CHAPTER V

AN ELEMENTARY PROOF OF $p(11n+6) \equiv 0$ MOD 11 AND SOME FURTHER IDENTITIES

5.1. INTRODUCTION.

Winquist [17] and Atkin and Swinnerton-Dyer [4] have given elementary proofs of

(5.1.1) $$p(11n + 6) \equiv 0 \mod 11.$$

In this chapter we show how the methods of Chapter 1 can be extended to derive (5.1.1). Our proof depends on an identity in Atkin and Swinnerton-Dyer's paper, which we prove using no more than Jacobi's triple product identity. Using ideas from Atkin and Hussain [5] we give an elementary proof of the following Theorem, due to Newman [13], for the case $p = 11$ and $r \leq 10$.

Theorem (5.1.2). Suppose that r is one of the numbers 2, 4, 6, 8, 10, 14, 26. Let p be a prime > 3 such that $r(p+1) \equiv 0 \pmod{24}$. Let $\Delta = r(p^2-1)/24$ and define $p_r(\alpha)$ as zero if α is not a non-negative integer. Then

$$p_r(np + \Delta) \equiv (-p)^{(r/2)-1} p_r(n/p).$$

We show that (5.1.1) follows easily from Theorem (5.1.2).

In section (5.5) we give an elementary derivation of the modular equation of eleventh order due to Fine [8], from which we are able to obtain the Ramanujan-type identity, stated below, for the generating function $\sum_{n\geq 0} p(11n+6)q^n$.

Theorem (5.1.3).

(5.1.4) $$\sum_{n\geq 0} p(11n+6)q^n = 11q^4(11^3 + 88A^* + A^{*2} + 11B^*)\frac{E(q^{11})^{11}}{E(q)^{12}},$$

where A^*, B^* are power series in q with integer coefficients defined by

$$(10A^* + 11^2)\,E(q^{11})^5 = \sum_{n\geq -2} p_5(11n+25)q^n$$

and

$$(14B^* + 112A^* + 11^3)E(q^{11})^7 = \sum_{n\geq -3} p_7(11n+35)q^n .$$

We note here that (5.1.4) is (3.25) in Fine's paper.

5.2. In this section we split Euler's function $E(q)$ according to the residue of the exponent (mod 11) and obtain relations analogous to (1.2.2). These are contained in the following Lemma.

Lemma (5.2.1).

$$E(q) = E(q^{121})[Q_0 - q\,Q_1 - q^2Q_2 - q^{15}Q_4 + q^5 + q^7Q_7],$$

where
$$Q_0 = \prod_{n\geq 1}\frac{(1-q^{121n-44})(1-q^{121n-77})}{(1-q^{121n-22})(1-q^{121n-99})},$$

$$Q_1 = \prod_{n\geq 1}\frac{(1-q^{121n-22})(1-q^{121n-99})}{(1-q^{121n-11})(1-q^{121n-110})},$$

$$Q_2 = \prod_{n\geq 1}\frac{(1-q^{121n-55})(1-q^{121n-66})}{(1-q^{121n-33})(1-q^{121n-88})},$$

$$Q_4 = \prod_{n\geq 1}\frac{(1-q^{121n-11})(1-q^{121n-110})}{(1-q^{121n-55})(1-q^{121n-66})},$$

$$Q_7 = \prod_{n\geq 1}\frac{(1-q^{121n-33})(1-q^{121n-88})}{(1-q^{121n-44})(1-q^{121n-77})},$$

and these satisfy

$$(5.2.2)\quad \begin{cases} Q_0Q_1^2 - Q_0^2Q_2 + q^{33}Q_4^2 + 2q^{22}Q_2Q_4Q_7 - 2q^{11}Q_1\,Q_7 = 0 \\ -Q_1Q_2^2 - q^{11}Q_2Q_7^2 + Q_0^2 + 2q^{11}Q_0Q_1Q_4 - 2q^{22}Q_4Q_7 = 0 \\ Q_0^2Q_7 - q^{22}Q_4Q_7^2 + Q_1^2 - 2q^{11}Q_1Q_2Q_4 - 2Q_0Q_2 = 0 \\ q^{22}Q_0Q_4^2 - q^{11}Q_2^2Q_4 + q^{11}Q_7^2 - 2Q_0Q_1Q_7 + 2Q_1Q_2 = 0 \\ Q_1^2Q_7 - q^{22}Q_1Q_4^2 + Q_2^2 - 2Q_0Q_2Q_7 - 2q^{11}Q_0Q_4 = 0\,, \end{cases}$$

$$(5.2.3)\qquad Q_0\,Q_1\,Q_2\,Q_4\,Q_7 = 1\,.$$

Proof. Here we assume Jacobi's triple product identity

$$\prod_{n\geq 1}(1 + tq^{2n-1})(1 + t^{-1}q^{2n-1})(1 - q^{2n}) = \sum_{-\infty}^{\infty} t^n q^{n^2}$$

from which can be deduced

$$E(q) = \sum_{-\infty}^{\infty} (-1)^n\, q^{\frac{1}{2}(3n^2-n)} \qquad \text{(Euler)},$$

$$E(q)^3 = \sum_{n\geq 0} (-1)^n(2n+1)q^{\frac{1}{2}(n^2+n)} \qquad \text{(Jacobi)},$$

and Watson's quintuple product identity

$$\prod_{n\geq 1}(1+tq^n)(1+t^{-1}q^{n-1})(1-t^2q^{2n-1})(1-t^{-2}q^{2n-1})(1-q^n)$$

$$= \prod_{n\geq 1}(1-t^3q^{3n-1})(1-t^{-3}q^{3n-2})(1-q^{3n}) + t^{-1}\prod_{n\geq 1}(1-t^3q^{3n-2})(1-t^{-3}q^{3n-1})(1-q^{3n})\,.$$

(For an elementary proof of the quintuple product identity, see Bailey [6].)

Write $E(q) = \sum_{i=0}^{10} E_i$,

where E_i contains those terms of $E(q)$ in which the power of q is congruent to i mod 11.

Since $\frac{1}{2}(3r^2 \pm r) \not\equiv 3, 6, 8, 9, 10 \bmod 11$, $E_3 = E_6 = E_8 = E_9 = E_{10} = 0$

and $E(q) = E_0 + E_1 + E_2 + E_4 + E_5 + E_7\,.$

$$E_0(q) = 1 + \sum_{\substack{r\geq 1 \\ \frac{1}{2}(3r^2-r)\equiv 0 \\ \text{mod } 11}} (-1)^r q^{\frac{1}{2}(3r^2-r)} + \sum_{\substack{r\geq 1 \\ \frac{1}{2}(3r^2+r)\equiv 0 \\ \text{mod } 11}} (-1)^r q^{\frac{1}{2}(3r^2+r)}$$

$$= 1 + \sum_{\substack{r\geq 1 \\ r\equiv 0,7 \\ \text{mod } 11}} (-1)^r q^{\frac{1}{2}(3r^2+r)} + \sum_{\substack{r\geq 1 \\ r\equiv 0,4 \\ \text{mod } 11}} (-1)^r q^{\frac{1}{2}(3r^2-r)}$$

$$= \sum_{-\infty}^{\infty} (-1)^{11n} q^{\frac{1}{2}(363n^2+11n)} + \sum_{-\infty}^{\infty} (-1)^{11n+4} q^{\frac{1}{2}(363n^2+253n+44)}$$

$$= \sum_{-\infty}^{\infty} \left(-q^{\frac{11}{2}}\right)^n \left(q^{\frac{363}{2}}\right)^{n^2} + q^{22} \sum_{-\infty}^{\infty} \left(-q^{\frac{253}{2}}\right)^n \left(q^{\frac{363}{2}}\right)^{n^2}$$

$$= \prod_{n\geq 1} \left(1 - q^{363n-176}\right)\left(1 - q^{363n-187}\right)\left(1 - q^{363n}\right)$$

$$= + q^{22} \prod_{n\geq 1} \left(1 - q^{363n-55}\right)\left(1 - q^{363n-308}\right)\left(1 - q^{363n}\right)$$

$$= \prod_{n\geq 1} \left(1+q^{121n-22}\right)\left(1+q^{121n-99}\right)\left(1-q^{242n-165}\right)\left(1-q^{242n-77}\right)\left(1-q^{121n}\right)$$

$$= E(q^{121}) \prod_{n\geq 1} \frac{(1-q^{242n-44})(1-q^{242n-198})(1-q^{242-165})(1-q^{242n-77})}{(1-q^{121n-22})(1-q^{121n-99})}$$

$$= E(q^{121}) \prod_{n\geq 1} \frac{(1-q^{121n-44})(1-q^{121n-77})}{(1-q^{121n-22})(1-q^{121n-99})} .$$

Similarly we obtain:

$$E_1(q) = - qE(q^{121}) \prod_{n\geq 1} \frac{(1-q^{121n-22})(1-q^{121n-99})}{(1-q^{121n-11})(1-q^{121n-110})} ,$$

$$E_2(q) = - q^2 E(q^{121}) \prod_{n\geq 1} \frac{(1-q^{121n-55})(1-q^{121n-66})}{(1-q^{121n-33})(1-q^{121n-88})} ,$$

$$E_4(q) = -q^{15} E(q^{121}) \prod_{n\geq 1} \frac{(1-q^{121n-11})(1-q^{121n-110})}{(1-q^{121n-55})(1-q^{121n-66})} ,$$

$$E_7(q) = q^7 E(q^{121}) \prod_{n\geq 1} \frac{(1-q^{121n-33})(1-q^{121n-88})}{(1-q^{121n-44})(1-q^{121n-77})} .$$

Now,

$$E_5(q) = \sum_{\substack{\frac{1}{2}(3r^2-r)\equiv 5 \\ \text{mod } 11}} (-1)^r q^{\frac{1}{2}(3r^2-r)}$$

$$= \sum_{\substack{r\equiv 2 \\ \text{mod } 11}} (-1)^r q^{\frac{1}{2}(3r^2-r)}$$

$$= \sum_{-\infty}^{\infty} (-1)^{11n+2} q^{\frac{1}{2}(363n^2+121n+10)}$$

$$= q^5 \sum_{-\infty}^{\infty} (-1)^n \left(q^{121}\right)^{\frac{1}{2}(3n^2+n)}$$

$$= q^5 E\left(q^{121}\right) .$$

Combining these results we obtain

$$E(q) = E(q^{121}) \; [Q_0 - qQ_1 - q^2Q_2 - q^{15}Q_4 + q^5 + q^7Q_7] \; ,$$

where the Q_i are defined as in the statement of the Lemma.

