s	1	12	4	11	0	8
s	10	9	5	2	3	7
	n'	1	m'	n	1'	m
	-1	m'	-n	1'	-m	n'
	m'	n	1'	m	n'	1
	-n	1'	-m	n'	-1	m'
	1'	m	n'	1	m'	n
	-m	n'	-1	m'	-n	1'

We emphasise that for any particular value of s the equation given in Theorem 2.2 is simply the equation given in Theorem 2.1 multiplied by 1, m', n, 1', m, or n'; the former equation, of degree 0 in the P(a), becomes an equation of degree -1 in the P(a). Although in Theorem 2.1 each $\Phi(s)$ is expressed in terms of only two variables, such as A and 1/K, the two variables are different for different values of s. In Theorem 2.2 six variables are needed, but they are the same for all the $\Phi(s)$, and moreover, unlike Theorem 2.1, the expressions are homogeneous in these variables.

3. In this paragraph all congruences are modulo 13. We state and prove:

THEOREM 3.1 We have

$\Phi(0) \equiv 6P(6)\Phi(6)/P(3)-5yP(0)/P(5)$,

$\Phi(1) \equiv 6P(2)\Phi(6)/P(1)+2yP(0)/P(6)$,

$$\Phi(2) \equiv -5P(3)\Phi(6)/P(2)+5P(0)P(5)/P(2)P(4),$$
$$\Phi(3) \equiv 5P(6)\Phi(6)/P(4)+4yP(0)P(3)/P(4)P(5),$$
$$\Phi(4) \equiv -6P(5)\Phi(6)/P(4)+6P(0)/P(2),$$
$$\Phi(5) \equiv -5y^{-1}P(5)\Phi(6)/P(1)+3y^{-1}P(0)P(4)/P(1)P(2),$$
$$\Phi(6) \equiv -2P(0)/f^2(y),$$
$$\Phi(7) \equiv 5yP(1)\Phi(6)/P(5)+2P(0)P(6)/P(3)P(5),$$
$$\Phi(8) \equiv -6yP(1)\Phi(6)/P(6)-4P(0)/P(3),$$
$$\Phi(9) \equiv -5P(4)\Phi(6)/P(6)-6P(0)P(2)/P(1)P(6),$$
$$\Phi(10) \equiv 5P(2)\Phi(6)/P(3)+yP(0)P(1)/P(3)P(6),$$
$$\Phi(11) \equiv 6P(3)\Phi(6)/P(5)+3P(0)/P(4),$$
$$\Phi(12) \equiv -6y^{-1}P(4)\Phi(6)/P(2)+y^{-1}P(0)/P(1).$$

We note that the form of these congruences is analogous to that of the corresponding results for q = 5, 7, and 11, given as Theorems 1, 2, and 3, in (ASD). There is a basic difference only in so far as $\Phi_{13}(6) \not\equiv 0$.

Now, the congruence for $\Phi(6)$ follows immediately from (1.18) $\{$since $f(y^{13})=P(0)\}$. Substituting for $\phi'(12)$ from (2.25) in the first equation of Theorem 2.2 we obtain

$$-y^3 f(y^{13})\Phi(12)/P(5) \equiv m/F + (6\ 1-m' + 22\ 1' + 99m)/F^2,$$

which may be written in the form

$$\Phi(12) \equiv -y^{-1}\frac{P(4)}{P(2)}\frac{P(0)}{f^2(y)} - \frac{y}{P(1)}\frac{P^3(0)}{f^4(y)}\frac{6\ 1-m'+22\ 1'+99m}{m\cdot 1'}.$$

Thus, comparing the congruence for $\Phi(12)$ in the theorem with this congruence $\{$using the congruence for $\Phi(6)\}$, we see that the former is valid if

$$y^{-2}f^4(y)/P^2(0) \equiv 1/l - 6/m' - 99/l' - 22/m$$

which equation may be written as

(3.1) $y^{-2}f^4(y)/P^2(0) \equiv -5/l + 3/m' - 6/n + 1/l' - 2/m - 4/n'$,

using (2.16) and (2.16) multiplied through by K. By a similar argument we may show that for each of the other five s of the group containing s = 12 the validity of the congruence in the theorem depends only on the validity of (3.1) multiplied through by some constant. Further, for the remaining six s we find, using the preceding process, that to prove the congruences in the theorem we need again only to show that (3.1) holds. We prove (3.1) as follows.

