SOME PROPERTIES OF PARTITIONS, WITH SPECIAL REFERENCE TO PRIMES OTHER THAN 5, 7 AND 11

by

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The work in this thesis
is original and entirely
my own, except where
otherwise stated.

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INTRODUCTION

The work in this thesis follows that done by Atkin and Swinnerton-Dyer [3], and Atkin and Hussain [2]. Constant reference is made to these papers, which we therefore denote by (ASD) and (AH) respectively. All unspecified notation is that of (ASD) together with the following additions.

We write

$$f(z) = \prod_{r=1}^{\infty} (1 - z^r).$$

Then

$$f(y) = P(0) \prod_{a=1}^{(q-1)/2} P(a),$$

$$f(y^q) = P(0),$$

$$1/f(x) = \sum_{n=0}^{\infty} p(n)\, x^n,$$

taking p(0) to be unity. {The above notation, with $q = 11$, is used in (AH).} Occasionally we need the congruence

$$f^q(y) \equiv f(y^q) \qquad (\text{mod.}\, q),$$

which follows from $(1 - y^r)^q \equiv 1 - y^{qr}$, modulo q. The enclosure of an ordered product of a number of variables in square brackets denotes a summation over all the <u>different</u> terms obtainable by permuting the variables cyclically in a typical term. In such a product one or more of the variables may have degree zero. Square brackets, then, replace capital sigma (which is rather an overworked symbol) as used on page 186 of (AH). It should be pointed out that in such a cyclic sum the number of terms is not necessarily the same as the number of variables. For example the following cyclic sum involving eight variables contains only two terms:

$$[a_1\ a_3\ a_5\ a_7] = a_1\ a_3\ a_5\ a_7 + a_2\ a_4\ a_6\ a_8 .$$

The symbol <b, c, d> is used to denote the following relation, proved in (ASD) (Lemma 4):

$$P^2(b)P(c+d)P(c-d)-P^2(c)P(b+d)P(b-d)+y^{c-d}P^2(d)P(b+c)P(b-c) = 0$$

if none of b, c, d, $b \pm c$, $c \pm d$, $b \pm d$, is divisible by q. Similarly <b, c, d, e> denotes the relation

$$P(b+e)P(b-e)P(c+d)P(c-d)-P(b+d)P(b-d)P(c+e)P(c-e)+$$
$$+y^{c-d}P(b+c)P(b-c)P(d+e)P(d-e) = 0$$

(none of $b \pm c$, $b \pm d$, $b \pm e$, $c \pm d$, $c \pm e$, $d \pm e$, divisible by q). The latter relation may be proved by the method used in (ASD) for the former (which is in fact <b, c, d, 0>), but is however given, in essence, in [14] {equation ($LVII_2$), page 160}. We note that either relation is homogeneous in the P(a).

The thesis is comprised of five Parts, which are to a large extent independent of one another and may in fact be read separately. The contents of these Parts are as follows.

Part 1, throughout which $q = 13$, is divided into four sections (§§ 1 to 4). In § 1 the process employed in § 11 of (AH) to express $\sum_{n=o}^{\infty} p(11n + 6)y^n$ in terms of simple functions of y is used to evaluate $\sum_{n=o}^{\infty} p(13n + 6)y^n$ in a form analogous to Ramanujan's results for $q = 5$ and $q = 7$; more elegance of method is possible in the case of $q = 13$. A secondary consequence of this process is the determination of what is in fact the simplest, non-homogeneous relation between the $P(a)$ for $q = 13$ {equation (1.17)}.* § 2 contains the evaluation of $\sum_{n=o}^{\infty} p(13n + s)y^n$, for all values of s($s = 0$ to 12) except $s = 6$, in a form which, while more complicated than for $s = 6$, involves only simple functions. In actual fact two such forms are given, but these are essentially equivalent. Simple congruences for $\sum_{n=o}^{\infty} p(13n + s)y^n$, $s = 0, 1, 2, \ldots, 12$, such as are given in (ASD) for $q = 5$, $q = 7$, and $q = 11$, are derived in § 3 from the results of § 2. A complete account

