2. We shall now find expressions for all the $\Phi(s)$ $(0 \leqslant s \leqslant 12,\ s \neq 6)$. Consider $\Phi(1)$. $y^{-7}f^{14}(y)\Phi(1)/f^{13}(y^{13})$ is the coefficient of x^8 in $y^{-7}f^{14}(y)/\{f^{14}(y^{13})(\alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1)\}$, a cyclically symmetric polynomial in α, β', γ, α', β, and γ'. Thus $y^{-7}f^{14}(y)\Phi(1)x^8/ - \alpha f^{13}(y^{13})$ is the coefficient of x^0 in a polynomial in α, β', γ, α', β, and γ', which although not cyclically symmetric, is a linear combination of terms $\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}$ (the indices here may be presumed non-negative because $-1/\alpha = \beta'\gamma\ \alpha'\ \beta\ \gamma'$), also, for any such term which occurs in the coefficient of x^0, (1.3) must hold. Hence, by Lemma 1.1, $y^{-6}f^{14}(y)P(1)\Phi(1)/f^{13}(y^{13})P(2)$ is equal to a linear combination of terms $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$. We define $\phi(s)$, the "normalised" form of $\Phi(s)$, in the following six cases:

$$\phi(1) = P(1)\ \Phi\ (1)/P(2),$$
$$\phi(12) = -yP(2)\ \Phi\ (12)/P(4),$$
$$\phi(4) = -P(4)\ \Phi\ (4)/P(5),$$
$$\phi(11) = P(5)\ \Phi\ (11)/P(3),$$
$$\phi(0) = P(3)\ \Phi\ (0)/P(6),$$
$$\phi(8) = -y^{-1}P(6)\ \Phi\ (8)/P(1).$$

Then we have shown that $yf(y^{13})\phi(1)F^7$ is equal to a linear combination of terms $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$. We can show, in a similar manner, that this is true if $\phi(1)$ is replaced by $\phi(s)$ for $s = 12, 4, 11, 0$, or 8, if we replace the multiplier $-1/\alpha$ by $-1/\beta'$, $-1/\gamma$, $-1/\alpha'$, $-1/\beta$, or $-1/\gamma'$, respectively. Further, given an expression for any $\phi(s)$ in the above list, we may obtain any other such $\phi(s)$ by interchanging the $\phi(s)$ (in the above order) and a, b', c, a', b, c', cyclically.

We define $\phi(s)$ in the remaining six cases as follows:

$$\phi(10) = P(3)\,\Phi(10)/P(2),$$

$$\phi(9) = -P(6)\,\Phi(9)/P(4),$$

$$\phi(5) = -yP(1)\,\Phi(5)/P(5),$$

$$\phi(2) = -P(2)\,\Phi(2)/P(3),$$

$$\phi(3) = P(4)\,\Phi(3)/P(6),$$

$$\phi(7) \doteq y^{-1}P(5)\,\Phi(7)/P(1).$$

We may show that the above result holds for these $\phi(s)$ by considering $y^{-7} f^{14}(y)/\{f^{14}(y^{13})(\alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1)\}$ multiplied by $\beta'\gamma\alpha'$, $\gamma\alpha'\beta$, $\alpha'\beta\gamma'$, $\beta\gamma'\alpha$, $\gamma'\alpha\beta'$, and $\alpha\beta'\gamma$, instead of $-1/\alpha$, $-1/\beta'$, $-1/\gamma$, $-1/\alpha'$, $-1/\beta$, and $-1/\gamma'$.