Clearly $\quad Q_0 \, Q_1 \, Q_2 \, Q_4 \, Q_7 = 1$.

Now,

$$\begin{aligned}
E(q)^3 &= (E_0 + E_1 + E_2 + E_4 + E_5 + E_7)^3 \\
&= (E_0^3 + 6E_2E_4E_5 + 3E_2^2E_7 + 6E_0E_4E_7 + 3E_1E_5^2) \\
&+ (E_4^3 + 6E_1E_4E_7 + 3E_0^2E_1 + 6E_0E_5E_7 + 3E_2E_5^2) \\
&+ (3E_0E_1^2 + 3E_0^2E_2 + 3E_4^2E_5 + 6E_2E_4E_7 + 6E_1E_5E_7) \\
&+ (E_1^3 + 6E_0E_1E_2 + 3E_0E_7^2 + 6E_2E_5E_7 + 3E_4E_5^2) \\
&+ (3E_0E_2^2 + 3E_1^2E_2 + 3E_0^2E_4 + 3E_4^2E_7 + 3E_1E_7^2 + E_5^3) \\
&+ (3E_1E_2^2 + 3E_0^2E_5 + 3E_2E_7^2 + 6E_0E_1E_4 + 6E_4E_5E_7) \\
&+ (E_2^3 + 6E_0E_2E_4 + 3E_1^2E_4 + 6E_0E_1E_5 + 3E_5^2E_7) \\
&+ (3E_1^2E_5 + 3E_0^2E_7 + 3E_4E_7^2 + 6E_1E_2E_4 + 6E_0E_2E_5) \\
&+ (3E_2^2E_4 + 3E_0E_4^2 + 3E_5E_7^2 + 6E_1E_2E_5 + 6E_0E_1E_7) \\
&+ (3E_1E_4^2 + 3E_2^2E_5 + 3E_1^2E_7 + 6E_0E_4E_5 + 6E_0E_2E_7) \\
&+ (E_7^3 + 6E_1E_4E_5 + 3E_2E_4^2 + 6E_1E_2E_7 + 3E_0E_5^2)
\end{aligned}$$

$$= \sum_{n\geq 0} (-1)^n (2n+1) q^{\frac{1}{2}(n^2+n)} .$$

Since $\frac{1}{2}(n^2 + n) \not\equiv 2, 5, 7, 8, 9 \mod 11$ we have

$$(5.2.4) \quad \begin{cases} 3E_0E_1^2 + 3E_0^2E_2 + 3E_4^2E_5 + 6E_2E_4E_7 + 6E_1E_5E_7 = 0 \\ 3E_1E_2^2 + 3E_0^2E_5 + 3E_2E_7^2 + 6E_0E_1E_4 + 6E_4E_5E_7 = 0 \\ 3E_1^2E_5 + 3E_0^2E_7 + 3E_4E_7^2 + 6E_0E_2E_5 + 6E_1E_2E_4 = 0 \\ 3E_2^2E_4 + 3E_oE_4^2 + 3E_5E_7^2 + 6E_0E_1E_7 + 6E_1E_2E_5 = 0 \\ 3E_1E_4^2 + 3E_1^2E_7 + 3E_2^2E_5 + 6E_0E_2E_7 + 6E_0E_4E_5 = 0 \end{cases} .$$

Substituting $E_0 = E(q^{121})Q_0$, $E_1 = -qQ_1E(q^{121})$, $E_2 = -q^2Q_2E(q^{121})$, $E_4 = -q^{15} Q_4E(q^{121})$, $E_5 = q^5E(q^{121})$ and $E_7 = q^7Q_7E(q^{121})$ we obtain (5.2.2).

The proof of the following Lemma is completely analogous to that of Lemma (1.2.5).

<u>Lemma (5.2.5).</u> If $\omega^{11} = 1$, $\omega \neq 1$, then

$$\prod_{i=0}^{10} E(\omega^i q) = \frac{E\left(q^{11}\right)^{12}}{E\left(q^{121}\right)} .$$

5.3. It appears that the identities in (5.2.2) and (5.2.3) are not sufficient to prove

$$(5.3.1) \qquad p(11n + 6) \equiv 0 \mod 11 \quad \text{for} \quad n \geq 0 .$$

In this section we will derive further identities involving Q_0, Q_1, Q_2, Q_4, Q_7, due to Atkin and Swinnerton-Dyer [4], which will later enable us to prove (5.3.1). Following Atkin and Swinnerton-Dyer we define

$$(5.3.2) \qquad P(a,x) = \prod_{n\geq 1} (1 - ax^{n-1})(1 - a^{-1} x^n) \quad \text{for} \quad a \neq 0, \ |x| < 1.$$

First we need some properties of P.

Lemma (5.3.3).

$$(5.3.4) \qquad P(a^{-1} x,x) = P(a, x) ,$$

$$(5.3.5) \qquad P(ax, x) = - a^{-1} \cdot P(a,x) ,$$

$$(5.3.6) \qquad P(a^{-1}, x) = - a^{-1} P(a,x) ,$$

$$(5.3.7) \qquad P(a, x^2) E(x^2) = \sum_{-\infty}^{\infty} (-1)^n a^n x^{n(n-1)} .$$

Proof. (5.3.4), (5.3.5) and (5.3.6) follow easily from (5.3.2). Jacobi's triple product identity is

$$\prod_{n\geq 1} (1 - ax^{2n-1})(1 - a^{-1} x^{2n-1})(1 - x^{2n}) = \sum_{-\infty}^{\infty} (- 1)^n a^n x^{n^2} ,$$

$$\prod_{n\geq 1} (1 - ax(x^2)^{n-1})(1 - (ax)^{-1} x^{2n})(1 - x^{2n}) = \sum_{-\infty}^{\infty} (- 1)^n a^n x^{n^2}$$

or

$$P(ax, x^2) E(x^2) = \sum_{-\infty}^{\infty} (- 1)^n a^n x^{n^2} ,$$

so that

$$P(a,x^2) E(x^2) = \sum_{-\infty}^{\infty} (- 1)^n a^n x^{n(n-1)} .$$

Lemma (5.3.8).

$$P(-x,\ x^2)\ P(bc,x)\ P(bc^{-1},x)\ -\ P^2(b,x)\ P(-c^2x,x^2)+\ bc^{-1}P^2(c,x)P(-b^2x,x^2)=0.$$

Proof. We define G by,

$$G(b,c,x)\ =\ P(-x,x^2)\ P(bc,x)\ P(bc^{-1},x)-\ P^2(b,x)P(-c^2x,x^2)+$$
$$+\ bc^{-1}\ P^2(c,x)\ P(-b^2x,x^2)\ .$$

From (5.3.5) we have

$$G(bx,c,x)\ =\ P(-x,x^2)\ P(bcx,x)\ P(bc^{-1}x,x)\ -\ P^2(bx,x)\ P(-c^2x,x^2)\ +$$
$$+\ bc^{-1}x\ P^2(c,x)\ P(-\ b^2x^3,x^2)$$
$$=\ P(-x,x^2)(-b^{-1}c^{-1})\ P(bc,x)(-b^{-1}c)\ P(bc^{-1},x)$$
$$-\ b^{-2}P(b,x)\ P(-c^2x,x^2)+\ bc^{-1}x\ P^2(c,x)(b^{-2}x^{-1})P(-b^2x,x^2)$$
$$=\ b^{-2}\ G(b,c,x)\ .$$

Now we expand $G(b,c,x)$ as a Laurent series:

$$G(b,c,x)\ =\ \sum_{-\infty}^{\infty} B_n(c,x)\ b^n\ .$$

We have

$$G(bx,c,x)\ =\ \sum_{-\infty}^{\infty} B_n(c,x)\ x^n b^n\ =\ b^{-2}\ G(b,c,x)$$
$$=\ \sum_{-\infty}^{\infty} B_n(c,x)\ b^{n-2}\ =\ \sum_{-\infty}^{\infty} B_{n+2}(c,x)b^n\ ,$$

so that

$$B_{n+2}(c,x)\ =\ x^n\ B_n(c,x)\quad \text{for all}\quad n\ .$$

It follows that for $n \geq 1$ we have

$$B_{2n}\ =\ x^{n(n-1)}\ B_0\ ,\qquad B_{2n-1}\ =\ x^{(n-1)^2}\ B_1\ ,$$
$$B_{-2n}\ =\ x^{n(n+1)}\ B_0\ ,\qquad B_{-(2n-1)}\ =\ x^{n^2}\ B_1\ .$$

