PART I

q = 13 throughout this Part

1. We write

$$\alpha = -x^{-5}P(2)/P(1),\quad \beta = -x^{-6}P(6)/P(3),\quad \gamma = x^{-2}P(5)/P(4),$$

$$\alpha' = -x^{5}P(3)/P(5),\quad \beta' = x^{-7}P(4)/P(2),\quad \gamma' = x^{15}P(1)/P(6);$$

then by (ASD), Lemma 6 (with $q = 13$) we have

$$(1.1)\quad x^{-7}f(x)/f(y^{13}) = \alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1.$$

In (1.1) we replace x by $\omega_r x$ where ω_r $(r = 1$ to $13)$ are the thirteenth roots of unity, and multiply together the thirteen resulting equations, obtaining

$$(1.2)\quad y^{-7}f^{14}(y)/f^{14}(y^{13}) = \prod_{r=1}^{13}(\alpha\omega_r^{-5}+\beta'\omega_r^{-7}+\gamma\omega_r^{-2}+\alpha'\omega_r^{5}+\beta\omega_r^{-6}+\gamma'\omega_r^{15}+1).$$

Now as ω_r runs through the thirteenth roots of unity so does ω_r^{-3}, so that the product on the right-hand side of (1.2) is equal to

$$\prod_{r=1}^{13}(\alpha\omega_r^{15}+\beta'\omega_r^{-5}+\gamma\omega_r^{-7}+\alpha'\omega_r^{-2}+\beta\omega_r^{5}+\gamma'\omega_r^{-6}+1),$$

and is thus unchanged if α, β', γ, α', β, and γ', are interchanged cyclically. The product is thus a linear combination of terms $[\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}]$ where i_1 to i_6 are non-negative integers, and considering the left-hand side of (1.2) such terms as occur can only involve x in terms of $y = x^{13}$. Thus if $\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}$ (or any other term of $[\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}]$) occurs we must have

$$(1.3)\quad -5i_1 - 7i_2 - 2i_3 + 5i_4 - 6i_5 + 15i_6 \equiv 0 \pmod{13}$$

(interchanging i_1, i_2, i_3, i_4, i_5, and i_6, cyclically gives the same congruence).

Now, writing

$$a = y^2P^2(1)/P(4)P(5), \qquad a' = y^{-1}P^2(5)/P(6)P(1),$$
$$b = -y^{-1}P^2(3)/P(1)P(2), \qquad b' = -yP^2(2)/P(5)P(3),$$
$$c = -P^2(4)/P(3)P(6), \qquad c' = y^{-1}P^2(6)/P(2)P(4),$$

it is easily verified that

$$\alpha^{13} = b'^{12}\, c^2\, a'^6\, b^7\, c'^4, \qquad \alpha'^{13} = b^{12}\, c'^2\, a^6\, b'^7\, b^4,$$
$$(1.4) \qquad \beta^{13} = c'^{12}\, a^2\, b'^6\, c^7\, a'^4, \qquad \beta'^{13} = c^{12}\, a'^2\, b^6\, c'^7\, a^4,$$
$$\gamma^{13} = a'^{12}\, b^2\, c'^6\, a^7\, b'^4, \qquad \gamma'^{13} = a^{12}\, b'^2\, c^6\, a'^7\, b^4.$$

It will be noticed that all of the equations (1.4) may be obtained from any one of them by interchanging a, b', c, a', b, c', and α, β', γ, α', β, γ', cyclically. By (1.4), since $ab'\,ca'\,bc' = -1$,

$$(\alpha^{i_1}\,\beta'^{i_2}\,\gamma^{i_3}\,\alpha'^{i_4}\,\beta^{i_5}\,\gamma'^{i_6})^{13} =$$
$$= (ab'\,ca'\,bc')^{\sigma}\, a^{\sigma_1}\, b'^{\sigma_2}\, c^{\sigma_3}\, a'^{\sigma_4}\, b^{\sigma_5}\, c'^{\sigma_6}$$

where $\sigma = 2i_1 + 4i_2 + 12i_3 + 18i_4 + 16i_5 + 8i_6$, an even integer, and

$$\sigma_1 = 4i_2+7i_3+6i_4+2i_5+12i_6, \qquad \sigma_2 = 4i_3+7i_4+6i_5+2i_6+12i_1,$$
$$\sigma_3 = 4i_4+7i_5+6i_6+2i_1+12i_2, \qquad \sigma_4 = 4i_5+7i_6+6i_1+2i_2+12i_3,$$
$$\sigma_5 = 4i_6+7i_1+6i_2+2i_3+12i_4, \qquad \sigma_6 = 4i_1+7i_2+6i_3+2i_4+12i_5;$$

moreover $\sigma + \sigma_1$ to $\sigma + \sigma_6$ are multiples of 13 by (1.3), hence we arrive at the following:

LEMMA 1.1. Any expression of the form $\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}$ for which (1.3) holds is of the form $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$ where j_1 to j_6 are non-negative integers.

By Lemma 1.1 every term occurring in the right-hand side of (1.2) is of the form $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$, and such terms occur in cyclically symmetrical sets of six terms each.

Further, $\Phi(6)$ is the coefficient of x^6 in $1/f(x)$ regarded as a polynomial of degree 12 in x with coefficients involving x in terms of $y = x^{13}$, so that $y^{-6}f^{14}(y)\Phi(6)/f^{13}(y^{13})$ is the coefficient of x^0 in $y^{-7}f^{14}(y)/\{f^{14}(y^{13})(\alpha+\beta'+\gamma+\alpha'+\beta+\gamma'+1)\}$. This is a cyclically symmetric polynomial of degree 12 in α, β', γ, α', β, and γ'; and the terms which give the coefficient of x^0 occur only in symmetrical sets of six expressible as $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$, as before. (This is not true for the coefficient of any power of x other than 0; the six terms of [a], for example, do not appertain to the same power of x.)