Thus we must now examine a $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$, rather than $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$. To do this we need certain preliminary results. Using (1.17), (1.15) can be written as

$$A + B + C + F + 4 = 0. \qquad (2.1)$$

Multiplying this equation by A, substituting for AB and CA from (1.12) and (1.14), and transposing we obtain

$$C = A^2 + (F + 3)A - 2. \tag{2.2}$$

Substituting this expression for C in (2.1), and transposing we have

$$B = -A^2 - (F + 4)A - F - 2. \tag{2.3}$$

Also, (1.17) can be written in the form

$$-K = -1/K + F + 3. \tag{2.4}$$

Thus, by virtue of (2.2), (2.3), and (2.4), any polynomial in A, B, C, 1/K, and -K, with integral coefficients, can be expressed as a polynomial in A, 1/K, and F, also with integral coefficients. Further, multiplying (2.3) by A, substituting for AB from (1.12), and transposing we obtain

$$A^3 = -(F + 4)A^2 - (F + 1)A + 1, \tag{2.5}$$

and, multiplying (2.4) by 1/K, and transposing we have

$$(1/K)^2 = (F + 3)/K + 1. \tag{2.6}$$

So, by virtue of (2.5) and (2.6), any polynomial in A, 1/K, and F, with integral coefficients, can be expressed as a linear combination of terms

$$F^h(e_1A^2/K + e_2A^2 + e_3A/K + e_4A + e_5/K + e_6) \tag{2.7}$$

where h is a non-negative integer and e_1 to e_6 are positive, negative, or zero, integers. We conclude that any polynomial in A, B, C, 1/K, and -K, with integral coefficients, is equal to a linear combination of terms (2.7).

We note here that by (1.5), (1.15), (1.16), and (1.17), A, B, and C, are the roots of the cubic equation

$$(2.8) \qquad z^3 + (F + 4)z^2 + (F + 1)z - 1 = 0;$$

that by (1.17), 1/K and -K are the roots of the quadratic equation

$$(2.9) \qquad z^2 - (F + 3)z - 1 = 0;$$

and that (2.5) and (2.6) follow from (2.8) and (2.9) respectively.

Now, using (1.6) to (1.11) any $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$ can be expressed as a polynomial in A, B, C, 1/K, and -K, with integral coefficients. Thus we arrive at

LEMMA 2.1 Any expression of the form $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$ is equal to a linear combination of terms (2.7). This statement remains valid if in (2.7) A is replaced by any one of A, B, C, and 1/K is replaced by either of 1/K, -K.

The latter sentence follows because of the cyclic properties of our relations.

We note that if we <u>define</u> F by (1.17) then Lemma 1.2 is a consequence of Lemma 2.1, for by Lemma 2.1 any $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$ is expressible as a linear combination of terms

$$F^h\{e_1(1/K-K)[A^2]+2e_2[A^2]+e_3(1/K-K)[A]+2e_4[A]+3e_5(1/K-K)+6e_6\},$$

and any such term, in view of (1.15), (1.16), and (1.17), is equal to a polynomial in 1/K - K with integral coefficients.

Now, we have shown that $yf(y^{13})\phi(1)F^7$ is equal to a linear combination of terms $a^{j_1}b^{j_2}c^{j_3}a'^{j_4}b'^{j_5}c'^{j_6}$, and hence by Lemma 2.1, to a linear combination of terms (2.7) where, for a reason which will appear in §3, we choose to replace A and 1/K by C and -K respectively. Also, given $\phi(1)$ in terms of C and -K we obtain all the $\phi(s)$ (s = 1, 12, 4, 11, 0, 8) immediately by interchanging $\phi(s)$ (in the order given), and A, B, C, and 1/K, -K, cyclically. We have exactly the same situation for the other six $\phi(s)$ (s = 10, 9, 5, 2, 3, 7) where, again for a reason which will appear in §3, we choose to express $\phi(10)$ in terms of C and -K. Thus if for each of the twelve values of s we choose variables from A, B, C, and 1/K, -K, according to the following tables

s	0	1	4	8	11	12
	A	C	B	B	C	A
	-K	-K	-K	1/K	1/K	1/K

Table 2.1

s	2	3	5	7	9	10
	C	A	B	B	A	C
	1/K	-K	-K	1/K	1/K	-K

Table 2.2

then $yf(y^{13})\phi(s)F^7$ is equal to a linear combination of terms (2.7) in each of which A and 1/K are replaced by variables appropriate to the particular value of s, and for each value of h the coefficients e_1 to e_6 in (2.7) are the same for all the s of one group of six. We find the values of e_1 to e_6

(for each value of h occurring) in the two distinct cases by comparing coefficients, as before.