Hence from (5.3.7) we have

$$G(b,c,x) = B_0(c,x)\left(1 + \sum_{n\geq 1} x^{n(n-1)}\, b^{2n} + x^{n(n+1)}\, b^{-2n}\right)$$

$$+ B_1(c,x)\left(\sum_{n\geq 1} x^{(n-1)^2}\, b^{2n-1} + x^{n^2}\, b^{-(2n-1)}\right)$$

$$= B_0(c,x)\ P(-b^2,x^2)\ E(x^2) + bB_1(c,x)\left(\sum_{n\geq 1} x^{(n-1)^2} b^{2(n-1)} + x^{n^2} b^{-2n}\right),$$

$$= B_0(c,x)P(-b^2,x^2)\ E(x^2) + bB_1(c,x)\left(1 + \sum_{n\geq 1} b^{2n}\, x^{n^2} + b^{-2n}\, x^{n^2}\right)$$

$$= B_0(c,x)\ P(-b^2,x^2)\ E(x^2) + bB_1(c,x)\ P(-b^2x,x^2)\ .$$

We will now show that $B_0(c,x)$, the coefficient of b^0 in $G(b,c,x)$, is identically zero. Now from (5.3.7) we have

$$P(bc,x^2) = \frac{1}{E(x^2)}\{\ldots b^{-3}c^{-3}x^{12} + b^{-2}c^{-2}x^{6} - b^{-1}c^{-1}x^{2} + 1 - bc + b^2c^2x^2$$

$$- b^3c^3x^6 + b^4c^4x^{12} + \ldots\},$$

$$P(bc^{-1},x^2) = \frac{1}{E(x^2)}\{\ \ldots\ b^{-3}c^{3}x^{12} + b^{-2}c^{2}x^{6} - b^{-1}c\ x^{2} + 1 - bc^{-1}$$

$$+ b^2c^{-2}x^2 - b^3c^{-3}x^6 + b^4c^{-4}x^{12} + \ldots\}$$

and the coefficient of b^0 in $P(bc,x^2)\ P(bc^{-1},\ x^2)$ is

$$\frac{1}{E(x^2)^2} = \{\ldots\ c^{-6}x^{18} + c^{-4}x^{8} + c^{-2}x^{2} + 1 + c^{2}x^{2} + c^{4}x^{8} + c^{6}x^{18}\}$$

$$= \frac{1}{E(x^2)^2}\ \sum_{-\infty}^{\infty} c^{2n}\ x^{2n^2}$$

so that the coefficient of b^0 in $P(bc,x)\ P(bc^{-1},x)$ is

$$\frac{1}{E(x)^2}\ \sum_{-\infty}^{\infty} c^{2n}\ x^{n^2} = \frac{E(x^2)}{E(x)^2}\ P(-c^2x,\ x^2)\ .$$

Now,

$$P^2(b,x^2) = \frac{1}{E(x^2)^2} \{\ldots b^{-4}x^{20} - b^{-3}x^{12} + b^{-2}x^6 - b^{-1}x^2 + 1 - b + b^2x^2 - b^3x^6 + b^4x^{12} - \ldots\}^2$$

so that the coefficient of b^0 in $P^2(b,x^2)$ is

$$\frac{1}{E(x^2)^2} \{\ldots x^{18} + x^8 + x^2 + 1 + x^2 + x^8 + x^{18} + \ldots\}$$

$$= \frac{1}{E(x^2)^2} \sum_{-\infty}^{\infty} x^{2n^2} .$$

It follows that the coefficient of b^0 in $P^2(b,x)$ is

$$\frac{1}{E(x)^2} \sum_{-\infty}^{\infty} x^{n^2} = \frac{E(x^2)}{E(x)^2} P(-x, x^2)$$

and the coefficient of b^0 in $G(b,c,x)$ is

$$B_0(c,x) = \frac{E(x^2)}{E(x)^2} \{ P(-x,x^2) P(-c^2x,x^2) - P(-c^2x,x^2) P(-x,x^2) \} = 0 .$$

Hence

$$G(b,c,x) = b B_1(c,x) P(-b^2x, x^2) .$$

Now, from (5.3.6) we have

$$G(1,c,x) = P(-x,x^2) P(c,x) P(c^{-1},x) + c^{-1} P^2(c,x) P(-x,x^2)$$

$$= P(-x,x^2) P(c,x) \{P(c^{-1},x) + c^{-1} P(c,x)\}$$

$$= 0 ,$$

but we also have

$$G(1,c,x) = B_1(c,x) P(-x, x^2) .$$

Hence $B_1(c,x) = 0$ since $P(-x, x^2) \neq 0$ for $|x| < 1$ and $G(b,c,x) = 0$ as required.

The following Lemma is (3.7) in Atkin and Swinnerton-Dyer's paper. Their proof relies on properties of complex functions. We give an elementary proof depending on Lemma (5.3.8).

Lemma (5.3.9).

$$P^2(a,x)\ P(bc,x)\ P(bc^{-1},x) - P^2(b,x)\ P(ac,x)\ P(ac^{-1},x) + bc^{-1}\ P^2(c,x)\ P(ab,x)\ P(ab^{-1},x) = 0 .$$

Proof. We define F by,

$$F(a,b,c,x) = P^2(a,x)\ P(bc,x)\ P(bc^{-1},x) - P^2(b,x)\ P(ac,x)\ P(ac^{-1},x) + bc^{-1}\ P^2(c,x)\ P(ab,x)\ P(ab^{-1},x) .$$

From (5.3.5) we easily find that

$$F(ax,b,c,x) = a^{-2}\ F(a,b,c,x) .$$

If we write $F(a,b,c,x)$ as the Laurent series:

$$F(a,b,c,x) = \sum_{-\infty}^{\infty} A_n(b,c,x)\ x^n$$

we find that

$$F(a,b,c,x) = A_0(b,c,x)\ P(-a^2,x^2)\ E(x^2) + aA_1(b,c,x)\ P(-a^2x,x^2) ,$$

by using methods analogous to those in the proof of Lemma (5.3.8). We will now show that $A_0(b,c,x)$, the coefficient of a^0 in $F(a,b,c,x)$ is identically zero. As in the proof of Lemma (5.3.8) we find that

the coefficient of a^0 in $P^2(a,x)$ is $\dfrac{E(x^2)}{E(x)^2}\ P(-x,\ x^2)$,

the coefficient of a^0 in $P(ac,x)\ P(ac^{-1},x)$ is $\dfrac{E(x^2)}{E(x)^2}\ P(-\ c^2x,\ x^2)$,

the coefficient of a^0 in $P(ab,x)\ P(ab^{-1},x)$ is $\dfrac{E(x^2)}{E(x)^2}\ P(-b^2x,\ x^2)$.

Hence, from Lemma (5.3.8) we have

$$A_0(b,c,x) = \frac{E(x^2)}{E(x)^2}\{P(-x,x^2)\ P(bc,x)\ P(bc^{-1},x) - P^2(b,x)\ P(-c^2x,\ x^2)$$
$$+ bc^{-1}\ P^2(c,x)\ P(-b^2x,\ x^2)\}$$
$$= 0\ ,$$

so that

$$F(a,b,c,x) = a\ A_1(b,c,x)\ P(-a^2x,\ x^2)\ .$$

Now from (5.3.6) we have

$$F(1,b,c,x) = -\ P^2(b,x)\ P(c,x)\ P(c^{-1},x) + bc^{-1}\ P^2(c,x)\ P(b,x)\ P(b^{-1},x)$$
$$= c^{-1}\ P^2(b,x)\ P^2(c,x) + bc^{-1}\ P^2(c,x)\ (-b^{-1})\ P^2(b,x)$$
$$= 0\ ,$$

but we also have

$$F(1,b,c,x) = A_1(b,c,x)\ P(-x,\ x^2)\ .$$

Hence $A_1(b,c,x) = 0$ since $P(-x,\ x^2) \neq 0$ for $|x| < 1$,

and $F(a,b,c,x) = 0$ as required.

Following Atkin and Swinnerton-Dyer we define

$$(5.3.10) \qquad P(r) = P(q^{11r},\ q^{121}) = \prod_{n\geq 1}(1 - q^{121n+11r-121})(1-q^{121n-11r})\ ,$$

where r is any non-zero integer. We easily find that

$$(5.3.11) \qquad P(11 - r) = P(r)\ .$$

The following Lemma is Lemma 4 of Atkin and Swinnerton-Dyer's paper.

<u>Lemma (5.3.12)</u>. Suppose that none of $r,s,t,$ $r \pm s,$ $s \pm t,$ $r \pm t$ is equal to zero. Then we have

$$P^2(r)\ P(s+t)\ P(s-t) - P^2(s)\ P(r+t)\ P(r-t) + q^{11(s-t)}P^2(t)P(r+s)P(r-s) = 0.$$

<u>Proof</u>. If we substitute $x = q^{121}$, $a = q^{11r}$, $b = q^{11s}$, $c = q^{11t}$ in Lemma (5.3.9) we obtain

$$P^2(q^{11r}, q^{121})\, P(q^{11(s+t)}, q^{121})\, P(q^{11(s-t)}, q^{121})$$
$$- P^2(q^{11s}, q^{121})\, P(q^{11(r+t)}, q^{121})\, P(q^{11(r-t)}, q^{121})$$
$$+ q^{11(s-t)}\, P^2(q^{11t}, q^{121})\, P(q^{11(r+s)}, q^{121})\, P(q^{11(r-s)}, q^{121}) = 0$$

or

$$P^2(r)\, P(s+t)\, P(s-t) - P^2(s)P(r+t)P(r-t) + q^{11(s-t)}P^2(t)P(r+s)P(r-s) = 0,$$

as required.