Writing

$$X = -5/l - 6/n - 2/m$$

we have, multiplying through by l and using (2.11) and (2.12),

$$lX = -5 - 6AB - 2A$$

which using (1.12) becomes

(3.2) $lX = 4A + 1.$

Similarly we may obtain

(3.3) $nX = - 3C - 4,$

(3.4) $mX = - B + 3.$

Multiplying together the last three equations we have

$$lnmX^3 \equiv - ABC + 3[AB] + 4[A] + 1,$$

and by (1.5), (1.15), (1.16), and (1.17), the right-hand side of this equation is congruent to $-F$ so that, squaring both sides of the equation,

$$l^2 n^2 m^2 X^6 \equiv y^{-2} f^4(y)/P^4(0);$$

but from the definitions of l, n, m, and K,

$$l^2 n^2 m^2 = y^7 P(0) K^3 / f(y),$$

hence

$$X^6 \equiv y^{-9} f^5(y)/P^5(0) K^3,$$

or since $f^{13}(y) \equiv P(0)$

$$X^2 \equiv y^{-3} f^6(y)/P^2(0) K,$$

where the value of the coefficient of the lowest power of y in the expansion of each side of this equation is examined to determine the appropriate root. By virtue of (1.17) we may write the last equation in the form

$$X^2 \equiv y^{-2} f^4(y)(1/K + 5)^2,$$

whence

(3.5) $\quad X \equiv y^{-1} f^2(y)(1/K + 5),$

where the sign of the coefficient of the lowest power of y on each side of this equation is examined to determine the appropriate root. Now, the right-hand side of (3.1) is congruent to $(5K + 1)(-5/l - 6/n - 2/m)$, i.e. to $(5K + 1)X$, and by (3.5) this is congruent to $y^{-1} f^2(y)(1/K - K - 3)$ which equals $y^{-2} f^4(y)/P^2(0)$ by (1.17). Thus (3.1) holds. This completes the proof of the theorem.

It would be possible to prove Theorem 3.1 by either of the methods used to prove Theorems 1 and 2, and Theorem 3, in (ASD). Indeed the congruences of Theorem 3.1 were originally derived from other more complicated congruences which were found by Dr. Atkin using the method of Theorems 1 and 2. It is because the above congruences for the $\Phi(s)$ were discovered before the identities given by Theorems 2.1 and 2.2 that I was able to assign convenient variables to particular $\Phi(s)$ for the purpose of these two theorems.

4. The values of the $r_{bc}(d)$ for $q = 11$ proved in (AH) were actually found empirically; for $q = 13$ we use a similar method.

Putting $b = 6, 5, 4, 3, 2, 1$, and 0, in equation (6.2) of (ASD) (with $q = 13$), and $b = 0$ and 3 in equation (6.3) of (ASD), we obtain respectively

(4.1)
$$S(6) = 0, \quad S(7) = -S(5), \quad S(8) = -S(4),$$
$$S(9) = -S(3), \quad S(10) = -S(2), \quad S(11) = -S(1),$$
$$S(12) = -S(0), \quad S(13) = -f(x)+S(0)+1, \quad S(16) = x^{-2}f(x)+S(3)+1,$$

and it is easily seen that there are essentially only six distinct $S(b)$, which we take to be $S(0)$ to $S(5)$.

We write

$$N_b = N_b(x) = \sum_{n=0}^{\infty} N(b, 13, n)x^n,$$

$$N_{bc} = N_b - N_c,$$

so that by (6.10) of (ASD)