Consider the case to which Table 2.1 applies. Let H be the highest value of h occurring, i.e. the highest value of h for which e_1 to e_6 are not all zero. Then $yf(y^{13})\phi(12)F^7$ is (without loss of generality) the sum of terms (2.7) with $0 \leqslant h \leqslant H$. Now, since A and $1/K$ (expanded as ascending power series in y) begin $y + \ldots$ and $y^{-1} + \ldots$ respectively, the lowest power of y occurring in the bracket of (2.7) is -1, and it occurs in the term e_5/K (and in none of the other five terms as it happens). Thus, writing E_1 to E_6 for the e_1 to e_6 appertaining to $h = H$, the lowest power of y in the aggregate of terms (2.7) is $-(H + 1)$ (since F begins $y^{-1}+\ldots$), and it occurs in the term $F^H E_5/K$ (only); but $yf(y^{13})\phi(12)F^7$ begins $-77y^{-5}+\ldots$, hence $E_5 = 0$ if $H + 1 > 5$. Applying this argument to all of the six $\phi(s)$, using the variables indicated in Table 2.1 in each case, we obtain (from s = 0, 1, 4, 8, 11, and 12, respectively)

$$E_6 = 0 \text{ if } H > 6,$$
$$E_2 - E_4 + E_6 = 0 \text{ if } H > 6,$$
$$E_2 = 0 \text{ if } H > 4,$$
$$E_1 = 0 \text{ if } H > 4,$$
$$E_1 - E_3 + E_5 = 0 \text{ if } H > 5,$$
$$E_5 = 0 \text{ if } H > 4;$$

when $s = 1$, or 11, $yf(y^{13})\phi(s)F^7$ is equal to an expression in which the lowest power of y occurs in three terms of the bracket prefixed by F^H. Thus if $H > 6$, E_1 to E_6 (found seriatim) are all zero, but this contradicts the definition of H, hence $H \leqslant 6$. We need only to notice that, from the case $s = 0$ above, $E_6 \neq 0$ if $H = 6$, to conclude that in fact $H = 6$.

It may be shown, by similar reasoning, that for the other group of $\phi(s)$, H is again 6.

For each group of $\phi(s)$ then we need to find the coefficients e_1 to e_6 for each h in the range $0 \leqslant h \leqslant 6$. Comparing coefficients of powers of y for the first 7 powers of y occurring in the expression for $yf(y^{13})\phi(s)F^7$ (for each s of the group in question) we obtain 42 equations relating the 42 unknown coefficients. It turns out that these equations are sufficient to determine the coefficients, in fact, in each of the two cases, the coefficients appear seriatim.

We state the results* in the form:

THEOREM 2.1 We have

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* In actual fact we checked the values of the coefficients found, in both cases, by comparing the coefficients of the eighth lowest power of y for $s = 8$ and $s = 7$.

$$yf(y^{13})\phi(12)=1/F \quad +(-56A/K-33A-1/K+99)/F^2+$$
$$+13(-6A^2/K-3A^2-109A/K-31A-9/K+159)/F^3+$$
$$+13^2(-11A^2/K-4A^2-85A/K-16A-11/K+105)/F^4+$$
$$+13^3(-7A^2/K-3A^2-34A/K-5A-5/K+37)/F^5+$$
$$+13^4(-2A^2/K-A^2-7A/K-A-1/K+7)/F^6+$$
$$+13^4(-3A^2/K-2A^2-8A/K-A-1/K+8)/F^7,$$

$$yf(y^{13})\phi(9)=(-39A+3)/F+(-39A^2+11A/K-985A-33/K+264)/F^2+$$
$$+13(2A^2/K-67A^2+13A/K-786A-83/K+348)/F^3+$$
$$+13^2(4A^2/K-46A^2+10A/K-334A-68/K+210)/F^4+$$
$$+13^3(3A^2/K-16A^2+4A/K-82A-28/K+68)/F^5+$$
$$+13^4(A^2/K-3A^2+A/K-11A-6/K+12)/F^6+$$
$$+13^4(2A^2/K-3A^2+A/K-8A-8/K+12)/F^7,$$

and these equations still hold if $\phi(12)$ or $\phi(9)$ is replaced by $\phi(s)$ for values of s occurring in Table 2.1 or Table 2.2 respectively provided that A is replaced by A, B, or C, and 1/K is replaced by 1/K or -K, according to these tables.