Lemma (5.3.13).

$$Q_2 - Q_0Q_7 + q^{22}\, Q_2Q_4^2Q_7 = 0,$$
$$Q_0Q_2^2Q_7 - Q_0Q_1 + q^{11}\, Q_7 = 0,$$
$$Q_2Q_7 - Q_1^2Q_2Q_4 + q^{11}\, Q_4 = 0,$$
$$Q_0 - Q_7^2Q_0Q_1 + q^{11}\, Q_1Q_4 = 0,$$
$$Q_0^2Q_1Q_4 - Q_1 + q^{11}\, Q_2Q_4 = 0,$$

where Q_0, Q_1, Q_2, Q_4, Q_7 are defined in Lemma (5.2.1).

Proof. From (5.3.10) and Lemma (5.2.1) we have

$$(5.3.14) \qquad Q_0 = \frac{P(4)}{P(2)},\ Q_1 = \frac{P(2)}{P(1)},\ Q_2 = \frac{P(5)}{P(3)},\ Q_4 = \frac{P(1)}{P(5)},\ Q_7 = \frac{P(3)}{P(4)} .$$

Taking $(r,s,t) = (5, 3, 1), (5, 4, 3), (5, 2, 1), (4, 3, 2), (4, 2, 1)$ in Lemma (5.3.12) we obtain

$$P^2(5)\, P(4)\, P(2) - P^2(3)\, P(4)\, P(5) + q^{22}\, P^2(1)\, P(2)\, P(3) = 0,$$
$$P^2(5)\, P(4)\, P(1) - P^2(4)\, P(2)\, P(3) + q^{11}\, P^2(3)\, P(1)\, P(2) = 0,$$
$$P^2(5)\, P(3)\, P(1) - P^2(2)\, P(4)\, P(5) + q^{11}\, P^2(1)\, P(3)\, P(4) = 0,$$
$$P^2(4)\, P(5)\, P(1) - P^2(3)\, P(2)\, P(5) + q^{11}\, P^2(2)\, P(1)\, P(4) = 0,$$
$$P^2(4)\, P(1)\, P(3) - P^2(2)\, P(3)\, P(5) + q^{11}\, P^2(1)\, P(2)\, P(5) = 0,$$

$$\frac{P^2(5)\ P(4)\ P(2)}{P^2(3)\ P(4)\ P(5)} - \frac{P^2(3)\ P(4)\ P(5)}{P^2(3)\ P(4)\ P(5)} + q^{22}\ \frac{P^2(1)\ P(2)\ P(3)}{P^2(3)\ P(5)\ P(4)} = 0\ ,$$

$$\frac{P^2(5)\ P(4)\ P(1)}{P^2(4)\ P(2)\ P(3)} - \frac{P^2(4)\ P(2)\ P(3)}{P^2(4)\ P(2)\ P(3)} + q^{11}\ \frac{P^2(3)\ P(1)\ P(2)}{P^2(4)\ P(3)\ P(2)} = 0\ ,$$

$$\frac{P^2(5)\ P(3)\ P(1)}{P^2(5)\ P(3)\ P(1)} - \frac{P^2(2)\ P(4)\ P(5)}{P^2(5)\ P(1)\ P(3)} + q^{11}\ \frac{P^2(1)\ P(3)\ P(4)}{P^2(5)\ P(3)\ P(1)} = 0\ ,$$

$$\frac{P^2(4)\ P(5)\ P(1)}{P^2(2)\ P(4)\ P(1)} - \frac{P^2(3)\ P(2)\ P(5)}{P^2(2)\ P(1)\ P(4)} + q^{11}\ \frac{P^2(2)\ P(1)\ P(4)}{P^2(2)\ P(1)\ P(4)} = 0\ ,$$

$$\frac{P^2(4)\ P(1)\ P(3)}{P^2(1)\ P(5)\ P(2)} - \frac{P^2(2)\ P(3)\ P(5)}{P^2(1)\ P(2)\ P(5)} + q^{11}\ \frac{P^2(1)\ P(2)\ P(5)}{P^2(1)\ P(2)\ P(5)} = 0\ ,$$

or

$$Q_2^2\ Q_1\ Q_4 - 1 + q^{22}\ Q_2^2\ Q_4^2\ Q_1\ Q_4\ Q_7 = 0\ ,$$

$$Q_2^2\ Q_7^2\ Q_0\ Q_2\ Q_4 - 1 + q^{11}\ Q_7^2\ Q_2\ Q_4 = 0\ ,$$

$$1 - Q_1^2\ Q_4^2\ Q_0\ Q_1\ Q_2 + q^{11} Q_4^2\ Q_0\ Q_1 = 0\ ,$$

$$Q_0^2\ Q_2\ Q_7 - Q_0^2\ Q_7^2\ Q_1\ Q_2\ Q_7 + q^{11} = 0\ ,$$

$$Q_0^2\ Q_1^2\ Q_4\ Q_0\ Q_7 - Q_1^2\ Q_0\ Q_7 + q^{11} = 0\ .$$

Multiplying the five equations above by Q_0Q_7, Q_0Q_1, Q_2Q_7, Q_1Q_4, Q_2Q_4 respectively and noting that $Q_0Q_1Q_2Q_4Q_7 = 1$ we obtain

$$Q_2 - Q_0\ Q_7 + q^{22}\ Q_2\ Q_4^2\ Q_7 = 0\ ,$$

$$Q_0\ Q_2^2\ Q_7 - Q_0\ Q_1 + q^{11}\ Q_7 = 0\ ,$$

$$Q_2\ Q_7 - Q_1^2\ Q_2\ Q_4 + q^{11}\ Q_4 = 0\ ,$$

$$Q_0 - Q_7^2\ Q_0\ Q_1 + q^{11}\ Q_1\ Q_4 = 0\ ,$$

$$Q_0^2\ Q_1\ Q_4 - Q_1 + q^{11}\ Q_2\ Q_4 = 0\ ,$$ as required.

5.4. We now proceed as in 1.3 and define

(5.4.1) $\alpha = q^{-5}Q_0,\ \beta = -q^{-4}Q_1,\ \gamma = -q^{10}Q_4,\ \delta = -q^{-3}Q_2,\ \varepsilon = q^{2}Q_7$.

From Lemma (5.2.1) we have

(5.4.2) $$\xi(q) = \frac{E(q)}{q^5\ E(q^{121})} = \alpha + \beta + \gamma + \delta + \varepsilon + 1\ .$$

From (5.2.2), (5.2.3) and Lemma (5.3.13) respectively we obtain

(5.4.3) $$\begin{cases} \alpha^2 + 2\gamma\varepsilon + \beta\delta^2 + \delta\varepsilon^2 + 2\alpha\beta\gamma = 0 \\ \beta^2 + 2\delta\alpha + \gamma\varepsilon^2 + \varepsilon\alpha^2 + 2\beta\gamma\delta = 0 \\ \gamma^2 + 2\varepsilon\beta + \delta\alpha^2 + \alpha\beta^2 + 2\gamma\delta\varepsilon = 0 \\ \delta^2 + 2\alpha\gamma + \varepsilon\beta^2 + \beta\gamma^2 + 2\delta\varepsilon\alpha = 0 \\ \varepsilon^2 + 2\beta\delta + \alpha\gamma^2 + \gamma\delta^2 + 2\varepsilon\alpha\beta = 0\ , \end{cases}$$

(5.4.4) $$\alpha\beta\gamma\delta\varepsilon = -1\ ,$$

(5.4.5) $$\begin{cases} \alpha + \beta\gamma + \alpha\beta\varepsilon^2 = 0 \\ \beta + \gamma\delta + \beta\gamma\alpha^2 = 0 \\ \gamma + \delta\varepsilon + \gamma\delta\beta^2 = 0 \\ \delta + \varepsilon\alpha + \delta\varepsilon\gamma^2 = 0 \\ \varepsilon + \alpha\beta + \varepsilon\alpha\delta^2 = 0\ . \end{cases}$$

We note here that each of the three sets of equations, above, is invariant under any power of the cyclic permutation $(\alpha\beta\gamma\delta\varepsilon)$. Multiplying the five equations in (5.4.5) by $\gamma\delta$, $\delta\varepsilon$, $\varepsilon\alpha$, $\alpha\beta$, $\beta\gamma$ respectively and noting that $\alpha\beta\gamma\delta\varepsilon = -1$, we obtain

$$(5.4.6)\quad \begin{cases} -\alpha + \beta\delta\varepsilon + \gamma\varepsilon\delta^2 = 0 \\ -\beta + \gamma\varepsilon\alpha + \delta\alpha\varepsilon^2 = 0 \\ -\gamma + \delta\alpha\beta + \varepsilon\beta\alpha^2 = 0 \\ -\delta + \varepsilon\beta\gamma + \alpha\gamma\beta^2 = 0 \\ -\varepsilon + \alpha\gamma\delta + \beta\delta\gamma^2 = 0 . \end{cases}$$

We shall use Σ to denote a sum obtained by permuting the typical term by powers of the cyclic permutation $(\alpha\beta\gamma\delta\varepsilon)$. For instance, we have

$$\sum \alpha\delta^2 = \alpha\delta^2 + \beta\varepsilon^2 + \gamma\alpha^2 + \delta\beta^2 + \varepsilon\gamma^2 .$$

We define

$$(5.4.7)\qquad Y = \sum \alpha^2\beta^3 , \qquad Z = \sum \alpha\,\gamma^6 .$$

Later we will show that for $1 \le i \le 10$,

$$H(\xi^i) = P(Y, Z) ,$$

where P is a polynomial, quadratic in Y and linear in Z, with integral coefficients and H is the operator that acts on a series in powers of q picking out those terms in which the power of q is congruent to 0 modulo 11.

