\alpha} = \begin{cases} \displaystyle\sum_{i\geq 1} x_{\alpha,i}\, q^n \frac{E(q^5)^{6i-k}}{E(q)^{6i}}, & \alpha \text{ odd}, \\ \displaystyle\sum_{i\geq 1} x_{\alpha,i}\, q^i \frac{E(q^5)^{6i}}{E(q)^{6i+k}}, & \alpha \text{ even}, \end{cases}$$

where

$$E(q) = \prod_{n\geq 1} (1-q^n),$$

$$\underset{\sim}{x}_1 = (m_{k,i})_{i\geq 1} = (m_{k,1}, m_{k,2}, \dots)$$

and for $\alpha \geqslant 1$,

(4.1.5)
$$\underset{\sim}{x}_{\alpha+1} = \begin{cases} \underset{\sim}{x}_\alpha A, & \alpha \text{ odd}, \\ \underset{\sim}{x}_\alpha B, & \alpha \text{ even}, \end{cases}$$

Here $A = (a_{i,j})_{i,j\geq 1}$ and $B = (b_{i,j})_{i,j\geq 1}$ are defined by

(4.1.6)
$$a_{i,j} = m_{6i,i+j}, \quad b_{i,j} = m_{6i+k,i+j},$$

where $M = (m_{i,j})_{i,j\geq 1}$ is defined as before. That is, the first five rows of M are

$$(4.1.7)\qquad \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2\times 5 & 5^3 & 0 & 0 & 0 & 0 & \cdots \\ 9 & 3\times 5^3 & 5^5 & 0 & 0 & 0 & \cdots \\ 4 & 22\times 5^2 & 4\times 5^5 & 5^7 & 0 & 0 & \cdots \\ 1 & 4\times 5^3 & 8\times 5^5 & 5^8 & 5^9 & 0 & \cdots \end{pmatrix}$$

and for $i \geq 6$, $m_{i,1} = 0$ and for $j \geq 2$,

$$(4.1.8)\qquad m_{i,j} = 25\, m_{i-1,j-1} + 25\, m_{i-2,j-1} + 15\, m_{i-3,j-1} + 5\, m_{i-4,j-1} + m_{i-5,j-1}\ .$$

It is clear that for the case $k = 1$ Theorem (4.1.3) reduces to Hirschhorn and Hunt's main result (Theorem (1.1.6)). Note first that with $k = 1$ the vectors $\underset{\sim}{x}_\alpha$ and the matrices A, B and M are the same as those in Hirschhorn and Hunt's paper.

Now

$$\sum_{\substack{n\equiv\delta_\alpha \\ \bmod 5^\alpha}} p_{-1}(n)\, q^{(n+(5^\alpha-\delta_\alpha))/5^\alpha} = \sum_{n\geq 0} p(5^\alpha n+\delta_\alpha)\, q^{n+1}$$

and with $k = 1$ (4.1.4) reduces to

$$\sum_{n\geq 0} p(5^\alpha n+\delta_\alpha)\, q^n = \begin{cases} \sum_{i\geq 1} x_{\alpha,i}\, q^{i-1} \dfrac{E(q^5)^{6i-1}}{E(q)^{6i}}, & \alpha \text{ odd}, \\[2ex] \sum_{i\geq 1} x_{\alpha,i}\, q^{i-1} \dfrac{E(q^5)^{6i}}{E(q)^{6i+1}}, & \alpha \text{ even}, \end{cases}$$

which is (1.1.7).

4.2. We now turn to a discussion of Theorem (4.1.1). Let $\nu(n)$ denote the exact power of 5 dividing n. To establish (4.1.2) for the case $p = 5$ we will find a lower bound for the following

$$\min_{\substack{m: \\ 24m\equiv k \\ \bmod 5^\alpha}} \nu(p_{-k}(m)) = \min_{\substack{n: \\ n\equiv\delta_\alpha k \\ \bmod 5^\alpha}} \nu(p_{-k}(n)) .$$

From (4.1.4) we have

$$\text{(4.2.1)} \qquad \min_{\substack{m: \\ 24m\equiv k \\ \bmod 5^\alpha}} \nu(p_{-k}(m)) \geqslant \min_{i\geq 1} \nu(x_{\alpha,i}) .$$

Suppose $n \equiv \delta_\alpha k \bmod 5^\alpha$ then for certain k

$$(n + (5^\alpha - \delta_\alpha)k)/5^\alpha > 1 \quad \text{for all} \quad n ,$$

so that for small i $x_{\alpha,i} = 0$. For instance, if $\alpha = 1$ and $k = 6$ then $\delta_\alpha = \delta_1 = 4$; $n \equiv \delta_\alpha k \equiv 4 \bmod 5$ and

$$(n + (5^\alpha - \delta_\alpha)k)/5^\alpha \geqslant (4 + (5-4)6)/5 = 2$$

so that $x_{1,1} = 0$. We shall now calculate the first non-zero entry of the vector $\underset{\sim}{x}_\alpha$. This depends on the smallest power of q in the expansion of either side of the equality in (4.1.4). So we are interested in the smallest non-negative integer n for which $n \equiv \delta_\alpha k \bmod 5^\alpha$. Suppose $n = k\delta_\alpha - 5^\alpha m$ then we want to find the largest possible m such that

$$k\delta_\alpha - 5^\alpha m \geqslant 0 .$$

So

$$k\delta_\alpha - 5^\alpha m \geqslant 0 \quad \text{and} \quad k\delta_\alpha - 5^\alpha(m+1) < 0 ,$$

$$\frac{k\delta_\alpha}{5^\alpha} - 1 < m \leqslant \frac{k\delta_\alpha}{5^\alpha}$$

or

$$m = \left[\frac{k\delta_\alpha}{5^\alpha}\right] \quad .$$

It follows that

$$\sum_{\substack{n\equiv\delta_\alpha k \\ \text{mod } 5^\alpha}} p_{-k}(n)q^{(n+(5^\alpha-\delta_\alpha)k)/5^\alpha} = \sum_{n\geq 0} p_{-k}\left(5^\alpha n+k\delta_\alpha - 5^\alpha\left[\frac{k\delta_\alpha}{5^\alpha}\right]\right) q^{n+k-\left[\frac{k\delta_\alpha}{5^\alpha}\right]} ,$$

so that (4.1.4) reduces to

(4.2.2) $$\sum_{n\geq 0} p_{-k}\left(5^\alpha n+k\delta_\alpha - 5^\alpha\left[\frac{k\delta_\alpha}{5^\alpha}\right]\right) q^n = \begin{cases} \sum_{i\geq 1} x_{\alpha,i} q^{i-k+\left[\frac{k\delta_\alpha}{5^\alpha}\right]} \dfrac{E(q^5)^{6i-k}}{E(q)^{6i}}, & \alpha \text{ odd}, \\ \sum_{i\geq 1} x_{\alpha,i} q^{i-k+\left[\frac{k\delta_\alpha}{5^\alpha}\right]} \dfrac{E(q^5)^{6i}}{E(q)^{6i+k}}, & \alpha \text{ even}, \end{cases}$$

which implies

$$x_{\alpha,i} = 0 \quad \text{for} \quad i < k - \left[\frac{k\delta_\alpha}{5^\alpha}\right]$$

and if $i = k - \left[\frac{k\delta_\alpha}{5^\alpha}\right]$ then $x_{\alpha,i}$ is the first non-zero entry of the vector $\underset{\sim}{x}_\alpha$.

If we now define

(4.2.3) $$d_\alpha = k - \left[\frac{k\delta_\alpha}{5^\alpha}\right] - 1 \quad \text{for} \quad \alpha \geqslant 1 ,$$

then

(4.2.4) $$x_{\alpha,i+d_\alpha} = 0 \quad \text{for} \quad i < 1 ,$$

and the following Theorem is equivalent to Theorem (4.1.3).

Theorem (4.2.5). For $\alpha \geqslant 1$,

$$(4.2.6)\quad \sum_{n\geq 0} p_{-k}\left(5^{\alpha}n+k\delta_{\alpha}-5^{\alpha}\left[\frac{k\delta_{\alpha}}{5^{\alpha}}\right]\right) q^{n} = \begin{cases} \sum_{i\geq 1} x_{\alpha,i+d_{\alpha}} q^{i-1} \dfrac{E(q^{5})^{6(i+d_{\alpha})-k}}{E(q)^{6(i+d_{\alpha})}}, & \alpha \text{ odd}, \\ \sum_{i\geq 1} x_{\alpha,i+d_{\alpha}} q^{i-1} \cdot \dfrac{E(q^{5})^{6(i+d_{\alpha})}}{E(q)^{6(i+d_{\alpha})+k}}, & \alpha \text{ even}, \end{cases}$$

where the $\underset{\sim}{x}_{\alpha}$ are defined by (4.1.5) - (4.1.8) and the d_{α} are defined in (4.2.3).

Now (4.2.1) becomes

$$(4.2.7)\quad \min_{\substack{m:\\ 24m\equiv k \\ \text{mod } 5^{\alpha}}} \nu\left(p_{-k}(m)\right) \geqslant \min_{i\geq 1} \left(\nu(x_{\alpha,i+d_{\alpha}})\right).$$

Before we can calculate a lower bound for the right-hand side of (4.2.7) we need some Lemmata. The following is Lemma (4.1) of Hirschhorn and Hunt's paper.