It is interesting to compare the powers of 13 occurring in the equations of this theorem with those occurring in the expression for $yf(y^{13})\Phi(6)$ given in (1.18).

We proceed to derive an alternative form of Theorem 2.1. Writing

$$l = y^2P(3)/P(6)P(5),\quad m = yP(4)/P(5)P(2),\quad n=-y^2P(1)/P(2)P(6),$$
$$l'= yP(2)/P(4)P(1),\quad m'=P(6)/P(1)P(3),\quad n'=-yP(5)/P(3)P(4),$$

we have immediately, from the definitions of A, B, C, and K,

(2.10) $\quad l/l' = m/m' = n/n' = K,$

which equations will be used without explicit mention, and

(2.11) to (2.13) $\quad l/m = A, \quad m/n = B, \quad n/l = C.$

We note that equations (2.10) do <u>not</u> remain valid if $1/K$, $-K$, and l, m', n, l', m, n', are interchanged cyclically, but that (2.10) to (2.13) all remain valid if A, B, C, and $1/K$, $-K$, are interchanged cyclically and l, m', n, l', m, and n', are interchanged according to either

(2.14) $\quad \begin{pmatrix} l & m' & n & l' & m & n' \\ m' & -n & l' & -m & n' & -l \end{pmatrix}$

or

(2.15) $\quad \begin{pmatrix} l & m' & n & l' & m & n' \\ -m' & n & -l' & m & -n' & l \end{pmatrix}.$

Substituting for A, B, and C, from (2.11) to (2.13), in (1.12) to (1.14) we obtain in each case

(2.16) $\quad 1/l + 1/m + 1/n = 0.$

Similarly (2.1) becomes

(2.17) $\quad l/m + m/n + n/l + F + 4 = 0.$

Now, (2.16) may be written as

(2.18) $\quad lm/n = -l - m,$

and (2.17) as

$$l^2/m = -lm/n - Fl - 4l - n$$

which using (2.18) becomes

(2.19) $\quad l^2/m = -Fl - 3l + m - n,$

and using (2.11) this equation may be written as

(2.20) $$mA^2 = -Fl - 3l + m - n$$

or, dividing through by K,

(2.21) $$mA^2/K = -Fl' - 3l' + m' - n'.$$

Also we have trivially from (2.11)

(2.22), and (2.23) $$mA = l, \quad mA/K = l'.$$

So, multiplying the first equation of Theorem 2.1 by m, and substituting for mA^2, mA^2/K, mA, and mA/K, from (2.20) to (2.23), we obtain $yf(y^{13})m\phi(12)$ as a sum of terms

(2.24) $$F^h(e_1' l + e_2' m' + e_3' n + e_4' l' + e_5' m + e_6' n').$$

We chose to take m with $\phi(12)$ for a reason which will appear in § 3. Now we have seen that the first equation of Theorem 2.1 still holds if we interchange $\phi(1)$, $\phi(12)$, $\phi(4)$, $\phi(11)$, $\phi(0)$, $\phi(8)$, and A, B, C, and $1/K$, $-K$, cyclically. Hence the above equation for $\phi(12)$ still holds if we interchange these $\phi(s)$ cyclically, and interchange l, m', n, l', m, and n', according to (2.14) or (2.15). We obtain a similar result for the other six $\phi(s)$ by multiplying the second equation of Theorem 2.1 by m. Thus multiplying $\phi(s)$ by l', m, n', l, m', and n, when s = 1, 12, 4, 11, 0, and 8, or 10, 9, 5, 2, 3, and 7, respectively, and denoting the result by $\phi'(s)$, so that