<u>Lemma (5.4.8).</u>

$$\sum \alpha\delta^2 = -4 ,$$

$$\sum \alpha\beta\gamma^2 = 3 ,$$

$$\sum \alpha\gamma\varepsilon^3 = 1 - Y ,$$

where $\alpha,\beta,\gamma,\delta,\varepsilon$ are defined by (5.4.1) and Y is defined by (5.4.7).

<u>Proof</u>. Multiplying the first equation in (5.4.5) by $\beta\gamma\delta\varepsilon = -\alpha^{-1}, \delta^2,$ respectively and summing we obtain

$$-5 + \sum \beta^2\gamma^2\delta\varepsilon - \sum \beta\varepsilon^2 = 0 ,$$

$$\sum \alpha\delta^2 + \sum \beta\gamma\delta^2 + \sum \alpha\beta\delta^2\varepsilon^2 = 0 ,$$

so that

$$\begin{aligned} \sum \alpha\beta\gamma^2 = \sum \beta\delta\gamma^2 &= - \sum \alpha\delta^2 - \sum \alpha\beta\delta^2\varepsilon^2 \\ &= - \sum \beta\varepsilon^2 - \sum \beta^2\gamma^2\delta\varepsilon \\ &= - 5 - 2 \sum 3\varepsilon^2 \\ &= - 5 - 2 \sum \alpha\delta^2 , \end{aligned}$$

or

(5.4.9) $$2 \sum \alpha\delta^2 + \sum \alpha\beta\gamma^2 = -5 .$$

Multiplying the fourth equation in (5.4.3) by α and summing we obtain

$$\sum \alpha\delta^2 + 2 \sum \gamma\alpha^2 + \sum \alpha\varepsilon\beta^2 + \sum \alpha\beta\gamma^2 + 2 \sum \delta\varepsilon\alpha^2 = 0$$

or

(5.4.10) $$3 \sum \alpha\delta^2 + 4 \sum \alpha\beta\gamma^2 = 0 .$$

From (5.4.9) and (5.4.10) it follows that

$$\sum \alpha\delta^2 = -4 \quad \text{and} \quad \sum \alpha\beta\gamma^2 = 3 ,$$ as required.

Multiplying the third equation in (5.4.3) by $\alpha\beta$ and summing we obtain

$$\sum \alpha\beta\gamma^2 + 2 \sum \alpha\varepsilon\beta^2 + \sum \beta\delta\alpha^3 + \sum \alpha^2\beta^3 - 10 = 0$$

or

$$\begin{aligned} \sum \alpha\gamma\varepsilon^3 = \sum \beta\delta\alpha^3 &= 10 - \sum \alpha^2\beta^3 - 3 \sum \alpha\beta\gamma^2 \\ &= 10 - Y - 9 \\ &= 1 - Y , \end{aligned}$$ as required.

Lemma (5.4.11). Any expression of the form $\Sigma\, \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e$ in which each term has powers of q congruent to 0 mod 11 and $a + b + c + d + e \leq 10$ can be written as a polynomial in Y and Z of the form

$$rY^2 + sY + tZ + u\ ,$$

where r, s, t, u are integers and Y, Z are defined by (5.4.7).

Proof. From (5.4.1) it is clear that any expression of the form $\alpha^a\beta^b\gamma^c\delta^d\varepsilon^e$ has powers of q congruent to 0 mod 11 if and only if

(5.4.12) $$6a + 7b + 10c + 8d + 2e \equiv 0 \mod 11\ ,$$

which is equivalent to

$$2^9a + 2^7b + 2^5c + 2^3d + 2e \equiv 0 \mod 11\ .$$

It follows that $\alpha^a\beta^b\gamma^c\delta^d\varepsilon^e$ has powers of q congruent to 0 mod 11 if and only if any one of

$$\beta^a\gamma^b\delta^c\varepsilon^d\alpha^e\ ,\quad \gamma^a\delta^b\varepsilon^c\alpha^d\beta^e\ ,\quad \delta^a\varepsilon^b\alpha^c\beta^d\gamma^e\ ,\quad \varepsilon^a\alpha^b\beta^c\gamma^d\delta^e\ ,$$

has powers of q congruent to 0 mod 11,

so that under the conditions of the Lemma $\Sigma\, \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e$ is well-defined. It is easy to verify that the only non-trivial sums of degree at most 5 are $\Sigma\, \alpha\delta^2$, $\Sigma\, \alpha\beta\gamma^2$, $\Sigma\, \alpha^2\beta^3$ and $\Sigma\, \alpha\gamma\varepsilon^3$. Hence from (5.4.7) and Lemma (5.4.8) it follows that the statement is true for the degree, $a + b + c + d + e \leq 5$.

We need some more equations. From the second equation in (5.4.5) and the third equation in (5.4.6) we have

(5.4.13) $$\alpha^2\beta\gamma = -\beta - \gamma\delta\ ,$$

(5.4.14) $$\alpha^2\beta\varepsilon = \gamma - \alpha\beta\delta\ .$$

From the third equation in (5.4.3), and the fourth equation in (5.4.6) and (5.4.5) we have

$$\alpha^2\gamma\delta = -\alpha\gamma\beta^2 - 2\delta\varepsilon\gamma^2 - \gamma^3 - 2\varepsilon\beta\gamma ,$$

$$\alpha^2\gamma\delta = (-\delta + \varepsilon\beta\gamma) - 2(-\delta - \varepsilon\alpha) - \gamma^3 - 2\beta\gamma\varepsilon ,$$

or

$$\alpha^2\gamma\delta = \delta + 2\varepsilon\alpha - \beta\gamma\varepsilon - \gamma^3 . \tag{5.4.15}$$

From the third and fifth equations in (5.4.5) we have

$$-1 + \gamma\delta\alpha^2\beta^2 - \alpha\delta^2 = 0 ,$$

$$\gamma\alpha^2 + \delta\varepsilon\alpha^2 + \gamma\delta\alpha^2\beta^2 = 0 ,$$

so that

$$\alpha^2\delta\varepsilon = -1 - \gamma\alpha^2 - \alpha\delta^2 . \tag{5.4.16}$$

We now proceed by induction. We assume the statement is true for all values of a, b, c, d, e with $a + b + c + d + e \le n$ where $5 \le n \le 9$, and consider any $\Sigma\, \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e$, satisfying (5.4.12) and $a + b + c + d + e = n + 1$.

Without loss of generality we may suppose $a \ge 2$. There are two cases:

<u>Case (I)</u>: Each term has at least three factors.

Suppose $b \ge 1$ and $c \ge 1$ then from (5.4.13) it follows that

$$\sum \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e \doteq \sum \alpha^2\beta\gamma(\alpha^{a-2}\beta^{b-1}\gamma^{c-1}\delta^d\varepsilon^e)$$

$$= -\sum \alpha^{a-2}\beta^b\gamma^{c-1}\delta^d\varepsilon^e - \sum \alpha^{a-2}\beta^{b-1}\gamma^c\delta^{d+1}\varepsilon^e .$$

But the right-hand side of the equation above has terms of degree less than $n + 1$ so we have

$$\sum \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e = P(Y,Z) .$$

Throughout we denote by $P(Y,Z)$ and $P_i(Y,Z)$, for $1 \le i \le 5$, polynomials of the form

$$rY^2 + sY + tZ + u ,$$

where r, s, t, u are integers.

Similarly from (5.4.14), (5.4.15), (5.4.16), respectively we obtain the desired result for $b \ge 1$ and $e \ge 1$, $c \ge 1$ and $d \ge 1$, $d \ge 1$ and $e \ge 1$, respectively,

Now suppose $b \ge 1$ and $d \ge 1$. We may assume $c = e = 0$. If $d \ge 2$ we have

$$\sum \alpha^a\beta^b\delta^d = \sum \alpha^d\delta^b\gamma^a = P(Y,Z) ,$$

for some polynomial P. So we may assume $d = 1$.