Lemma (4.2.8).

$$\nu(m_{i,j}) \geqslant [\tfrac{1}{2}(5j-i-1)],$$

where the $m_{i,j}$ are defined by (4.1.7) and (4.1.8).

As an immediate consequence we have the following Lemma.

Lemma (4.2.9)

$$\nu(a_{i,j}) \geqslant [\tfrac{1}{2}(5j-i-1)], \qquad \nu(b_{i,j}) \geqslant [\tfrac{1}{2}(5j-k-i-1)] .$$

Proof.

By (4.1.6) and Lemma (4.2.8),

$$\nu(a_{i,j}) = \nu(m_{6i,i+j}) \geqslant [\tfrac{1}{2}(5(i+j) - 6i - 1)] = [\tfrac{1}{2}(5j-i-1)],$$

$$\nu(b_{i,j}) = \nu(m_{6i+k,i+j}) \geqslant [\tfrac{1}{2}(5(i+j) - (6i+k) - 1)] = [\tfrac{1}{2}(5j-k-i-1)] .$$

The following is Lemma (2.11) of Hirschhorn and Hunt's paper.

Lemma (4.2.10). δ_α, the reciprocal modulo 5^α of 24, satisfies

$$\delta_1 = 4$$

and for $\alpha \geqslant 1$,

$$\delta_{\alpha+1} = \begin{cases} 4 \times 5^\alpha + \delta_\alpha, & \alpha \text{ odd,} \\ 3 \times 5^\alpha + \delta_\alpha, & \alpha \text{ even.} \end{cases}$$

Lemma (4.2.11). For $\alpha \geqslant 1$,

$$d_{\alpha+1} = \begin{cases} [\frac{d_\alpha}{5}] & , \alpha \text{ odd,} \\ [\frac{d_\alpha+k}{5}] & , \alpha \text{ even,} \end{cases}$$

where the d_α are defined by (4.2.3).

Proof.

$$(4.2.12) \qquad d_\alpha = k - \left[\frac{k\delta_\alpha}{5^\alpha}\right] - 1 = k + \left[\frac{-k\delta_\alpha - 1}{5^\alpha}\right] = \left[\frac{(5^\alpha - \delta_\alpha)k - 1}{5^\alpha}\right] .$$

Now suppose α is odd. From Lemma (4.2.10) we have

$$d_{\alpha+1} = \left[\frac{(5^{\alpha+1} - \delta_{\alpha+1})k - 1}{5^{\alpha+1}}\right] = \left[\frac{1}{5}\left(\frac{(5^{\alpha+1} - 4\times 5^\alpha - \delta_\alpha)k - 1}{5^\alpha}\right)\right]$$

$$= \left[\frac{1}{5}\left(\frac{(5^\alpha - \delta_\alpha)k - 1}{5^\alpha}\right)\right] = \left[\frac{d_\alpha}{5}\right] .$$

Now suppose α is even. From Lemma (4.2.10) we have

$$d_{\alpha+1} = \left[\frac{(5^{\alpha+1} - \delta_{\alpha+1})k - 1}{5^{\alpha+1}}\right] = \left[\frac{1}{5}\left(\frac{(5^{\alpha+1} - 3\times 5^\alpha - \delta_\alpha)k - 1}{5^\alpha}\right)\right]$$

$$= \left[\frac{1}{5}\left(\frac{(5^\alpha - \delta_\alpha)k - 1}{5^\alpha} + k\right)\right] = \left[\frac{d_\alpha + k}{5}\right] .$$

We will now investigate $\nu(x_{\alpha,i+d_\alpha})$ for the first few values of α.

From Lemma (4.2.8) we have

$$\nu(x_{1,i+d_1}) = \nu(m_{k,i+d_1}) \geqslant [\frac{5(i+d_1) - k - 1}{2}] = [\frac{5i + 5d_1 - k - 1}{2}] .$$

Suppose $k = 5r + s$, where $0 < s \leqslant 5$, then from (4.2.12) we have

$$d_1 = [\frac{(5-\delta_1)k - 1}{5}] = [\frac{k - 1}{5}] = [\frac{5r + s - 1}{5}] = r ,$$

$$5k_1 - k = 5r - (5r+s) = - s$$

and

$$\nu(x_{1,i+d_1}) = \nu(m_{k,i+d_1}) \geqslant [\frac{5i - s - 1}{2}]$$

so that this lower bound depends only on i and the residue of k modulo 5. Putting $i = 1$ we obtain

$$\nu(x_{1,1+d_1}) \geqslant \max \{0, [\frac{4 - s}{2}] \} .$$

The right-hand side of this inequality is equal to 1, when $k \equiv 1$ or 2 mod 5, and 0, when $k \equiv 3, 4,$ or 5 mod 5. So if we define

(4.2.13) $$\theta(\ell) = \begin{cases} 1 & \text{if } \ell \equiv 1 \text{ or } 2 \bmod 5, \\ 0 & \text{if } \ell \equiv 3, 4 \text{ or } 5 \bmod 5, \end{cases}$$

then we have

(4.2.14) $$\nu(x_{1,1+d_1}) \geqslant \theta(k) ,$$

or

(4.2.15) $$\nu\left(m_{k,[\frac{k+4}{5}]}\right) \geqslant \theta(k) , \quad \text{since} \quad 1 + d_1 = 1 + [\frac{k-1}{5}] = [\frac{k+4}{5}] .$$

Later we will investigate when we have equality in (4.2.15). Suppose $k \equiv s \bmod 5$ and $0 < s \leqslant 5$ then if $s = 1$ or 2 we have

$$\nu(x_{1,i+d_1}) \geq [\frac{5i - s - 1}{2}] = 1 + [\frac{5i - s - 3}{2}] \geq \theta(k) + [\frac{5i - 5}{2}]$$

and if $s = 3, 4$ or 5 we have

$$\nu(x_{1,i+d_1}) \geq [\frac{5i - s - 1}{2}] \geq \theta(k) + [\frac{5i - 6}{2}] \quad .$$

Hence

$$(4.2.16) \qquad \nu(x_{1,i+d_1}) \geq \theta(k) + \max\{0, [\frac{5i - 6}{2}]\} \quad .$$

Now from (4.1.5) and (4.2.4) we have

$$x_{2,j+d_2} = \sum_{i\geq 1} x_{1,i}\, a_{i,j+d_2} = \sum_{i\geq 1} x_{1,i+d_1}\, a_{i+d_1,j+d_2}$$

so

$$\nu(x_{2,j+d_2}) \geq \min_{i\geq 1} \{ \nu(x_{1,i+d_1}) + \nu(a_{i+d_1,j+d_2})\}$$

$$\geq \min_{i\geq 1}\big(\theta(k) + \max\{0, [\frac{5i - 6}{2}]\}$$

$$+ \max\{0, [\tfrac{1}{2}(5j - i + 5d_2 - d_1 - 1)]\}$$

The minimum is attained at $i = 1$ (since increasing i by 1 increases the second term by at least 2, and decreases the last term by at most 1), so that

$$\nu(x_{2,j+d_2}) \geq \theta(k) + \max\{0, [\tfrac{1}{2}(5j + 5d_2 - d_1 - 2)]\}$$

Suppose $d_1 = 5r + s$ where $0 \leq s \leq 4$ then from Lemma (4.2.11) it follows that

$$5d_2 - d_1 = 5[\frac{d_1}{5}] - d_1 = 5r - (5r+s) = -s$$

and

$$\nu(x_{2,j+d_2}) \geq \theta(k) + \max\{0, [\tfrac{1}{2}(5j - s - 2)]\}$$

so that this lower bound depends only on i and the residue of d_1 modulo 5. Putting $i = 1$ we obtain

$$\nu(x_{2,1+d_2}) \geq \max\{0, [\frac{3-s}{2}]\} \quad .$$

The right-hand side of this inequality is equal to 1, when $s = 0$ or 1, and 0, when $s = 2, 3,$ or 4 so that

$$\nu(x_{2,1+d_2}) \geqslant \theta(k) + \theta(d_1+1)$$

Suppose $d_1 \equiv s \mod 5$ and $0 \leqslant s \leqslant 4$ then if $s = 0$, or 1 we have

$$\nu(x_{2,i+d_2}) \geqslant \theta(k) + [\tfrac{1}{2}(5i-s-2)] \geqslant \theta(k) + 1 + [\tfrac{1}{2}(5i-s-4)]$$

$$\geqslant \theta(k) + \theta(d_1+1) + [\tfrac{1}{2}(5i-5)] \ ,$$

and if $s = 2, 3,$ or 4 we have

$$\nu(x_{2,i+d_2}) \geqslant \theta(k) + [\tfrac{1}{2}(5i-s-2)] \geqslant \theta(k) + \theta(d_1+1) + [\tfrac{1}{2}(5i-6)] \ .$$

Hence

(4.2.17) $$\nu(x_{2,i+d_1}) \geqslant \theta(k) + \theta(d_1+1) + \max\{0,[\tfrac{1}{2}(5i-6)]\} \ .$$

If we now define A_α as follows

$$A_1 = \theta(k) \quad \text{and for} \quad \alpha \geqslant 1 \ ,$$

(4.2.18) $$A_{\alpha+1} = \begin{cases} A_\alpha + \theta(d_\alpha+1) \ , & \alpha \ \text{odd,} \\ A_\alpha + \theta(d_\alpha+k+1) \ , & \alpha \ \text{even,} \end{cases}$$

then we have just verified the first two cases of Lemma (4.2.20), stated below. Before we can prove this for general α we need one more Lemma.