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*Neither of the two results of § 1 is new (see text), although such an elementary, algebraic method has not previously been employed. This section is due in its entirety to Dr. Atkin.

of Dyson's rank functions for the cases $q = 5$, $q = 7$, and $q = 11$, is given in (ASD) and (AH). In particular the values of the $r_{bc}(d)$ are obtained for each of these q. We find the values of the $r_{bc}(d)$ for $q = 13$ in § 4, by a method akin to that used for $q = 11$. They are of a somewhat different form than for $q = 11$ and rather more complicated, but are, on the other hand, <u>all</u> of the same nature, similar to that of the expressions given by Theorem 2.2 for $\sum_{n=0}^{\infty} p(13n + s)y^n$ $(s \neq 6)$. In the case of $q = 11$, the $r_{bc}(6)$ and the remaining $r_{bc}(d)$ have values <u>not</u> of the same nature. We note here that in Theorem 4.1, which gives the $r_{bc}(d)$ for $q = 13$, $p(0)$ must be taken to be zero {see (ASD), page 86}. It is of interest to observe that there is a set of linear congruence relations, (4.41), between the $r_{bc}(d)$ for a given value of d when $q = 13$, corresponding to (AH), equations (9.16), for $q = 11$.

Parts 2 (§ 5) and 3(§ 6) contain the evaluations of $\sum_{n=0}^{\infty} p(17n + 5)y^n$ and $\sum_{n=0}^{\infty} p(19n + 4)y^n$ respectively, again by the method used in (AH), § 11, for $\sum_{n=0}^{\infty} p(11n + 6)y^n$. In each case both the process and the result are more elegant than for $q = 11$, but less so than for $q = 13$. Simple congruences for $\sum_{n=0}^{\infty} p(17n + 5)y^n$ and $\sum_{n=0}^{\infty} p(19n + 4)y^n$ are derived from these results. The apparently simplest, non-homogeneous relation between the $P(a)$ for $q = 17$ and $q = 19$ is embodied in Theorem 5.1 (third equation) and Theorem 6.1 (fifth equation) respectively.

In part 4 (§ 7) an alternative expression for $\sum_{n=0}^{\infty} p(11n + 6)y^n$ is derived from that given in (AH) {equation (11.9)}, and we then conjecture similar expressions for $\sum_{n=0}^{\infty} p(11n + s)y^n$ ($s = 0$ to 10) when $s \neq 6$. (Such similarity does not obviously exist in the case of $q = 13$.) We make no attempt to prove our conjecture, which is almost certainly valid, in this thesis. The form of the expressions concerned is quite different from either of the forms obtained for $q = 13$ in § 2.* It is worthwhile to note that equation (7.1) is, in effect, what appears to be the simplest, non-homogeneous relation between the $P(a)$ for $q = 11$, and to pause at this point in order to state together the simplest relations for all prime q as far as $q = 19$. The relations for $q = 5$† and $q = 7$ follow immediately from [7] (Kolberg), equations (4.15) and (5.20) respectively, if, for both $q = 5$ and $q = 7$, the g_s of this paper

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* Kolberg has obtained certain expressions for $\sum_{n=0}^{\infty} p(5n+s)y^n$, $s=0, 1, 2, 3$, {[7], equations (4.17) to (4.20)}, and $\sum_{n=0}^{\infty} p(7n+s)y^n$, $s=0, 1, 2, 3, 4, 6$, {[7], equations (5.23) to (5.27), and (5.29)}. The former decomposition is due originally to Ramanujan [12].

† This relation appears in [12] (Ramanujan).