From the third equation in (5.4.3) we have

$$\alpha\beta^2 = -\alpha^2\delta - 2\gamma\delta\varepsilon - \gamma^2 - 2\beta\varepsilon ,$$

so that if $b \ge 3$ we have

$$\begin{aligned}\sum \alpha^a\beta^b\delta &= \sum (\alpha\beta^2)\,\alpha^{a-1}\beta^{b-2}\delta \\ &= -\sum \alpha^{a+1}\beta^{b-2}\delta^2 - 2\sum \alpha^{a-1}\beta^{b-2}\gamma\delta^2\varepsilon - \sum \alpha^{a-1}\beta^{b-2}\gamma^2\delta \\ &\qquad - 2\sum \alpha^{a-1}\beta^{b-1}\delta\varepsilon \\ &= -\sum \alpha^2\gamma^{a+1}\delta^{b-2} + 2\sum \alpha^{a-2}\beta^{b-3}\delta - \sum \alpha^{a-1}\beta^{b-2}\gamma^2\delta - 2\sum \alpha^{a-1}\beta^{b-1}\delta\varepsilon \\ &= P(Y,Z), \quad \text{for some polynomial } P.\end{aligned}$$

We may assume $b = 1$, since if $b = 2$, $d = 1$, $c = e = 0$ then from (5.4.12) we have

$$6a + 14 + 8 \equiv 0 \mod 11$$

or

$$a \equiv 0 \mod 11, \text{ which is impossible.}$$

For $b = d = 1, c = e = 0$ we find $a \equiv 3 \mod 11$ which implies $a = 3$, but $a + b + c + d + e = 5 = n + 1$, contradiction. Hence we have shown that the statement is true for $b \geq 1$ and $d \geq 1$. Now suppose suppose $c \geq 1$ and $e \geq 1$. We may assume $b = d = 0$. If $c \geq 2$ we have

$$\sum \alpha^a \gamma^c \varepsilon^e = \sum \alpha^c \gamma^e \delta^a = P(Y,Z) ,$$

for some polynomial P.

If $e \geq 2$ we have

$$\sum \alpha^a \gamma^c \varepsilon^e = \sum \alpha^e \beta^a \delta^c = P(Y,Z) ,$$

for some polynomial P.

So we may assume $c = e = 1$. In this case we find that $a = 9$, which is impossible.

<u>CASE (II)</u>: Each term has only two factors.

We have the following table:

Degree	Terms with two factors				
6	$\alpha^2 \delta^4$	$\beta^2 \varepsilon^4$	$\gamma^2 \alpha^4$	$\delta^2 \beta^4$	$\varepsilon^2 \gamma^4$
7	$\alpha^5 \beta^2$	$\beta^5 \gamma^2$	$\gamma^5 \delta^2$	$\delta^5 \varepsilon^2$	$\varepsilon^5 \alpha^2$
	$\alpha^6 \delta$	$\beta^6 \varepsilon$	$\gamma^6 \alpha$	$\delta^6 \beta$	$\varepsilon^6 \gamma$
8	$\alpha \beta^7$	$\beta \gamma^7$	$\gamma \delta^7$	$\delta \varepsilon^7$	$\varepsilon \alpha^7$
9	$\alpha^8 \beta$	$\beta^8 \gamma$	$\gamma^8 \delta$	$\delta^8 \varepsilon$	$\varepsilon^8 \alpha$
	$\alpha^3 \delta^6$	$\beta^3 \varepsilon^6$	$\gamma^3 \alpha^6$	$\delta^3 \beta^6$	$\varepsilon^3 \gamma^6$
10	$\alpha^4 \beta^6$	$\beta^4 \gamma^6$	$\gamma^4 \delta^6$	$\delta^4 \varepsilon^6$	$\varepsilon^4 \alpha^6$
	$\alpha^7 \delta^3$	$\beta^7 \varepsilon^3$	$\gamma^7 \alpha^3$	$\delta^7 \beta^3$	$\varepsilon^7 \gamma^3$

We will now show that each of the expressions,

$$\sum \alpha^2\delta^4 \ , \ \sum \alpha^5\beta^2 \ , \ \sum \alpha^6\delta \ , \ \sum \alpha\beta^7 \ , \ \sum \alpha^8\beta \ , \ \sum \alpha^3\delta^6 \ ,$$
$$\sum \alpha^4\beta^6 \ , \ \sum \alpha^7\delta^3 \ ,$$

can be written as $P(Y,Z)$, for some polynomial P .

We have

$$\sum \alpha^2\delta^4 = \left(\sum \alpha\delta^2\right)^2 - 2\sum \alpha\beta\delta^2\varepsilon^2 - 2\sum \alpha\beta^2\delta^3$$
$$= P(Y,Z) \ , \quad \text{for some polynomial } P \ ,$$

since from Lemma (5.4.8), $\left(\sum \alpha\delta^2\right)^2 = 16$, and the latter two expressions are of degree 6 and they have terms with at least three factors.

$$\sum \alpha^6\delta = \sum \alpha\gamma^6 = Z \ , \quad \text{by definition.}$$

Multiplying the third equation in (5.4.3) by α^4 we obtain

$$\alpha^4\gamma^2 + 2\alpha^4\beta\varepsilon + \alpha^6\delta + \alpha^5\beta^2 + 2\alpha^4\gamma\delta\varepsilon = 0 \ ,$$

so that

$$\sum \alpha^5\beta^2 = -\sum \alpha^6\delta - 2\sum \alpha^4\gamma\delta\varepsilon - \sum \alpha^4\gamma^2 - 2\sum \alpha^4\beta\varepsilon$$
$$= -Z + P(Y,Z) \ , \quad \text{for some polynomial } P \ ,$$

since the last three expressions are either of degree less than 7 or are of degree 7 with terms consisting of at least three factors.

Similarly the desired result for $\Sigma\, \alpha\beta^7$ follows from

$$\beta^5\gamma^2 + 2\beta^6\varepsilon + \alpha^2\beta^5\delta + \alpha\beta^7 + 2\beta^5\gamma\delta\varepsilon = 0 \ ,$$

which is the third equation in (5.4.3) multiplied by β^5 .

Multiplying the first equation in (5.4.3) by $\alpha^6\beta$ we obtain

$$\alpha^8\beta + 2\alpha^6\beta\gamma\varepsilon + \alpha^6\beta^2\delta^2 + \alpha^6\beta\delta\varepsilon^2 + 2\alpha^7\beta^2\gamma = 0 \ ,$$

so that

$$(5.4.17) \qquad \sum \alpha^8\beta = -2\sum \alpha^6\beta\gamma\varepsilon - \sum \alpha^6\beta^2\delta^2 - \sum \alpha^6\beta\delta\varepsilon^2 - 2\sum \alpha^7\beta^2\gamma .$$

We will now show that each expression on the right-hand side of (5.4.17) can be written as a polynomial in Y and Z, of the desired type. Now, $\Sigma\, \alpha^6\beta\gamma\varepsilon = P_1(Y,Z)$ for some polynomial P_1, since it has the same degree as $\Sigma\, \alpha^8\beta$ and it consists of terms with at least three factors.

We have

$$(5.4.18) \qquad \left(\sum \alpha^3\beta\delta\right)^2 = \sum \alpha^6\beta^2\delta^2 - 2\sum \alpha^2\beta^3 + 2\sum \alpha^4\beta\gamma^3\delta^2 .$$

From (5.4.13) it follows that

$$\begin{aligned}\sum \alpha^4\beta\gamma^3\delta^2 &= -\sum \alpha^2\beta\gamma^2\delta^2 - \sum \alpha^2\gamma^3\delta^3 \\ &= P_2(Y,Z) , \quad \text{for some polynomial } P_2 ,\end{aligned}$$

since each expression has degree less than 9, the degree of $\Sigma\, \alpha^8\beta$.

Now, from Lemma (5.4.8)

$$\left(\sum \alpha^3\beta\delta\right)^2 = \left(\sum \alpha\gamma\varepsilon^3\right)^2 = 1 - 2Y + Y^2$$

and $$\sum \alpha^6\beta^2\delta^2 = P_3(Y,Z) , \quad \text{for some polynomial } P_3 .$$

Now, from (5.4.13) we have

$$\begin{aligned}\sum \alpha^6\beta\delta\varepsilon^2 &= \sum \alpha^2\beta^6\gamma\varepsilon \\ &= -\sum \beta^6\varepsilon - \sum \beta^5\gamma\delta\varepsilon \\ &= P_4(Y,Z) , \quad \text{for some polynomial } P_4 .\end{aligned}$$

Similarly, $\sum \alpha^7\beta^2\gamma = P_5(Y,Z)$, for some polynomial P_5.

Hence, $\sum \alpha^8\beta = P(Y,Z)$, for some polynomial P.

Multiplying the third equation in (5.4.3) by $\alpha\delta^5$ we obtain

$$\alpha\gamma^2\delta^5 + 2\alpha\beta\delta^5\varepsilon + \alpha^3\delta^6 + \alpha^2\beta^2\delta^5 + 2\alpha\gamma\delta^6\varepsilon = 0 ,$$

so that

$$\sum \alpha^3\delta^6 = -\sum \alpha^2\beta^2\delta^5 - 2\sum \alpha\gamma\delta^6\varepsilon - \sum \alpha\gamma^2\delta^5 - 2\sum \alpha\beta\delta^5\varepsilon$$

$$= P(Y,Z) , \quad \text{for some polynomial } P ,$$

since each expression on the right-hand side is either of degree less than 9 or is of degree 9 with terms consisting of at least three factors. We can argue similarly to obtain the desired result for

$\sum \alpha^4\beta^6$ and $\sum \alpha^7\gamma^3$.

This completes the proof of Lemma (5.4.11).

As an immediate consequence we have the following Lemma.

<u>Lemma (5.4.19).</u> For each i, $1 \le i \le 10$, there exist integers r, s, t, u such that

$$H(\xi^i) = rY^2 + sY + tZ + u ,$$

where Y, Z are defined by (5.4.7).

<u>Proof.</u> (5.4.2) is

$$\xi = \frac{E(q)}{q^5 E(q^{121})} = \alpha + \beta + \gamma + \delta + \varepsilon + 1 ,$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ are defined by (5.4.1).