<u>Lemma (4.2.19).</u> For $\alpha \geqslant 1$, d_α satisfies the following inequalities

$$5d_{\alpha+1} - d_\alpha \geqslant \begin{cases} 2\,\theta(d_\alpha+1) - 4, & \alpha \ \text{odd,} \\ 2\,\theta(d_\alpha+k+1) + k - 4 \ , & \alpha \ \text{even} \end{cases}$$

where θ is defined by (4.2.13) and the d_α are defined by (4.2.3).

Proof.

Suppose α is odd and $d_\alpha = 5r + s$ where $0 \leqslant s \leqslant 4$ then from Lemma (4.2.11) we have

$$5d_{\alpha+1} - d_\alpha - 2\theta(d_\alpha+1) = 5[\frac{5r+s}{5}] - (5r+s) - 2\theta(5r+s+1)$$

$$= -s - 2\theta(s+1)$$

$$= \begin{cases} -s-2, & \text{if } s = 0 \text{ or } 1 \\ -s, & \text{if } s = 2, 3, \text{ or } 4 \end{cases}$$

$$\geqslant -4, \text{ as required.}$$

Now suppose α is even and $d_\alpha + k = 5r + s$ where $0 \leqslant s \leqslant 4$ then from Lemma (4.2.11) we have

$$5d_{\alpha+1} - (d_\alpha+k) - 2\theta(d_\alpha+k+1) = 5[\frac{5r+s}{5}] - (5r+s) - 2\theta(5r+s+1)$$

$$= -s - 2\theta(s+1)$$

$$\geqslant -4, \text{ as required.}$$

Lemma (4.2.20). For $\alpha, i \geqslant 1$,

(4.2.21)
$$\nu(x_{\alpha,i+d_\alpha}) \geqslant A_\alpha + \max\{0, [\frac{5i-6}{2}]\},$$

where the A_α are defined by (4.2.18).

Proof. From (4.2.16) we have that (4.2.21) is true for $\alpha = 1$. We now proceed by induction on α. Suppose α is odd, and

$$\nu(x_{\alpha,i}) \geqslant A_\alpha + \max\{0, [\frac{5i-6}{2}]\}.$$

We have

$$x_{\alpha+1,j+d_{\alpha+1}} = \sum_{i\geq 1} x_{\alpha,i}\, a_{i,j+d_{\alpha+1}} = \sum_{i\geq 1} x_{\alpha,i+d_\alpha}\, a_{i+d_\alpha,j+d_{\alpha+1}},$$

so from Lemma (4.2.19) it follows that

$$\nu(x_{\alpha+1,j+d_{\alpha+1}}) \geq \min_{i\geq 1} \{\nu(x_{\alpha,i+d_\alpha}) + \nu(a_{i+d_\alpha,j+d_{\alpha+1}})\}$$

$$\geq \min_{i\geq 1} \{A_\alpha + \max\{0,[\tfrac{5i-6}{2}]\} + [\tfrac{1}{2}(5(j+d_{\alpha+1}) - (i+d_\alpha)-1)]\}$$

$$= \min_{i\geq 1} \{A_\alpha + \max\{0,[\tfrac{5i-6}{2}]\} + [\tfrac{1}{2}(5j-i+5d_{\alpha+1}-d_\alpha-1)]\}$$

$$= A_\alpha + [\tfrac{1}{2}(5j + 5d_{\alpha+1} - d_\alpha - 2)]$$

$$\geq A_\alpha + [\tfrac{1}{2}(5j + 2\theta(d_\alpha+1) - 6)]$$

$$= A_\alpha + \theta(d_\alpha+1) + [\tfrac{5j-6}{2}]$$

$$= A_{\alpha+1} + [\tfrac{5j-6}{2}] .$$

Further,

$$\nu(x_{\alpha+1,1+d_{\alpha+1}}) \geq \min\{\{A_\alpha + \nu(a_{1+d_\alpha,1+d_{\alpha+1}})\}, \min_{i\geq 2}\{A_\alpha + [\tfrac{5i-6}{2}] + [\tfrac{1}{2}(4-i+5d_{\alpha+1}-d_\alpha)]\}\}$$

or

$$\text{(4.2.22)}\quad \nu(x_{\alpha+1,1+d_{\alpha+1}}) \geq \min\{\{A_\alpha + \nu(m_{6(d_\alpha+1),\, d_\alpha+d_{\alpha+1}+2})\}, \min_{i\geq 2}\{A_\alpha+[\tfrac{5i-6}{2}] + [\tfrac{1}{2}(4-i+5d_{\alpha+1}-d_\alpha)]\}\} .$$

By Lemma (4.2.11),

$$[\frac{6(d_\alpha+1)+4}{5}] = [\frac{5d_\alpha + d_\alpha + 10}{5}] = d_\alpha + [\frac{d_\alpha}{5}] + 2 = d_\alpha + d_{\alpha+1} + 2 ,$$

So from (4.2.15) it follows that the first term in (4.2.22) is greater than or equal to $A_\alpha + \theta(6(d_\alpha+1)) = A_\alpha + \theta(d_\alpha+1)$.

The minimum of the latter term is attained when $i = 2$ so from Lemma (4.2.19) we have

$$\nu(x_{\alpha+1,1+d_{\alpha+1}}) \geqslant \min\{A_\alpha + \theta(d_\alpha+1),\ A_\alpha + 2 + [\tfrac{1}{2}(2 + 5d_{\alpha+1} - d_\alpha)]\}$$

$$\geqslant \min\{A_\alpha + \theta(d_\alpha+1),\ A_\alpha + \theta(d_\alpha+1) + 1\}$$

$$= A_\alpha + \theta(d_\alpha+1) = A_{\alpha+1} .$$

Hence,

$$\nu(x_{\alpha+1,j+d_{\alpha+1}}) \geqslant A_{\alpha+1} + \max\{0, [\frac{5j-6}{2}]\} .$$

Now suppose α is even, and

$$\nu(x_{\alpha,i+d_\alpha}) \geqslant A_\alpha + \max\{0, [\frac{5i-6}{2}]\} .$$

We have

$$x_{\alpha+1,j+d_{\alpha+1}} = \sum_{i\geq 1} x_{\alpha,i}\, b_{i,j+d_{\alpha+1}} = \sum_{i\geq 1} x_{\alpha,i+d_\alpha}\, b_{i+d_\alpha,j+d_{\alpha+1}} ,$$

so from Lemma (4.2.19) it follows that

$$\nu(x_{\alpha+1,j+d_{\alpha+1}}) \geqslant \min_{i\geq 1} \{\nu(x_{\alpha,i+d_\alpha}) + \nu(b_{i+d_\alpha,j+d_{\alpha+1}})\}$$

$$\geqslant \min_{i\geq 1} \{A_\alpha + \max\{0, [\frac{5i-6}{2}]\} + [\tfrac{1}{2}\, 5(j+d_{\alpha+1}) - (i+d_\alpha)-k-1)]\}$$

$$= \min_{i\geq 1} \{A_\alpha + \max\{0,[\frac{5i-6}{2}]\} + [\tfrac{1}{2}(5j - i + 5d_{\alpha+1}-d_\alpha-k-1)]\}$$

$$= A_\alpha + [\tfrac{1}{2}(5j + 5d_{\alpha+1} - d_\alpha - k - 2)]$$

$$\geqslant A_\alpha + [\tfrac{1}{2}(5j + 2\theta(d_\alpha+k+1) - 6)]$$

$$= A_\alpha + \theta(d_\alpha+k+1) + [\frac{5j-6}{2}]$$

$$= A_{\alpha+1} + [\frac{5j-6}{2}] .$$

Further,

$$\nu(x_{\alpha+1,1+d_{\alpha+1}}) \geqslant \min\{\{A_\alpha + \nu(b_{1+d_\alpha,1+d_{\alpha+1}})\}, \min_{i\geq 2} \{A_\alpha + [\frac{5i-6}{2}] + [\tfrac{1}{2}(4 - i + 5d_{\alpha+1} - d_\alpha - k)]\}\}$$

or

$$(4.2.23) \qquad \nu\left(x_{\alpha+1,1+d_{\alpha+1}}\right) \geq \min\left\{\{ A_\alpha + \nu\left(m_{6(1+d_\alpha)+k,d_\alpha+d_{\alpha+1}+2}\right)\} \right.,$$

$$\left. \min_{i\geq 2} \{A_\alpha + [\tfrac{5i-6}{2}] + [\tfrac{1}{2}(4 - i + 5d_{\alpha+1}-d_\alpha - k)]\}\right\} .$$

By Lemma (4.2.11),

$$\left[\frac{6(1+d_\alpha) + k + 4}{5}\right] = 2 + \left[\frac{5d_\alpha + d_\alpha + k}{5}\right] = 2 + d_\alpha + \left[\frac{d_\alpha+k}{5}\right] = d_\alpha + d_{\alpha+1} + 2,$$

so from (4.2.15) it follows that the first term in (4.2.23) is greater than or equal to $A_\alpha + \theta(6(1+d_\alpha) + k) = A_\alpha + \theta(d_\alpha+k+1)$. The minimum of the latter term is attained when $i = 2$ so from Lemma (4.2.19) we have

$$\nu\left(x_{\alpha+1,1+d_{\alpha+1}}\right) \geq \min\left\{A_\alpha + \theta(d_\alpha+k+1),\ A_\alpha + 2 + [\tfrac{1}{2}(2+5d_{\alpha+1} - d_\alpha - k)]\right\}$$

$$\geq \min\left\{A_\alpha + \theta(d_\alpha+k+1),\ A_\alpha + \theta(d_\alpha + k+1) + 1\right\}$$

$$= A_\alpha + \theta(d_\alpha + k + 1) = A_{\alpha+1} .$$

Hence,

$$\nu\left(x_{\alpha+1,j+d_{\alpha+1}}\right) \geq A_{\alpha+1} + \max\left\{0,\ [\tfrac{5j-6}{2}]\right\} .$$

This completes the proof of Lemma (4.2.20).