$\{$defined by $f(x) = \sum_{s=0}^{q-1} g_s x^s$, $g_s = g_s(y)\}$ are expressed in terms of the $P(a)$ by means of (ASD), Lemma 6. We have, remembering that $f(y)/f(y^q) = \prod_{a=1}^{(q-1)/2} P(a)$,

$q = 5$: $f^6(y)/f^6(y^5) = P^5(2)/P^5(1) - 11y - y^2P^5(1)/P^5(2)$,

$q = 7$: $f^4(y)/f^4(y^7) + 8y = P^2(2)P(3)/P^2(1) + yP^2(3)P(1)/P^2(2) - y^2P^2(1)P(2)/P^2(3)$,

$q = 11$: $f^2(y)/f^2(y^{11}) = P^2(5)P(4) - y^2P^2(1)P(3) - yP^2(2)P(5) - yP^2(4)P(1) - yP^2(3)P(2)$,

$q = 13$: $f^2(y)/f^2(y^{13}) = P(2)P(5)P(6)/P(1)P(3)P(4) - 3y - y^2P(1)P(3)P(4)/P(2)P(5)P(6)$,

$q = 17$: $f^2(y)/f^2(y^{17}) = P(2)P(8)P(6)P(7) - yP(6)P(7)P(1)P(4) - y^2P(1)P(4)P(3)P(5) - yP(3)P(5)P(2)P(8)$,

$q = 19$: $f(y)/f(y^{19}) = 1/P(2)P(3)P(5) - y/P(1)P(7)P(8) - y^2/P(4)P(6)P(9)$.

The results for $q = 11$, $q = 17$, and $q = 19$, seem to be new.

Parts 1 to 4 involve only elementary algebra. In Part 5 (§ 8) recourse is made to the theory of the elliptic modular functions. We show that there exists, for $q = 13$, a polynomial relation between $xf^2(y)/f^2(x)$ and $x^7f(y^{13})/f(x)$, of degrees at most 7 and at most 13 in these variables respectively.* Then, working for convenience in terms of $y^{-1}f^2(y)/f^2(y^{13})$ and $x^{-7}f(x)/f(y^{13})$ as new variables, we show by elementary algebra that the relation (of degrees at most 7 and at most 13 in the new variables respectively**) is irreducible and that the coefficients involved have, in pairs, a certain symmetry. The relation is evaluated (in terms of the new variables) by comparing coefficients of powers of x in the expansions of the quantities involved, use being made of the symmetry mentioned above to facilitate the calculation. The result could also be obtained by using the expressions for $x^{-7}f(x)/f(y^{13})$ and $y^{-1}f^2(y)/f^2(y^{13})$ in terms of the P(a) {equations (1.1) and (1.17)}, and the homogeneous relations between the P(a) previously described in this Introduction, but this would be comparatively tedious.†

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* It is in fact shown that there is a corresponding result or "modular equation" for all prime q, in which the degree of the function corresponding to $x^7 f(y^{13})/f(x)$ is at most q in the cases q = 5, q = 7, and q = 13, and is at most a greater integral multiple of q otherwise. We are indebted to Dr. Morris Newman of the National Bureau of Standards, Washington, D.C., who communicated the proof to us. The relations for q = 5 and q = 7 have been obtained, in essence, by Watson {[15], page 105, formula (3.2), and page 118, (5.2)}, although the former is due originally to Weber {[16], page 256, formula (27)}.

** The degrees are in fact 7 and 13.

† I hope to publish in the near future firstly a paper on the work of Part 5 and secondly, in conjunction with Dr. Atkin, a paper "Some properties of the coefficients of modular forms modulo powers of 13", depending upon the first.

We take this opportunity to observe that it would probably be possible to use the theory of modular functions to obtain expressions for $\sum_{n=0}^{\infty} p(17n + 5)y^n$ and $\sum_{n=0}^{\infty} p(19n + 4)y^n$ more easily than in this thesis, and indeed to obtain corresponding results for still greater values of q (this would otherwise be a very tedious matter), but that a further development of the theory would be needed.

Tabulated values of p(n) (as far as n = 1000), needed at various points in the thesis, are to be found in [5]. The table of the coefficients of powers of f(x) computed by Newman [11] is also required.

Finally, we note that a table of notation {not including that of (ASD) or (AH)} and a list of references are given at the end of the thesis. Some letters occur more than once in the text in different senses (this is purposeful where analogous processes are carried out for different values of q), but the contexts are so different as to give no danger of confusion.