Now,

$\xi^i = (\alpha + \beta + \gamma + \delta + \varepsilon + 1)^i$ is a sum of expressions of the form

$$\sum \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e = \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e + \beta^a\gamma^b\delta^c\varepsilon^d\alpha^e + \gamma^a\delta^b\varepsilon^c\alpha^d\beta^e + \delta^a\varepsilon^b\alpha^c\beta^d\gamma^e + \varepsilon^a\alpha^b\beta^c\gamma^d\delta^e ,$$

where $a + b + c + d + e \le 10$.

From the remarks at the beginning of the proof of Lemma (5.4.11) we have that

$H(\xi^i)$ is a sum of expressions of the form

$\sum \alpha^a \beta^b \gamma^c \delta^d \varepsilon^e$, where a, b, c, d, e satisfies (5.4.12) and $a + b + c + d + e \le 10$. Hence from Lemma (5.4.11) we have

$$H(\xi^i) = rY^2 + sY + tZ + u,$$

for some integers r, s, t, u.

5.5 In this section we prove a special case of Theorem (5.1.2), from which we are able to show that

$$p(11n + 6) \equiv 0 \mod 11, \quad \text{for} \quad n \ge 0.$$

We also derive the modular equation of eleventh order due to Fine [8]. We define

(5.5.1) $$A = -16 - Y, \qquad B = 2 - Z,$$

where Y, Z are defined by (5.4.7).

We now introduce the operators H_i, $0 \le i \le 10$, which act on a series of powers of q and simply pick out these terms in which the power of q is congruent to i modulo 11, so that $H_0 = H$.

Lemma (5.5.2).

$$H(\xi) = H(\xi^2) = 1, \qquad H(\xi^7) = 14B + 112A + 11^3,$$
$$H(\xi^3) = H(\xi^4) = -11, \qquad H(\xi^8) = -11^3,$$
$$H(\xi^5) = 10A + 11^2, \qquad H(\xi^9) = -9A^2 + 1386A - 99B + 11^4,$$
$$H(\xi^6) = 11^2, \qquad H(\xi^{10}) = 11^4.$$

Proof. From Lemma (5.4.8) we easily find that

$$H(\alpha + \ldots + \varepsilon) = 0 ,$$

$$H((\alpha + \ldots + \varepsilon)^2) = 0 ,$$

$$H((\alpha + \ldots + \varepsilon)^3) = 3 \sum \alpha\delta^2 = -12 ,$$

$$H((\alpha + \ldots + \varepsilon)^4) = 12 \sum \alpha\beta\gamma^2 = 36 ,$$

$$\begin{aligned} H((\alpha + \ldots + \varepsilon)^5) &= 10 \sum \alpha^2\beta^3 + 20 \sum \alpha\gamma\varepsilon^3 + 120\, \alpha\beta\gamma\delta\varepsilon \\ &= 10Y + 20(1-Y) - 120 \\ &= -100 - 10Y . \end{aligned}$$

Since H is a linear operator we have

$$H(\xi) = H(\alpha + \ldots + \varepsilon) + H(1) = 1 ,$$

$$H(\xi^2) = H((\alpha + \ldots + \varepsilon)^2) + 2H((\alpha + \ldots + \varepsilon)) + H(1) = 1 ,$$

$$H(\xi^3) = -12 + 1 = -11 ,$$

$$H(\xi^4) = 36 - 48 + 1 = -11 ,$$

$$\begin{aligned} H(\xi^5) &= (-100 - 10Y) + 5 \times 36 - 10 \times 12 + 1 \\ &= -10Y - 39 = -10(-16 - A) - 39 = 10A + 11^2 . \end{aligned}$$

From (5.4.3) we have

$$H_1(\alpha + \ldots + \varepsilon) = 0 ,$$

$$H_1((\alpha + \ldots + \varepsilon)^2) = \alpha^2 + 2\gamma\varepsilon ,$$

$$H_1((\alpha + \ldots + \varepsilon)^3) = 3\beta\delta^2 + 3\delta\varepsilon^2 + 6\alpha\beta\gamma ,$$

so

$$\begin{aligned} H_1(\xi^3) &= H_1((\alpha + \ldots + \varepsilon)^3 + 3(\alpha + \ldots + \varepsilon)^2 + 3(\alpha + \ldots + \varepsilon) + 1) \\ &= 3\alpha^2 + 6\gamma\varepsilon + 3\beta\delta^2 + 3\delta\varepsilon^2 + 6\alpha\beta\gamma = 0 . \end{aligned}$$

Similarly,

$$H_3(\xi^3) = 3\beta^2 + 6\delta\alpha + 3\gamma\varepsilon^2 + 3\varepsilon\alpha^2 + 6\beta\gamma\delta = 0 ,$$

$$H_4(\xi^3) = 3\varepsilon^2 + 6\beta\delta + 3\alpha\gamma^2 + 3\gamma\delta^2 + 6\varepsilon\alpha\beta = 0 ,$$

$$H_5(\xi^3) = 3\delta^2 + 6\alpha\gamma + 3\varepsilon\beta^2 + 3\beta\gamma^2 + 6\delta\varepsilon\alpha = 0 ,$$

$$H_9(\xi^3) = 3\gamma^2 + 6\varepsilon\beta + 3\delta\alpha^2 + 3\alpha\beta^2 + 6\gamma\delta\varepsilon = 0 .$$

Hence, $H(\xi^6) = (H(\xi^3))^2 = 11^2$.

We could use similar methods to evaluate $H(\xi^i)$, for $i \geq 7$, but this would be extremely tedious. We now calculate the first five terms of $H(\xi^i)$ as a series of powers of $x = q^{11}$, for $i \geq 7$, using Newman's [14] table for the coefficients of $E(q)^i$.

Now,

$$\xi = \frac{E(q)}{q^5 E(q^{121})}$$

so

$$(5.5.3) \quad H(\xi^7) = \frac{q^{-35}H_2(E(q)^7)}{E(q^{121})^7} = 14x^{-3} + 112x^{-2} + 238x^{-1} - 181 + 588x + \ldots ,$$

$$(5.5.4) \quad H(\xi^8) = \frac{q^{-40}H_7(E(q)^8)}{E(q^{121})^8} = 0x^{-3} + 0x^{-2} + 0x^{-1} - 11^3 + 0 + \ldots ,$$

$$(5.5.5) \quad H(\xi^9) = \frac{q^{-45}H_1(E(q)^9)}{E(q^{121})^9} = -9x^{-4} - 135x^{-3} + 1566x^{-2} + 3015x^{-1} - 2423 + \ldots ,$$

$$(5.5.6) \quad H(\xi^{10}) = \frac{q^{-50}H_6(E(q)^{10})}{E(q^{121})^{10}} = 0x^{-4} + 0x^{-3} + 0x^{-2} + 0x^{-1} + 11^4 + \ldots .$$

We have calculated the first eleven terms of Y^2, Y, Z as a series of powers of $x = q^{11}$.

$$(5.5.7) \quad Y = \sum \alpha^2\beta^2 = -x^{-2} - 2x^{-1} - 4 - 5x - 8x^2 - x^3 - 7x^4 + 11x^5 - 10x^6 + 12x^7 + 18x^8 + \ldots ,$$

$$(5.5.8) \quad Z = \sum \alpha\gamma^6 = -x^{-3} + 0x^{-2} - x^{-1} + 14 - 2x - 2x^2 - 16x^3 - 16x^4 - 18x^5 + 46x^6 + 31x^7 + \ldots ,$$

$$(5.5.9) \quad Y^2 = x^{-4} + 4x^{-3} + 12x^{-2} + 26x^{-1} + 52 + 74x + 107x^2 + 94x^3 + 106x^4 + 14x^5 - x^6 + \ldots .$$

From Lemma (5.4.19) we have

$$H(\xi^7) = rY^2 + sY + tZ + u ,$$

for some integers $r, s, t, u,$ so that

$$r = 0, \quad t = -14, \quad s = -112, \quad 14t - 4s + u = -181$$

and $u = -433$. Therefore,

$$H(\xi^7) = -112(-16 - A) - 14(2 - B) - 433$$

$$= 14B + 112A + 11^3 .$$

Similarly,

$$H(\xi^8) = rY^2 + sY + tZ + u , \quad \text{so that}$$

$r = s = t = 0$ and $u = -11^3$, as required.

$$H(\xi^9) = rY^2 + sY + tZ + u , \quad \text{so that}$$

$r = -9$, $4r - t = -135$, $12r - s = 1566$

and $52r + 14t - 4s + u = -2423$.

Hence, $r = -9$, $t = 99$, $s = -1674$, $u = -10037$ and

$$H(\xi^9) = -9(A+16)^2 - 1674(-16 - A) + 99(2 - B) + 10037$$

$$= -9A^2 + 1386A - 99B + 11^4 .$$

$$H(\xi^{10}) = rY^2 + sY + tZ + u , \quad \text{so that}$$

$r = s = t = 0$ and $u = 11^4$, as required.

Theorem (5.5.10). Suppose r is one of the numbers 2, 4, 6, 8, 10, and define $p_r(\alpha)$ as zero if α is not a non-negative integer, Then

$$p_r(11n + 5r) = (-11)^{(r/2)-1} p_r(n/11) .$$

Proof. From Lemma (5.5.1) we have

$$H(\xi^r) = (-11)^{(r/2)-1} ,$$

$$H(q^{-5r} E(q)^r) = (-11)^{(r/2)-1} E(q^{121})^r ,$$

$$\sum_{n\geq 0} p_r(11n+5r)q^{11n} = (-11)^{(r/2)-1} \sum_{n\geq 0} p_r(n)q^{121n}$$

or $\sum_{n\geq 0} p_r(11n+5r)q^n = (-11)^{(r/2)-1} \sum_{n\geq 0} p_r(n/11)q^n$, as required.