4.3. By (4.2.7) and Lemma (4.2.20) we have

(4.3.1) If $24\,m \equiv k \mod 5^{\alpha}$ then $p_{-k}(m) \equiv 0 \mod 5^{A_\alpha}$.

We are interested when this is best possible. Under certain conditions we will show that

$$\nu\left(x_{\alpha,1+d_\alpha}\right) = A_\alpha$$

and since

$$\nu\left(x_{\alpha,i+d_\alpha}\right) > A_\alpha \quad \text{for} \quad i \geqslant 2\ ,$$

this implies that (4.3.1) is best possible. To do this we will calculate the exact power of 5 that divides the first non-zero element of each row in the matrix M. It can be shown that the first non-zero element in the i-th row is $m_{i,[\frac{i+4}{5}]}$.

The following Lemma is an improvement on (4.2.15).

Lemma (4.3.2). If $j = [\frac{i+4}{5}]$ then

$$\nu\left(m_{i,j}\right) > \theta(i) \quad \text{if} \quad i \equiv 11 \text{ or } 17 \mod 25,$$

$$\nu\left(m_{i,j}\right) = \theta(i) \quad \text{otherwise.}$$

Proof.

For $i \geqslant 1$, define $M_i = m_{i,[\frac{i+4}{5}]}$.

We will now show that,

(4.3.3) $$M_{5n-4} = \frac{5}{24}\left(125n^4 + 50n^3 - 305n^2 + 178n - 24\right),$$

(4.3.4) $$M_{5n-3} = \frac{5}{6}\left(25n^3 - 19n + 6\right),$$

(4.3.5) $$M_{5n-2} = \frac{1}{2}\left(25n^2 - 5n - 2\right),$$

(4.3.6) $$M_{5n-1} = 5n - 1$$

and

(4.3.7) $$M_{5n} = 1\ .$$

From (4.1.8) it can be easily shown that

(4.3.8) $M_{5n+4} = 25M_{5n} + 25M_{5n-1} + 15M_{5n-2} + 5M_{5n-3} + M_{5n-4}$,

(4.3.9) $M_{5n+3} = 25M_{5n} + 15M_{5n-1} + 5M_{5n-2} + M_{5n-3}$,

(4.3.10) $M_{5n+2} = 15M_{5n} + 5M_{5n-1} + M_{5n-2}$,

(4.3.11) $M_{5n+1} = 5M_{5n} + M_{5n-1}$,

(4.3.12) $M_{5n+5} = M_{5n}$.

It is easily verified that (4.3.3) - (4.3.7) hold for $n = 1$.

We now proceed by induction on n. Suppose (4.3.3) - (4.3.7) hold for n. Then from (4.3.8) - (4.3.12) we have

$$M_{5n+5} = M_{5n} = 1,$$

$$M_{5n+4} = 5M_{5n} + M_{5n-1} = 5 + 5n - 1 = 5(n+1) - 1,$$

$$\begin{aligned} M_{5n+3} &= 15M_{5n} + 5M_{5n-1} + M_{5n-2} \\ &= 15 + 5(5n-1) + \tfrac{1}{2}(25n^2 - 5n - 2) \\ &= \frac{1}{2}(25n^2 + 45n + 18) \\ &= \frac{1}{2}(25(n+1)^2 - 5(n+1) - 2), \end{aligned}$$

$$\begin{aligned} M_{5n+2} &= 25M_{5n} + 15M_{5n-1} + 5M_{5n-2} + M_{5n-3} \\ &= 25 + 15(5n-1) + \frac{5}{2}(25n^2 - 5n-2) + \frac{1}{6}(125n^3 - 95n + 30) \\ &= \frac{1}{6}(125n^3 + 375n^2 + 280n + 60) \\ &= \frac{5}{6}(25(n+1)^3 - 19(n+1) + 6), \end{aligned}$$

$$\begin{aligned} M_{5n+1} &= 25M_{5n} + 25M_{5n-1} + 15M_{5n-2} + 5M_{5n-3} + M_{5n-4} \\ &= 25 + 25(5n-1) + \frac{15}{2}(25n^2-5n-2) + \frac{5}{6}(125n^3 - 95n + 30) \\ &\quad + \frac{1}{24}(625n^4 + 250n^3 - 1525n^2 + 890n - 120) \\ &= \frac{1}{24}(625n^4 + 2750n^3 + 2975n^2 + 1090n + 120) \\ &= \frac{5}{24}(125(n+1)^4 + 50(n+1)^3 - 305(n+1)^2 + 178(n+1) - 24). \end{aligned}$$

Hence (4.3.3) - (4.3.7) are true for all n.

Now suppose $j = [\frac{i+4}{5}]$. If $i \equiv 3, 4$ or $5 \bmod 5$ then from (4.3.5), (4.3.6) and (4.3.7) we have

$$\nu(m_{i,j}) = \theta(i) = 0.$$

From (4.3.3) we have

$$M_{5n-4} \equiv 5(2n-1) \bmod 25,$$

or

$$\nu(m_{i,j}) > \theta(i) = 1 \quad \text{if} \quad i \equiv 11 \bmod 25$$

and

$$\nu(m_{i,j}) = \theta(i) = 1 \quad \text{if} \quad i \equiv 1 \bmod 5 \quad \text{and } i \not\equiv 11 \bmod 25.$$

From (4.3.4) we have

$$M_{5n-3} \equiv 5(n+1) \bmod 25,$$

or

$$\nu(m_{i,j}) > \theta(i) = 1 \quad \text{if} \quad i \equiv 17 \bmod 25$$

and

$$\nu(m_{i,j}) = \theta(i) = 1 \quad \text{if} \quad i \equiv 2 \bmod 5 \quad \text{and} \quad i \not\equiv 17 \bmod 25.$$

Lemma (4.3.13). If

(4.3.14) none of k, $6d_\alpha + 6$ (α odd), $6d_\alpha + k + 6$ (α even) are congruent to 11 or 17 mod 25

then for $\alpha \geq 1$,

$$\nu(x_{\alpha,1+d_\alpha}) = A_\alpha .$$

Proof. In Lemma (4.2.20) we showed that

$$\nu(x_{\alpha,1+d_\alpha}) \geqslant A_\alpha .$$

The proof of Lemma (4.3.13) follows analogously except if condition (4.3.14) is satisfied we have equality instead of inequality at all stages of the argument. We will now suppose that this condition is satisfied.

The case $\alpha = 1$ follows immediately from Lemma (4.3.2) since

$$\nu(x_{1,1+d_1}) = \nu\left(m_{k,[\frac{k+4}{5}]}\right) .$$

Suppose α is odd, and the statement is true for α. Then by Lemma (4.3.2) the first term on the right-hand side of (4.2.22) is equal to $A_\alpha + \theta(6d_\alpha+6) = A_\alpha + \theta(d_\alpha+1) = A_{\alpha+1}$.

We also have

$$\nu\left(x_{\alpha,i+d_\alpha} a_{i+d_\alpha,1+d_{\alpha+1}}\right) \geqslant A_\alpha + \theta(d_\alpha+1) + 1 = A_{\alpha+1} + 1 \quad \text{for} \quad i \geqslant 2,$$

but

$$\nu\left(x_{\alpha,1+d_\alpha} a_{1+d_\alpha,1+d_{\alpha+1}}\right) = A_{\alpha+1} ,$$

so that

$$\nu\left(x_{\alpha+1,1+d_{\alpha+1}}\right) = A_{\alpha+1} .$$

Now suppose α is even, and the statement is true for α. Then by Lemma (4.3.2) the first term on the right-hand side of (4.2.23) is equal to $A_\alpha + \theta\left(6(1+d_\alpha) + k\right) = A_\alpha + \theta\left(d_\alpha + k + 1\right) = A_{\alpha+1}$. We also have

$$\nu\left(x_{\alpha,i+d_\alpha} b_{i+d_\alpha,1+d_{\alpha+1}}\right) \geqslant A_\alpha + \theta(d_\alpha+k+1) + 1 \quad \text{for} \quad i \geqslant 2 ,$$

but

$$\nu\left(x_{\alpha,1+d_\alpha} b_{1+d_\alpha,1+d_{\alpha+1}}\right) = A_{\alpha+1} ,$$

so that

$$\nu\left(x_{\alpha+1,1+d_{\alpha+1}}\right) = A_{\alpha+1} .$$

Lemma (4.3.13) follows by induction on α.

From the remarks at the beginning of this section and the Lemma above we have the following Theorem.