(2.25)
$$\phi'(1) = y\Phi(1)/P(4), \quad \phi'(10) = yP(3)\Phi(10)/P(4)P(1),$$
$$\phi'(12) = -y^2\Phi(12)/P(5), \quad \phi'(9) = -yP(6)\Phi(9)/P(5)P(2),$$
$$\phi'(4) = y\Phi(4)/P(3), \quad \phi'(5) = y^2P(1)\Phi(5)/P(3)P(4),$$
$$\phi'(11) = y^2\Phi(11)/P(6), \quad \phi'(2) = -y^2P(2)\Phi(2)/P(6)P(5),$$

$$\phi'(0) = \Phi(0)/P(1), \qquad \phi'(3) = P(4)\Phi(3)/P(1)P(3),$$

$$\phi'(8) = y\Phi(8)/P(2), \qquad \phi'(7) = -yP(5)\Phi(7)/P(2)P(6),$$

we may re-state Theorem 2.1 in the form:

THEOREM 2.2 We have

$$\begin{aligned} yf(y^{13})\phi'(12) = m/F + (&6l - m' + 22l' + 99m)/F^2 + \\ +13(&30l - 15m' + 3n + 52l' + 156m + 6n')/F^3 + \\ +13^2(&35l - 22m' + 4n + 39l' + 101m + 11n')/F^4 + \\ +13^3(&17l - 12m' + 3n + 13l' + 34m + 7n')/F^5 + \\ +13^4(&4l - 3m' + n + 2l' + 6m + 2n')/F^6 + \\ +13^4(&5l - 4m' + 2n + l' + 6m + 3n')/F^7, \end{aligned}$$

$$\begin{aligned} yf(y^{13})\phi'(9) = 3m/F + (&3l - 33m' + 39n - 15l' + 225m)/F^2 + \\ +13(&13l - 81m' + 67n - 45l' + 281m - 2n')/F^3 + \\ +13^2(&12l - 64m' + 46n - 41l' + 164m - 4n')/F^4 + \\ +13^3(&5l - 25m' + 16n - 18l' + 52m - 3n')/F^5 + \\ +13^4(&l - 5m' + 3n - 4l' + 9m - n')/F^6 + \\ +13^4(&l - 6m' + 3n - 5l' + 9m - 2n')/F^7, \end{aligned}$$

and these equations still hold if $\phi'(12)$ or $\phi'(9)$ is replaced by $\phi'(s)$ for values of s occurring in the first or the second row of the following table respectively provided that l, m', n, l', m, and n', are interchanged according to this table:

s	1	12	4	11	0	8
s	10	9	5	2	3	7
	n'	l	m'	n	l'	m
	$-l$	m'	$-n$	l'	$-m$	n'
	m'	n	l'	m	n'	l
	$-n$	l'	$-m$	n'	$-l$	m'
	l'	m	n'	l	m'	n
	$-m$	n'	$-l$	m'	$-n$	l'

We emphasise that for any particular value of s the equation given in Theorem 2.2 is simply the equation given in Theorem 2.1 multiplied by l, m', n, l', m, or n'; the former equation, of degree 0 in the P(a), becomes an equation of degree −1 in the P(a). Although in Theorem 2.1 each $\Phi(s)$ is expressed in terms of only two variables, such as A and $1/K$, the two variables are different for different values of s. In Theorem 2.2 six variables are needed, but they are the same for all the $\Phi(s)$, and moreover, unlike Theorem 2.1, the expressions are homogeneous in these variables.

3. In this paragraph all congruences are modulo 13. We state and prove:

THEOREM 3.1 We have

$\Phi(0) \equiv 6P(6)\Phi(6)/P(3)-5yP(0)/P(5)$,

$\Phi(1) \equiv 6P(2)\Phi(6)/P(1)+2yP(0)/P(6)$,