Similarly we also find that

$$\sum_{n\geq 0} p_1(11n+5)q^n = \sum_{n\geq 0} p_1(n/11)q^n ,$$

$$\sum_{n\geq 0} p_3(11n+15)q^n = -11 \sum_{n\geq 0} p_3(n/11)q^n .$$

Further,

$$\sum_{n\geq 0} p_5(11n+3)q^{11n+3} = q^{25}(10A + 11^2)\ E(q^{121})^5 ,$$

$$\sum_{n\geq -2} p_5(11n+25)q^{11n} = (10A + 11^2)\ E(q^{121})^5$$

or

(5.5.11)
$$\sum_{n\geq -2} p_5(11n+25)q^n = (10A^* + 11^2)\ E(q^{11})^5 ,$$

where $A^*(q) = A(q^{1/11})$.

Similarly,

(5.5.12) $$\sum_{n\geq -3} p_7(11n+35)q^n = (14B^* + 112A^* + 11^3)\ E(q^{11})^7 ,$$

where $B^*(q^{1/11})$.

Here we note that A^*, B^* are respectively α, β in Fine's paper. Also if we compare (5.5.11), (5.5.12) respectively to the third and fourth equation of (46) in Atkin [1] we find that Atkin's G_2, G_3 are related to A^*, B^* by

$$G_2 = A^* , \qquad G_3 = B^* - 3A^* .$$

<u>Theorem (5.5.13).</u> For $n \geq 0$,

$$p(11n+6) \equiv 0 \mod 11 .$$

Proof. From Theorem (5.5.10) we have

$$p_{10}(11n + 6) = 11^4 \, p_{10}((n-4)/11) ,$$

so that

$$H_6(E(q)^{10}) \equiv 0 \mod 11 .$$

Now,

$$\sum_{n\geq 0} p(n)\, q^n = \frac{1}{E(q)} = \frac{E(q)^{10}}{E(q)^{11}} \equiv \frac{E(q)^{10}}{E(q^{11})} \mod 11 ,$$

since $(1 - q)^{-11} \equiv (1 - q)^{-1} \mod 11$.

It follows that

$$\sum_{n\geq 0} p(11n + 6) q^{11n+6} \equiv \frac{H_6(E(q)^{10})}{E(q^{11})} \equiv 0 \mod 11, \text{ as required.}$$

Lemma (5.5.14) (Fine's Modular Equation, Fine [8], (3.21)).

$$(5.5.15) \quad \frac{E(q^{11})^{12}}{q^{55}E(q^{121})^{12}} = \xi^{11} - 11\xi^{10} + 5 \times 11\xi^9 - 11^2\xi^8 - 11^2\xi^7$$
$$+ 11(11^2-2A)\xi^6 - 11^2(11-2A)\xi^5$$
$$- 11(11^3+126A+2B)\xi^4 + 11^2(5 \times 11^2+ 38A + 3B)\xi^3$$
$$- 11(11^4 + 72 \times 11A - A^2 + 9 \times 11B)\xi^2$$
$$+ 11^2(11^3 + 8 \times 11A + A^2 + 11B)\xi .$$

Proof. For $0 \leq i \leq 10$ define $\xi_i(q) = \xi(\omega^i q)$ where $\omega^{11} = 1$, $\omega \neq 1$. then

$$H(\xi^j) = \frac{1}{11} \sum_{i=0} \xi^j(\omega^i q) = \frac{1}{11} \sum_{i=0} \xi_i^j .$$

Therefore from Lemma (5.5.2) we have

$$P_1 = \xi_0 + \xi_1 + \ldots + \xi_{10} = 11 ,$$

$$P_2 = \xi_0^2 + \xi_1^2 + \ldots + \xi_{10}^2 = 11 ,$$

$$P_3 = \xi_0^3 + \xi_1^3 + \ldots + \xi_{10}^3 = - 11^2 ,$$

$$P_4 = \xi_0^4 + \xi_1^4 + \ldots + \xi_{10}^4 = -11^2 ,$$

$$P_5 = \xi_0^5 + \xi_1^5 + \ldots + \xi_{10}^5 = 11(10A + 11^2) ,$$

$$P_6 = \xi_0^6 + \xi_1^6 + \ldots + \xi_{10}^6 = 11^3 ,$$

$$P_7 = \xi_0^7 + \xi_1^7 + \ldots + \xi_{10}^7 = 11(14B + 112A + 11^3) ,$$

$$P_8 = \xi_0^8 \quad \xi_1^8 + \ldots + \xi_{10}^8 = -11^4 ,$$

$$P_9 = \xi_0^9 \quad \xi_1^9 + \ldots + \xi_{10}^9 = 11(-9A^2 + 1386A - 99B + 11^4)$$

$$P_{10} = \xi_0^{10} \quad \xi_1^{10} + \ldots + \xi_{10}^{10} = 11^5 .$$

From standard formulae it follows that

$$S_1 = \sum \xi_i = 11 ,$$

$$S_2 = \sum_{i<j} \xi_i \xi_j = 5 \times 11 ,$$

$$S_3 = \sum_{i<j<k} \xi_i \xi_j \xi_k = 11^2 ,$$

$$S_4 = \sum_{i<j<k<\ell} \xi_i \xi_j \xi_k \xi_\ell = -11^2 ,$$

$$S_5 = -11(11^2 - 2A) ,$$

$$S_6 = -11^2(11 - 2A) ,$$

$$S_7 = 11(11^3 + 126A + 2B) ,$$

$$S_8 = 11^2(5 \times 11^2 + 38A + 2B) ,$$

$$S_9 = 11(11^4 + 72 \times 11A - A^2 + 9 \times 11B) ,$$

$$S_{10} = 11^2(11^3 + 8 \times 11A + A^2 + 11B) .$$

From Lemma (5.2.5) we have

$$S_{11} = \prod_{i=0}^{10} \xi_i = \prod_{i=0}^{10} \xi(\omega^i q)$$

$$= \prod_{i=0}^{10} \frac{E(\omega^i q)}{q^5 \omega^{5i} R(q^{121})} = \frac{E(q^{11})^{12}}{q^{55} E(q^{121})^{12}} .$$

Hence the ξ_i are the roots of the equation

$$X^{11} - 11X^{10} + 5 \times 11X^9 - 11^2X^8 + \ldots + 11^2(11^3+8\times 11A+A^2+11B)X - \frac{E(q^{11})^{12}}{q^{55}E(q^{121})^{12}} = 0,$$

but $\xi = \xi_0$ and the Lemma is proved.

<u>5.6.</u> We are now in a position to prove Theorem (5.1.3).

From Lemma (5.5.2) and (5.5.15) we have

$$H(\xi^{-1}) \frac{E(q^{11})^{12}}{q^{55}E(q^{121})^{12}}$$

$$= H(\xi^{10}) - 11H(\xi^9) + 55H(\xi^8) - 11^2H(\xi^7) - 11^2H(\xi^6)$$

$$+ 11(11^2-2A) H(\xi^5) - 11^2(11-2A) H(\xi^4) - 11(11^3+126A + 2B) H(\xi^3)$$

$$+ 11^2(5 \times 11^2 + 38A + 2B) H(\xi^2) - 11(11^4 + 72 \times 11A - A^2 + 99B) H(\xi)$$

$$+ 11^2(11^3 + 88A + A^2 + 11B)$$

$$= 11^4 - 11(-9A^2 + 1386A - 99B + 11^4) - 5 \times 11^4 - 11^2(14B + 112A + 11^3)$$

$$- 11^4 + 11(11^2 - 2A)(10A + 11^2) + 11^3(11 - 2A) + 11^2(11^3 + 126A + 2B)$$

$$+ 11^2(5 \times 11^2 + 38A + 2B) - 11(11^4 + 72 \times 11A - A^2 + 99B)$$

$$+ 11^2(11^3 + 88A + A^2 + 11B)$$

$$= 11(11^3 + 88A + A^2 + 11B) ,$$

or

$$(5.6.1) \qquad H(\xi^{-1}) = 11(11^3 + 88A + A^2 + 11B)\, q^{55} \frac{E(q^{121})^{12}}{E(q^{11})^{12}} .$$

Now from (5.4.2) we have

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{E(q)} = \frac{\xi^{-1}}{q^5 E(q^{121})} .$$

Picking out those powers of q congruent to 6 mod 11 we obtain

$$\sum_{n\geq 0} p(11n+6)q^{11n+6} = \frac{H(\xi^{-1})}{q^5 E(q^{121})}$$

$$= 11q^{50}(11^3 + 88A + A^2 + 11B) \frac{E(q^{121})^{11}}{E(q^{11})^{12}} ,$$

$$\sum_{n\geq 0} p(11n+6)q^{11n} = 11q^{44}(11^3 + 88A + A^2 + 11B) \frac{E(q^{121})^{11}}{E(q^{11})^{12}} .$$

so that

$$(5.6.2) \qquad \sum_{n\geq 0} p(11n+6)q^n = 11q^4(11^3 + 88A^* + A^{*2} + 11B^*) \frac{E(q^{11})^{11}}{E(q)^{12}} .$$

We note here that (5.6.2) is (3.25) in Fine's paper.

Also, A^* and B^* are uniquely determined by (5.5.11) and (5.5.12).