<u>Theorem (4.3.15)</u>. If $\alpha, k > 0$ and $24m \equiv k \mod 5^\alpha$ then

$$P_{-k}(m) \equiv 0 \mod 5^{A_\alpha} ,$$

where the A_α are defined by (4.2.18). Further, this congruence is best possible if none of k, $6d_\alpha + 6$ (α odd), $6d_\alpha + k + 6$ (α even) are congruent to 11 or 17 mod 25, where the d_α are defined by (4.2.3).

<u>4.4.</u> We are now in a position to prove Theorem (4.1.1) for $p = 5$. If $24m \equiv k \bmod 5^\alpha$ then from the previous Theorem we already have

$$p_{-k}(m) \equiv 0 \mod 5^{A_\alpha} .$$

In this section we will show that

$$A_\alpha = B\frac{\alpha}{2} + \varepsilon , \tag{4.4.1}$$

where $B = B(k)$ depends only on the residue of k modulo 24 and $\varepsilon = \varepsilon(k) = 0(\log k)$.

If α is odd then $\delta_\alpha = \frac{1}{24}(19 \times 5^\alpha + 1)$ so that (4.2.12) becomes

$$d_\alpha = \left[\frac{(5^\alpha - \delta_\alpha)k - 1}{5^\alpha}\right] = \left[\frac{k(5^{\alpha+1})-1)}{24 \times 5^\alpha} - \frac{1}{5^\alpha}\right] \tag{4.4.2}$$

or

$$d_\alpha = \left[\frac{5k - (k+24)/5^\alpha}{24}\right] = \left[\frac{5k-1}{24}\right] = K, \quad \text{say}, \tag{4.4.3}$$

provided $\frac{k+24}{5^\alpha} \leqslant 1$ or $\alpha \geqslant \log_5 (k+24) = \frac{\ln(k+24)}{\ln 5}$.

If α is even then $\delta_\alpha = \frac{1}{24}(23 \times 5^\alpha + 1)$ so that (4.2.12) becomes

$$d_\alpha = \left[\frac{(5^\alpha - \delta_\alpha)k - 1}{5^\alpha}\right] = \left[\frac{k(5^\alpha - 1)}{24 \times 5^\alpha} - \frac{1}{5^\alpha}\right] \tag{4.4.4}$$

or

$$d_\alpha = \left[\frac{k - (k+24)/5^\alpha}{24}\right] = \left[\frac{k-1}{24}\right] = \left[\frac{K}{5}\right] = L, \quad \text{say}, \tag{4.4.5}$$

provided $\alpha \geqslant \frac{\ln(k+24)}{\ln 5}$.

Before proceeding we note some errors in section 2.5 of Atkin's paper. Atkin's k_n and ℓ_n are related to d_α by

$$k_n = d_{2n} \quad \text{and} \quad \ell_n = d_{2n-1}$$

so from (4.4.3) and (4.4.5) Atkin's equations should read

$$k_n = \left[\frac{5k - (k+24)/5^{2n-1}}{24}\right] = \left[\frac{5k - 1}{24}\right] = K ,$$

provided $2n - 1 \geqslant \dfrac{\ln(k+24)}{\ln 5}$,

and

$$\ell_n = \left[\frac{k - (k+24)/5^{2n}}{24}\right] = \left[\frac{k - 1}{24}\right] = \left[\frac{K}{5}\right] = L ,$$

provided $2n \geqslant \dfrac{\ln(k+24)}{\ln 5}$.

We now define

$$B = B(k) = \theta(K+1) + \theta(L + k + 1).$$

Since increasing k by 24 increases $K + 1$ by 5 and $L + k + 1$ by 25, the values of $\theta(K + 1)$ and $\theta(L + k + 1)$ remain the same, so that B depends only on the residue of k modulo 24; examination of cases gives the list for $p = 5$ in Theorem (4.1.1).

In our notation Atkin deduces that

$$A_\alpha \geqslant B \frac{\alpha}{2} + O(\log k),$$

but it is as easy to show that

$$A_\alpha = B \frac{\alpha}{2} + O(\log k),$$

which is (4.4.1). Let α_0 be the smallest integer that satisfies

$$\alpha_0 \geqslant \frac{\ln(k + 24)}{\ln 5} .$$

Let,

$$r_0 = \begin{cases} \dfrac{\alpha_0 + 1}{2}, & \alpha_0 \text{ odd,} \\ \\ \dfrac{\alpha_0 + 2}{2}, & \alpha_0 \text{ even .} \end{cases}$$

In either case from (4.4.3) and (4.4.5) it follows that

$$d_{2r-1} = K \quad \text{and} \quad d_{2r} = L, \quad \text{provided} \quad r \geqslant r_0 .$$

Now from (4.2.18) it is easy to see that for $\beta \geqslant r_0$,

$$A_{2\beta-1} = \theta(k) + \sum_{r=1}^{\beta-1} \theta(d_{2r-1} + 1) + \theta(d_{2r} + k + 1)$$

$$= \theta(k) + \sum_{r=1}^{r_{0-1}} \theta(d_{2r-1}+1) + \theta(d_{2r}+k+1) + \sum_{r=r_0}^{\beta-1} \theta(K+1) + \theta(L+k-1)$$

$$= B(\beta - r_0) + \theta(k) + \sum_{r=1}^{r_0-1} \theta(d_{2r-1}+1) + \theta(d_{2r}+k+1) ,$$

so that,

$$0 \leqslant A_{2\beta-1} - B(\beta - r_0) \leqslant 2r_0 - 1 ,$$

or

(4.4.6) $$|A_{2\beta-1} - B\beta| \leqslant 2r_0 - 1 + Br_0 .$$

Since $A_{2\beta} = A_{2\beta-1} + \theta(d_{2\beta-1} + 1)$, we also have

(4.4.7) $$|A_{2\beta} - B\beta| \leqslant 2r_0 + Br_0 .$$

Combining (4.4.6) and (4.4.7) we have

$$A_\alpha = B[\frac{\alpha+1}{2}] + \varepsilon,$$

where

$$|\varepsilon(k)| \leqslant (B(k)+2)r_0 \leqslant \tfrac{1}{2}(B(k)+2)(\alpha_0+2) \leqslant \tfrac{1}{2}\{\max_{1\le k\le 24} B(k)+2\}([\ell n(k+24)/\ell n5]+3) ,$$

which implies (4.4.1). This completes the proof of Theorem (4.1.1) for $p = 5$.

4.5. We now return to further discussion on the best possible nature of Theorem (4.1.1). Let $d = d(k, \alpha)$ be the largest power of 5 which divides $p_{-k}(m)$ for all m with $24m \equiv 1 \mod 5^{\alpha}$. From Theorem (4.3.15) we have

$$d \geqslant B \frac{\alpha}{2} + O(\log k),$$

and if

$$(4.5.1) \quad \left\{ \begin{array}{l} \text{none of } k,\ 6d_{\alpha} + 6\ (\alpha \text{ odd}),\ 6d_{\alpha} + k + 6\ (\alpha \text{ even}) \\ \text{are congruent to } 11 \text{ or } 17 \mod 25, \end{array} \right.$$

$$d = A_{\alpha} = B \frac{\alpha}{2} + O(\log k).$$

Atkin has found that for each residue class modulo 24 there exists a k satisfying (4.5.1). If we define

$$(4.5.2) \qquad e_{\alpha} = \begin{cases} 6d_{\alpha} + 6, & \alpha \text{ odd}, \\ 6d_{\alpha} + k + 6, & \alpha \text{ even}, \end{cases}$$

then we require $k, e_{\alpha} \not\equiv 11$ and $17 \mod 25$ for $\alpha \geqslant 1$.

The smallest such k in each residue class mod 24 are

$k = 1, 2, 3, 4, 53, 6, 7, 8, 9, 10, 59, 12, 13, 14, 15, 16, 41,$
$18, 19, 20, 21, 22, 23, 24.$

We have the following table:

(4.5.3)