Thus $y^{-7}f^{14}(y)/f^{14}(y^{13})$ and $y^{-6}f^{14}(y)\Phi(6)/f^{13}(y^{13})$ are each equal to a linear combination of terms $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$. We now write

$A = yP(2)P(3)/P(4)P(6)$, $\quad B = -y^{-1}P(4)P(6)/P(1)P(5)$,

$C = -P(1)P(5)/P(2)P(3)$; $\quad K = yP(1)P(3)P(4)/P(2)P(5)P(6)$.

Then

(1.5) $ABC = 1.$

<4, 2, 1>, <6, 3, 1>, <5, 4, 3>, <6, 5, 3>, <5, 4, 2>, and <6, 2, 1>, give, respectively,

(1.6) to (1.8) $a = A - K, \quad b = B - K, \quad c = C - K,$

(1.9) to (1.11) $a' = A + 1/K, \ b' = B + 1/K, \ c' = C + 1/K;$

all of the equations (1.6) to (1.11) may be obtained from any one of them by interchanging a, b', c, a', b, c', and A,B,C, and 1/K - K, cyclically. Also, <5, 3, 2, 1> gives

(1.12) to (1.14) $AB + A + 1 = 0, \quad BC + B + 1 = 0, \quad CA + C + 1 = 0,$

which equations are equivalent by virtue of (1.5), and <5, 2, 1> gives

$$a + b' = CA,$$

which using (1.6), (1.10), and (1.12) to (1.14), becomes

(1.15) $A + B + C = -1/K + K - 1,$

(1.16) $AB + BC + CA = 1/K - K - 2.$

We are now in a position to prove

LEMMA 1.2 Any expression of the form $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$ is equal to a polynomial in 1/K - K with integral coefficients.

Using (1.6) to (1.11), any $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$ can be expressed as a polynomial in A, B, C, 1/K, and -K, with integral coefficients, cyclically symmetric in A, B, C, and 1/K, - K. This polynomial is a linear combination of terms

$\{(1/K)^h + (-K)^h\}[A^\lambda\ B^\mu\ C^\nu]$ where h, λ, μ, and ν, are non-negative integers, for if a term $(1/K)^h[A^\lambda\ B^\mu\ C^\nu]$ occurs so does the term $(-K)^h[A^\lambda\ B^\mu\ C^\nu]$, and vice versa. Further, by Newton's formula for sums of powers of the roots of a polynomial equation in one variable, $(1/K)^h + (-K)^h$ can be expressed as a polynomial in the coefficients of the quadratic equation $z^2 - (1/K - K)z - 1 = 0$ having roots $1/K$ and $-K$, i.e. as a polynomial in $1/K - K$ with integral coefficients. We now assert that any $[A^\lambda\ B^\mu\ C^\nu]$ is also equal to a polynomial in $1/K - K$ with integral coefficients. Assume that this is true for all values of λ, μ, and ν, with $\lambda + \mu + \nu \leq \tau$ where $\tau \geq 1$, and consider any $[A^\lambda\ B^\mu\ C^\nu]$ with $\lambda + \mu + \nu = \tau + 1$. If any two of λ, μ, and ν, are non-zero we can express $[A^\lambda\ B^\mu\ C^\nu]$ as a linear combination of similar sums with $\lambda + \mu + \nu \leq \tau$ by using (1.12) to (1.14); and so by the induction hypothesis it is equal to a polynomial in $1/K - K$ with integral coefficients. Also, using Newton's formula, $[A^\lambda]$ can be expressed as a polynomial in $1/K - K$ with integral coefficients, by (1.5), (1.15), and (1.16).

Thus if our assertion is true for $\lambda + \mu + \nu \leq \tau$ it is true for all λ, μ, and ν, with $\lambda + \mu + \nu = \tau + 1$; but it is clearly true for $\tau = 1$, hence it is true for all values of τ by the strong form of mathematical induction. This completes the proof of Lemma 1.2.

Writing

$$F = y^{-1}f^2(y)/f^2(y^{13}),$$

we have shown that F^7 is equal to a linear combination of terms $[a^{j_1} b'^{j_2} c^{j_3} a'^{j_4} b^{j_5} c'^{j_6}]$, and hence, by Lemma 1.2, to a polynomial in $1/K - K$ with integral coefficients. Further, this polynomial is of degree 7 since the lowest powers of y in the expansions of F^7 and $1/K - K$ as ascending power series in y are -7 and -1 respectively. By comparing coefficients of powers of y as far as y^0 we find that

$$F^7 = (1/K - K - 3)^7,$$

or, since F and K are real for real y,

(1.17) $$F = 1/K - K - 3.^*$$

Similarly $y^{-6}f^{14}(y)\,\Phi(6)/f^{13}(y^{13})$ is equal to a polynomial of degree 6 in $1/K-K$ with integral coefficients, or by (1.17), in F. Comparing coefficients as far as y^0 we find that

(1.18) $$yf(y^{13})\,\Phi(6) = 11/F + 36.13/F^2 + 38.13^2/F^3 + + 20.13^3/F^4 + 6.13^4/F^5 + 13^5/F^6 + 13^5/F^7$$

on dividing through by F^7. (1.18) was first found by Zuckermann [17], using the theory of the elliptic modular functions.

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* Dr. Atkin points out that this identity is given (in a different notation) on page 326 of [13] (Ramanujan).