k	d_α					e_α (mod 25)					A_α					
	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=6$
1	0	0	0	0	0	6	7	6	7	6	1	2	3	4	5	6
2	0	0	0	0	0	6	8	6	8	6	1	2	2	3	3	4
3	0	0	0	0	0	6	9	6	9	6	0	1	1	2	2	3
4	0	0	0	0	0	6	10	6	10	6	0	1	1	2	2	3
5	0	0	1	0	1	6	11	12	11	12	0	1	2	3	4	5
6	1	0	1	0	1	12	12	12	12	12	1	2	3	4	5	6
7	1	0	1	0	1	12	13	12	13	12	1	2	2	3	3	4
8	1	0	1	0	1	12	14	12	14	12	0	1	1	2	2	3
9	1	0	1	0	1	12	15	12	15	12	0	1	1	2	2	3
10	1	0	2	0	2	12	16	18	16	18	0	1	2	2	3	3
11	2	0	2	0	2	18	17	18	17	18	1	1	2	2	3	3
12	2	0	2	0	2	18	18	18	18	18	1	1	1	1	1	1
13	2	0	2	0	2	18	19	18	19	18	0	0	0	0	0	0
14	2	0	2	0	2	18	20	18	20	18	0	0	0	0	0	0
15	2	0	3	0	3	18	21	24	21	24	0	0	1	1	2	2
16	3	0	3	0	3	24	22	24	22	24	1	1	2	2	3	3
17	3	0	3	0	3	24	23	24	23	24	1	1	1	1	1	1
18	3	0	3	0	3	24	24	24	24	24	0	0	0	0	0	0
19	3	0	3	0	3	24	0	24	0	24	0	0	0	0	0	0
20	3	0	4	0	4	24	1	5	1	5	0	0	1	1	2	2
21	4	0	4	0	4	5	2	5	2	5	1	1	2	2	3	3
22	4	0	4	0	4	5	3	5	3	5	1	1	1	1	1	1
23	4	0	4	0	4	5	4	5	4	5	0	0	0	0	0	0
24	4	0	4	0	4	5	5	5	5	5	0	0	0	0	0	0
25	4	0	5	1	5	5	6	11	12	11	0	0	1	2	3	4
26	5	1	5	1	5	11	13	11	13	11	1	2	2	3	3	4
27	5	1	5	1	5	11	14	11	14	11	1	2	2	3	3	4
28	5	1	5	1	5	11	15	11	15	11	0	1	1	2	2	3
29	5	1	6	1	6	11	16	17	16	17	0	1	2	3	4	5
30	5	1	6	1	6	11	17	17	17	17	0	1	2	3	4	5
31	6	1	6	1	6	17	18	17	18	17	1	2	2	3	3	4
32	6	1	6	1	6	17	19	17	19	17	1	2	2	3	3	4
33	6	1	6	1	6	17	20	17	20	17	0	1	1	2	2	3
34	6	1	7	1	7	17	21	23	21	23	0	1	2	2	3	3
35	6	1	7	1	7	17	22	23	22	23	0	1	2	2	3	3
36	7	1	7	1	7	23	23	23	23	23	1	1	1	1	1	1
37	7	1	7	1	7	23	24	23	24	23	1	1	1	1	1	1
38	7	1	7	1	7	23	0	23	0	23	0	0	0	0	0	0
39	7	1	8	1	8	23	1	4	1	4	0	0	1	1	2	2
40	7	1	8	1	8	23	2	4	2	4	0	0	1	1	2	2
41	8	1	8	1	8	4	3	4	3	4	1	1	1	1	1	1
42	8	1	8	1	8	4	4	4	4	4	1	1	1	1	1	1
43	8	1	8	1	8	4	5	4	5	4	0	0	0	0	0	0
44	8	1	9	1	9	4	6	10	6	10	0	0	1	1	2	2
45	8	1	9	1	9	4	7	10	7	10	0	0	1	1	2	2
46	9	1	9	1	9	10	8	10	8	10	1	1	1	1	1	1
47	9	1	9	1	9	10	9	10	9	10	1	1	1	1	1	1

cont'd...

(4.5.3) cont'd...

k	d_α $\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	e_α (mod 25) $\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	A_α $\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$	$\alpha=6$
48	9	1	9	1	9	10	10	10	10	10	0	0	0	0	0	0
49	9	1	10	2	10	10	11	16	17	16	0	0	1	2	3	4
50	9	1	10	2	10	10	12	16	18	16	0	0	1	2	2	3
51	10	2	10	2	10	16	19	16	19	16	1	2	2	3	3	4
52	10	2	10	2	10	16	20	16	20	16	1	2	2	3	3	4
53	10	2	11	2	11	16	21	22	21	22	0	1	2	3	4	5
54	10	2	11	2	11	16	22	22	22	22	0	1	2	3	4	5
55	10	2	11	2	11	16	23	22	23	22	0	1	1	2	2	3
56	11	2	11	2	11	22	24	22	24	22	1	2	2	3	3	4
57	11	2	11	2	11	22	0	22	0	22	1	2	2	3	3	4
58	11	2	12	2	12	22	1	3	1	3	0	1	2	2	3	3
59	11	2	12	2	12	22	2	3	2	3	0	1	2	2	3	3
60	11	2	12	2	12	22	3	3	3	3	0	1	1	1	1	1
61	12	2	12	2	12	3	4	3	4	3	1	1	1	1	1	1
62	12	2	12	2	12	3	5	3	5	3	1	1	1	1	1	1
63	12	2	13	2	13	3	6	9	6	9	0	0	1	1	2	2
64	12	2	13	2	13	3	7	9	7	9	0	0	1	1	2	2
65	12	2	13	2	13	3	8	9	8	9	0	0	0	0	0	0
66	13	2	13	2	13	9	9	9	9	9	1	1	1	1	1	1
67	13	2	13	2	13	9	10	9	19	9	1	1	1	1	1	1
68	13	2	14	2	14	9	11	15	11	15	0	0	1	1	2	2
69	13	2	14	2	14	9	12	15	12	15	0	0	1	1	2	2
70	13	2	14	2	14	9	13	15	13	15	0	0	0	0	0	0
71	14	2	14	2	14	15	14	15	14	15	1	1	1	1	1	1
72	14	2	14	2	14	15	15	15	15	15	1	1	1	1	1	1

Now for $k \leqslant 72$ $\quad \dfrac{\log(k+24)}{\log 5} \leqslant \dfrac{\log 96}{\log 5} < 3$.

So from (4.4.3) and (4.4.5) we have for $\alpha \geqslant 3$

$$(4.5.4) \qquad d_\alpha = \begin{cases} \left[\dfrac{5k-1}{24}\right], & \alpha \text{ odd}, \\ \\ \left[\dfrac{k-1}{24}\right], & \alpha \text{ even}, \end{cases}$$

and for condition (4.5.1) to hold we need only check c_α (mod 25) for $1 \leqslant \alpha \leqslant 4$. This gives us the required values for k .

When

$k \equiv 12, 13, 14, 17, 18, 19, 22, 23, 24 \mod 24$ we have $B = 0$ and (4.4.1) is

$$A_\alpha = O(\log k) .$$

This tells us very little about the power of 5 dividing $p_{-k}(m)$ when $24m \equiv k \mod 5^\alpha$. We would rather find a lower bound for the A_α . The smallest k in each residue class mod 24 that satisfy (4.5.1) and $A_\alpha = 0$ for all α , are

$$k = 13, 14, 65, 19, 70, 23, 24.$$

These are the values of k marked by an asterisk in Atkin's paper.

The last line of page 74 of Atkin's paper should read (in our notation)

(4.5.5) for $k \equiv 12 \mod 24$, $A_\alpha \geqslant 1$ <u>for sufficiently large α</u> although $B = 0$,

(see Lemma (4.5.8) below). Note that we don't necessarily have $A_\alpha \geqslant 1$ always. For instance if $k = 60$ then $A_1 = \theta(k) = 0$. We will now verify (4.5.5) for the first five cases of k. For ease of calculation we define

(4.5.6)
$$f_\alpha = \begin{cases} d_\alpha + 1, & \alpha \text{ odd}, \\ d_\alpha + k + 1, & \alpha \text{ even}. \end{cases}$$

Note that $e_\alpha \equiv f_\alpha \bmod 5$. From (4.2.18) it is clear that

(4.5.7)
$$A_\alpha = \theta(k) + \sum_{\beta<\alpha} \theta(f_\beta) = \theta(k) + \sum_{\beta<\alpha} \theta(e_\beta) .$$

Hence (4.5.5) is equivalent to

for $k \equiv 12 \bmod 24$, $f_\beta \equiv 1$ or $2 \bmod 5$ for some β, or $\theta(k) = 1$.

Now suppose $k = 24w + 12$ for some w where $0 \leqslant w \leqslant 4$.

We have the following table

	f_β (mod 5)				
w	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$
0	3	3	3	3	3
1	3	3	3	3	3
2	2	3	3	3	3
3	2	3	3	3	3
4	2	3	3	3	3

For $w = 0,1$ $A_\alpha = \theta(k) = 1$ for $\alpha \geqslant 1$.

For $1 < w \leqslant 4$ $f_1 \equiv 2 \bmod 5$ and $A_\alpha \geqslant \theta(f_1) = 1$ for $\alpha \geqslant 2$.

<u>Lemma (4.5.8).</u>

If $k \equiv 12 \bmod 24$ then $A_\alpha \geqslant 1$ for sufficiently large α, where the A_α are defined by (4.2.18).

<u>Proof</u>. If $k = 12$ we already have $A_\alpha = 1$ for $\alpha \geqslant 1$.

Now suppose $k \equiv 12 \bmod 24$ and $k > 12$, then there exists some positive integer r with $k = 24r + 12$

where $r = w5^{\alpha^*} + v$ for some integers α^*, w, v

that satisfy $\alpha^* \geq 0,\ 1 \leq w \leq 4,\ 0 \leq v < 5^{\alpha^*}$.

We have already shown the statement is true for $\alpha^* = 0$ so we may assume $\alpha^* \geq 1$. Suppose α^* is odd.

From (4.4.2) we have

$$d_{\alpha^*} = \left[\frac{k(5^{\alpha^*+1} - 1)/24 - 1}{5^{\alpha^*}}\right]$$

$$= \left[\frac{(r+\tfrac{1}{2})(5^{\alpha^*+1} - 1) - 1}{5^{\alpha^*}}\right]$$

$$= 5r + \left[\frac{(5^{\alpha^*+1})/2 - r - 1}{5^{\alpha^*}}\right]$$

$$\equiv \left[\frac{12.5^{\alpha^*-1} + (5^{\alpha^*-1}-1)/2 - r - 1}{5^{\alpha^*}}\right] \mod 5,$$

since $5^{\alpha^*+1} - 1 = (24+1)5^{\alpha^*-1} - 1$

or $\dfrac{5^{\alpha^*+1}-1}{2} = 12.5^{\alpha^*-1} + \dfrac{5^{\alpha^*-1}-1}{2}$,

$$\equiv \left[\frac{(2-w)5^{\alpha^*} + 2.5^{\alpha^*-1} + (5^{\alpha^*-1}-1)/2 - v - 1}{5^{\alpha^*}}\right] \mod 5$$

$$\equiv (2 - w) + \left[\frac{(5^{\alpha^*}-1)/2 - v - 1}{5^{\alpha^*}}\right] \mod 5 .$$

Now, if $v \leq \dfrac{5^{\alpha^*} - 1}{2} - 1$ then $0 \leq (5^{\alpha^*}-1)/2 - v - 1 < 5^{\alpha}$

and if $v > \dfrac{5^{\alpha^*}-1}{2} - 1$ then $-5^{\alpha^*} < (5^{\alpha^*}-1)/2 - v - 1 < 0$

so that mod 5 we have

$$f_{\alpha^*} = d_{\alpha^*} + 1 \equiv \begin{cases} 2, & \text{if } w = 1 \text{ and } v \leq (5^{\alpha^*}-1)/2 - 1, \\ 1, & \text{if } w = 1 \text{ and } v > (5^{\alpha^*}-1)/2 - 1, \\ 1, & \text{if } w = 2 \text{ and } v \leq (5^{\alpha^*}-1)/2 - 1. \end{cases}$$

Hence from (4.5.7) we have

$$A_\alpha \geqslant 1 \quad \text{for} \quad \alpha \geqslant \alpha^* + 1,$$

for $w = 2$ and $v \leqslant (5^{\alpha^*} - 1)/2 - 1$ or $w = 1$.

For the other values of w and v we consider d_{α^*+1}.

From (4.4.4) we have

$$\begin{aligned}
d_{\alpha^*+1} + k &= \left[\frac{k(5^{\alpha^*+1} - 1)/24 - 1}{5^{\alpha^*+1}}\right] + k \\
&= \left[\frac{5^{\alpha^*+1}(24r+12) + (r+\tfrac{1}{2})(5^{\alpha^*+1} - 1) - 1}{5^{\alpha^*+1}}\right] \\
&= 25r + \left[\frac{12.5^{\alpha^*+1} + (5^{\alpha^*+1} - 1)/2 - r - 1}{5^{\alpha^*+1}}\right] \\
&\equiv 2 + \left[\frac{12.5^{\alpha^*-1} + (5^{\alpha^*-1} - 1)/2 - r - 1}{5^{\alpha^*+1}}\right] \\
&\equiv 2 + \left[\frac{(2-w)5^{\alpha^*} + 2.5^{\alpha^*-1} + (5^{\alpha^*-1} - 1)/2 - v - 1}{5^{\alpha^*+1}}\right] \mod 5 \\
&\equiv 2 + \left[\frac{(2-w)5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - v - 1}{5^{\alpha^*+1}}\right] \mod 5.
\end{aligned}$$

Now, if $w = 2$ and $v > \frac{5^{\alpha^*} - 1}{2} - 1$ or $w = 3,4$ then

$$-5^{\alpha^*+1} \leqslant (2-w)5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - v - 1 < 0,$$

so that

$$f_{\alpha^*+1} = d_{\alpha^*+1} + k + 1 \equiv 2 \mod 5$$

and

$$A_\alpha \geqslant 1 \quad \text{for} \quad \alpha \geqslant \alpha^* + 2.$$

Hence the statement is true for α^* odd.

Now suppose α^* is even. From (4.4.2) we have

$$d_{\alpha^*+1} = \left[\frac{k(5^{\alpha^*+2} - 1)/24 - 1}{5^{\alpha^*+1}}\right]$$

$$= \left[\frac{(r + \tfrac{1}{2})(5^{\alpha^*+2} - 1) - 1}{5^{\alpha^*+1}}\right]$$

$$= 5r + \left[\frac{(5^{\alpha^*+2} - 1)/2 - r - 1}{5^{\alpha^*+1}}\right]$$

$$\equiv \left[\frac{12.5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - r - 1}{5^{\alpha^*+1}}\right] \mod 5$$

$$\equiv 2 + \left[\frac{(2-w)5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - v - 1}{5^{\alpha^*+1}}\right] \mod 5 .$$

Now, if $w = 2$ and $v > \frac{5^{*} - 1}{2} - 1$ or $w = 3,4$ then

$$- 5^{\alpha^*+1} < (2 - w)5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - v - 1 < 0$$

so that

$$f_{\alpha^*+1} + 1 = d_{\alpha^*+1} + 1 \equiv 2 \mod 5$$

and

$$A_\alpha \geqslant 1 \quad \text{for} \quad \alpha \geqslant \alpha^* + 2 .$$

For the other values of w and v we consider d_{α^*} .

From (4.4.4) we have

$$d_{\alpha^*} + k = \left[\frac{k(5^{\alpha^*} - 1)/24 - 1}{5^{\alpha^*}}\right] + k$$

$$= \left[\frac{5^{\alpha^*}(24r + 12) + (r + \tfrac{1}{2})(5^{\alpha^*} - 1) - 1}{5^{\alpha^*}}\right]$$

$$= 25r + \left[\frac{12.5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - r - 1}{5^{\alpha^*}}\right]$$

$$\equiv \left[\frac{2.5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - r - 1}{5^{\alpha^*}}\right] \mod 5$$

$$\equiv \left[\frac{(2 - w).5^{\alpha^*} + (5^{\alpha^*} - 1)/2 - v - 1}{5^{\alpha^*}}\right] \mod 5$$

$$\equiv (2 - w) + \left[\frac{(5^{\alpha^*} - 1)/2 - v - 1}{5^{\alpha^*}}\right] \mod 5 .$$

As before we find that mod 5 we have

$$f_{\alpha^*} = d_{\alpha^*} + k + 1 \equiv \begin{cases} 2, & \text{if } w = 1 \text{ and } v \leqslant (5^{\alpha^*}-1)/2 - 1, \\ 1, & \text{if } w = 1 \text{ and } v > (5^{\alpha^*}-1)/2 - 1, \\ 1, & \text{if } w = 2 \text{ and } v \leqslant (5^{\alpha^*}-1)/2 - 1 . \end{cases}$$

Hence from (4.5.7) we have

$$A_\alpha \geqslant 1 \quad \text{for} \quad \alpha \geqslant \alpha^* + 1 ,$$

for $w = 2$ and $v \leqslant (5^{\alpha^*} - 1)/2 - 1$ or $w = 1$.

This completes the proof of Lemma (4.5.8).

4.6. We now determine some exact results for $1 \leqslant k \leqslant 24$.

If $24m \equiv k \mod 5^\alpha$ then $p_{-k}(m) \equiv 0 \mod 5^{A_\alpha}$.

This best possible unless $k = 5, 11, 17$.

Now $\dfrac{\log(k + 24)}{\log 5} \leqslant \dfrac{\log 48}{\log 5} < 3$.

So from (4.4.3) and (4.4.5) we have for $\alpha \geqslant 3$

(4.6.1) $$d_\alpha = \begin{cases} \left[\dfrac{5k - 1}{24}\right], & \alpha \text{ odd}, \\ \left[\dfrac{k - 1}{24}\right], & \alpha \text{ even}. \end{cases}$$

In fact from (4.5.3) we observe that (4.6.1) holds for $\alpha \geqslant 2$.

Hence for $\alpha \geqslant 2$ (4.5.7) is

$$A_\alpha = \theta(k) + \theta(e_1) + \sum_{2\le\beta<\alpha} \theta(e_\beta) ,$$

$$A_\alpha = \begin{cases} \theta(k) + \theta(e_1) + \frac{\alpha-2}{2} (\theta(e_2) + \theta(e_3)) , & \alpha \text{ even,} \\ \theta(k) + \theta(e_1) + \frac{\alpha-1}{2}\theta(e_2) + \frac{\alpha-3}{2}\theta(e_3), & \alpha \text{ odd,} \end{cases}$$

or

$$(3.6.2) \qquad A_\alpha = \begin{cases} \theta(k) + \theta(e_1) , & \text{if } \theta(e_2) = \theta(e_3) = 0 , \\ \theta(k) + \theta(e_1) + [\frac{\alpha-2}{2}] , & \text{if } \theta(e_2) = 0, \theta(e_3) = 1 , \\ \theta(k) + \theta(e_1) + [\frac{\alpha-1}{2}] , & \text{if } \theta(e_2) = 1, \theta(e_3) = 0 , \\ \theta(k) + \theta(e_1) + \alpha - 2 , & \text{if } \theta(e_2) = \theta(e_3) = 1 . \end{cases}$$

Hence from (4.5.3) we have the following best possible congruances when $24m \equiv k \mod 5^\alpha$:

For

$k = 1, 6$ $\quad p_{-k}(m) \equiv 0 \mod 5^\alpha$,

$k = 2, 7$ $\quad p_{-k}(m) \equiv 0 \mod 5^{[\alpha/2]+1}$,

$k = 3, 4, 8, 9$ $\quad p_{-k}(m) \equiv 0 \mod 5^{[\alpha/2]}$,

$$k = 10 \qquad p_{-k}(m) \equiv \begin{cases} 0 \mod 5^0 , & \alpha = 1 , \\ 0 \mod 5^{[\frac{\alpha+1}{2}]} , & \alpha > 1 , \end{cases}$$

$k = 16, 21$ $\quad p_{-k}(m) \equiv 0 \mod 5^{[\frac{\alpha+1}{2}]}$,

$k = 12, 22$ $\quad p_{-k}(m) \equiv 0 \mod 5^1$,

$k = 13, 14, 18, 19, 23, 24$ $\quad p_{-k}(m) \equiv 0 \mod 5^0$,

$k = 15, 20$ $\quad p_{-k}(m) \equiv 0 \mod 5^{[\frac{\alpha-1}{2}]}$.

Now for $k = 5$ we find $A_\alpha = \alpha - 1$ and

(4.6.3) $$p_{-5}(m) \equiv 0 \mod 5^{\alpha-1} \quad \text{when} \quad 24m \equiv 5 \mod 5^\alpha$$

but from (4.5.3) we have

$$e_\alpha = 6d_\alpha + k + 6 = 11 \quad \text{for} \quad \alpha \text{ even}$$

so that (4.6.3) is not best possible.

The following result is stated in Atkin's paper without proof.

If $24m \equiv 5 \mod 5^\alpha$ then $p_{-5}(m) \equiv 0 \mod 5^{[(3\alpha-3)/2]}$ and this is best possible.

This follows from the Lemma stated below.

<u>Lemma (4.6.4)</u>. For $k = 5$ and $\alpha \geqslant 1$,

$$\nu(x_{\alpha,1+d_\alpha}) = [\tfrac{3\alpha-3}{2}]$$

and

$$\nu(x_{\alpha,i+d_\alpha}) \geqslant [\tfrac{3\alpha-3}{2}] + [\tfrac{5i-6}{2}] \quad \text{for} \quad i \geqslant 2 .$$

<u>Proof</u>. For $k = 5$ we find that $d_1 = 0$ and for $\alpha \geqslant 2$,

$$d_\alpha = \begin{cases} 0, & \alpha \text{ even,} \\ 1, & \alpha \text{ odd.} \end{cases}$$

From (4.1.7) and (4.2.16) we have

$$\nu(x_{1,1+d_1}) = \nu(x_{1,1}) = \nu(m_{5,1}) = \nu(1) = 0 ,$$

$$\nu(x_{1,i+d_1}) \geqslant [\tfrac{5i-6}{2}] ,$$

so that the statement is true for $\alpha = 1$.

We have

$$x_{2,j+d_2'} = x_{2,j} = \sum_{i\geq j} x_{1,i}\, a_{i,j}$$

so

$$\nu(x_{2,j+d_2}) \geqslant \min_{i\geq 1} \{\nu(x_{1,i}) + \nu(a_{i,j})\}$$

$$\geqslant \min_{i\geq 1} \{ \max\{0, [\tfrac{5i-6}{2}]\} + [\tfrac{1}{2}(5j - i - 1)] \}$$

$$\geqslant [\tfrac{5j-2}{2}] = 1 + [\tfrac{5j-4}{2}] .$$

Now $a_{1,1} = m_{6,2} = 63 \times 5$ so

$$\nu(x_{2,1+d_2}) = \nu(x_{1,1}\, a_{1,1}) = \nu(x_{1,1}) + \nu(a_{1,1}) = 1 ,$$

since

$$\nu(x_{1,i}\, a_{i,1}) \geqslant \max\{0, [\tfrac{5i-6}{2}]\} + [\tfrac{4-i}{2}] \geqslant 3 \quad \text{for} \quad i \geqslant 2 .$$

Hence the statement is true for $\alpha = 2$.

We now proceed by induction on α . Suppose $\alpha \geqslant 2$, α is even and

$$\nu(x_{\alpha,1+d_\alpha}) = [\tfrac{3\alpha-3}{2}] ,$$

$$\nu(x_{\alpha,i+d_\alpha}) \geqslant [\tfrac{3\alpha-3}{2}] + [\tfrac{5i-6}{2}] \quad \text{for} \quad i \geqslant 2 .$$

We have

$$x_{\alpha+1,j+d_{\alpha+1}} = x_{\alpha+1,j+1} = \sum_{i\geq 1} x_{\alpha,i}\, b_{i,j+1}$$

so

$$\nu(x_{\alpha+1,j+d_{\alpha+1}}) \geqslant \min_{i\geq 1} \{\nu(x_{\alpha,i}) + \nu(b_{i,j+1})\}$$

$$\geqslant \min_{i\geq 1} \{[\tfrac{3\alpha-3}{2}] + \max\{0, [\tfrac{5i-6}{2}]\} + [\tfrac{1}{2}(5(j+1)-k-i-1)] \}$$

$$= \min_{i\geq 1} \{[\tfrac{3\alpha-3}{2}] + \max\{0, [\tfrac{5i-6}{2}]\} + [\tfrac{1}{2}(5j - i - 1)]\}$$

$$\geqslant [\tfrac{3\alpha-3}{2}] + [\tfrac{5j-2}{2}] = 2 + [\tfrac{3\alpha-3}{2}] + [\tfrac{5j-6}{2}]$$

$$= [\tfrac{3\alpha-1}{2}] + [\tfrac{5j-6}{2}]$$

$$= [\tfrac{3(\alpha+1) - 3}{2}] + [\tfrac{5j-6}{2}] , \quad \text{since } \alpha \text{ is even.}$$

Now $b_{1,2} = m_{11,3} = 77 \times 5^2$ so

$$\nu(x_{\alpha+1,1+d_{\alpha+1}}) = \nu(x_{\alpha,1}\, b_{1,2}) = \nu(x_{\alpha,1}) + \nu(b_{1,2})$$

$$= [\tfrac{3\alpha-3}{2}] + 2 = [\tfrac{3(\alpha+1) - 3}{2}]$$

since

$$\nu(x_{\alpha,i}\, b_{i,2}) \geqslant [\tfrac{3\alpha-3}{2}] + \max\{0, [\tfrac{5i-6}{2}]\} + [\tfrac{4-i}{2}]$$

$$\geqslant 3 + [\tfrac{3\alpha-3}{2}] = 1 + [\tfrac{3(\alpha+1) - 3}{2}] , \quad \text{for } i \geqslant 2 .$$

Now suppose $\alpha \geqslant 3$, α is odd and

$$\nu(x_{\alpha,1+d_\alpha}) = [\tfrac{3\alpha-3}{2}] ,$$

$$\nu(x_{\alpha,i+d_\alpha}) \geqslant [\tfrac{3\alpha-3}{2}] + [\tfrac{5i-6}{2}] \quad \text{for } i \geqslant 2 .$$

We have

$$x_{\alpha+1,j+d_{\alpha+1}} = x_{\alpha+1,j} = \sum_{i\geq 1} x_{\alpha,i}\, a_{i,j} = \sum_{i\geq 1} x_{\alpha,i+1}\, a_{i+1,j}$$

so

$$\nu(x_{\alpha+1,j+d_{\alpha+1}}) \geqslant \min_{i\geq 1} \{\nu(x_{\alpha,i+1}) + \nu(a_{i+1,j})\}$$

$$\geqslant \min_{i\geq 1} \{[\tfrac{3\alpha-3}{2}] + \max\{0, [\tfrac{5i-6}{2}]\} + [\tfrac{1}{2}(5j - (i+1) - 1)]\}$$

$$\geqslant [\tfrac{3\alpha-3}{2}] + [\tfrac{5j-3}{2}] = 1 + [\tfrac{3\alpha-3}{2}] + [\tfrac{5j-5}{2}]$$

$$\geqslant [\frac{3\alpha-1}{2}] + \max\{0, [\frac{5j-6}{2}]\}$$

$$= [\frac{3(\alpha+1) - 3}{2}] + \max\{0, [\frac{5j-6}{2}]\} \quad , \quad \text{since } \alpha \text{ is odd.}$$

Now $a_{2,1} = m_{12,3} = 104 \times 5$ so

$$\nu(x_{\alpha+1,1+d_{\alpha+1}}) = \nu(x_{\alpha,2}\, a_{2,1}) = \nu(x_{\alpha,2}) + \nu(a_{2,1})$$

$$= 1 + [\frac{3\alpha-3}{2}] = [\frac{3(\alpha+1) - 3}{2}] \quad ,$$

since

$$\nu(x_{\alpha,i+1}\, a_{i+1,1}) \geqslant [\frac{3\alpha-3}{2}] + \max\{0, [\frac{5i-6}{2}]\} + [\frac{4-i}{2}]$$

$$\geqslant 3 + [\frac{3\alpha-3}{2}] = 2 + [\frac{3(\alpha+1) - 3}{2}] \quad \text{for } i \geqslant 2 .$$

This completes the proof of Lemma (4.6.4).

Similarly for $k = 11$ we can obtain for $\alpha \geqslant 1$,

$$\nu(x_{\alpha,1+d_\alpha}) = 2[\frac{\alpha+1}{2}]$$

and

$$\nu(x_{\alpha,i+d_\alpha}) \geqslant 2[\frac{\alpha+1}{2}] + [\frac{5i-6}{2}] \quad \text{for } i \geqslant 2 ,$$

so that

if $24m \equiv 11 \bmod 5^\alpha$ then $p_{-11}(m) \equiv 0 \bmod 5^{2[(\alpha+1)/2]}$

and this is best possible.

For $k = 17$ we easily find that

if $24m \equiv 17 \bmod 5^\alpha$ then $p_{-17}(m) \equiv 0 \bmod 5^2$

and this is best possible.