the methods used to prove Theorems 1 and 2, and Theorem 3, in (ASD). Indeed the congruences of Theorem 3.1 were originally derived from other more complicated congruences which were found by Dr. Atkin using the method of Theorems 1 and 2. It is because the above congruences for the $\Phi(s)$ were discovered before the identities given by Theorems 2.1 and 2.2 that I was able to assign convenient variables to particular $\Phi(s)$ for the purpose of these two theorems.

4. The values of the $r_{bc}(d)$ for $q = 11$ proved in (AH) were actually found empirically; for $q = 13$ we use a similar method.

Putting $b = 6, 5, 4, 3, 2, 1$, and 0, in equation (6.2) of (ASD) (with $q = 13$), and $b = 0$ and 3 in equation (6.3) of (ASD), we obtain respectively

$$S(6) = 0, \qquad S(7) = -S(5), \qquad S(8) = -S(4),$$

$$S(9) = -S(3), \qquad S(10) = -S(2), \qquad S(11) = -S(1), \tag{4.1}$$

$$S(12) = -S(0), \qquad S(13) = -f(x)+S(0)+1, \qquad S(16) = x^{-2}f(x)+S(3)+1,$$

and it is easily seen that there are essentially only six distinct $S(b)$, which we take to be $S(0)$ to $S(5)$.

We write

$$N_b = N_b(x) = \sum_{n=0}^{\infty} N(b, 13, n)x^n,$$

$$N_{bc} = N_b - N_c,$$

so that by (6.10) of (ASD)

$$N_{bc}=\sum_{d=0}^{12} r_{bc}(d)x^{d}. \qquad (4.2)$$

Then by (2.13) and (6.1) of (ASD), and (4.1) above,

$$(4.3)\quad \begin{aligned}
f(x)N_{01} &= \{S(0)+S(13)\}-\{S(1)+S(12)\}=-f(x)+3S(0)-S(1)+1,\\
f(x)N_{12} &= \{S(1)+S(12)\}-\{S(2)+S(11)\}=-S(0)+2S(1)-S(2),\\
f(x)N_{23} &= \{S(2)+S(11)\}-\{S(3)+S(10)\}=-S(1)+2S(2)-S(3),\\
f(x)N_{34} &= \{S(3)+S(10)\}-\{S(4)+S(9)\}=-S(2)+2S(3)-S(4),\\
f(x)N_{45} &= \{S(4)+S(9)\}-\{S(5)+S(8)\}=-S(3)+2S(4)-S(5),\\
f(x)N_{56} &= \{S(5)+S(8)\}-\{S(6)+S(7)\}=-S(4)+2S(5),
\end{aligned}$$

and putting m = 2, 6, 3, 1, 5, and 4, in (6.7) of (ASD) we obtain using (4.1) the following expressions for S(0) to S(5), respectively.

$$S(0)=f(x)\{y^{2}\frac{\Sigma(2,0)}{P(0)}+1\}-g(2)-1+P^{2}(0)\{x\frac{P(3)P(6)}{P(1)P(2)P(5)}-x^{2}\frac{y}{P(3)}-$$

$$-x^{5}\frac{P(4)P(5)}{P^{2}(2)P(6)}-x^{9}y\frac{P(1)P(6)}{P(2)P(4)P(5)}+x^{12}\frac{P(5)}{P(2)P(6)}\},$$

$$S(1)=f(x)\{x^{4}y^{4}\frac{\Sigma(6,0)}{P(0)}\}-g(6)+P^{2}(0)\{-x^{3}\frac{P(3)P(5)}{P(1)P(2)P(6)}+$$

$$+x^{4}y^{2}\frac{P(2)}{P(5)P(6)}-x^{5}\frac{y}{P(4)}-x^{6}y^{3}\frac{P(1)P(2)}{P(5)P^{2}(6)}+x^{9}\frac{P(4)P(5)}{P(2)P(3)P(6)}\},$$

$$S(2)=f(x)\{x^{12}y^{2}\frac{\Sigma(3,0)}{P(0)}\}+g(3)+P^{2}(0)\{-xy^{2}\frac{P(1)}{P(3)P(4)}-x^{4}\frac{P(4)P(5)}{P(1)P(3)P(6)}-$$

$$(4.4)\qquad -x^{8}y\frac{P(1)P(6)}{P^{2}(3)P(4)}+x^{11}\frac{1}{P(2)}+x^{12}\frac{P(2)P(4)}{P(1)P(3)P(5)}\},$$

$$S(3)=f(x)\{-x^{11}y^{-1}-x^{11}\frac{\Sigma(1,0)}{P(0)}\}-g(1)-1+P^{2}(0)\{x^{3}\frac{P(4)}{P(1)P(3)}+x^{7}\frac{y}{P(5)}-$$

$$-x^{10}\frac{P(3)P(5)}{P(1)P(4)P(6)}+x^{11}y^{-1}\frac{P(2)P(4)}{P^{2}(1)P(3)}-x^{12}y^{-1}\frac{P(3)P(6)}{P(1)P(2)P(4)}\},$$

$$S(4)=f(x)\{-xy^{4}\frac{\Sigma(5,0)}{P(0)}\}-g(5)+P^{2}(0)\{-xy^{2}\frac{P(2)P(4)}{P(3)P(5)P(6)}+$$

$$+x^2y\,\frac{P(3)P(6)}{P(2)P^2(5)} + x^3y^3\,\frac{P(1)P(2)}{P(4)P(5)P(6)} - x^6\,\frac{1}{P(1)} + x^{10}\,\frac{P(6)}{P(2)P(5)}\Big\},$$

$$S(5)=f(x)\Big\{-x^8y^3\,\frac{\Sigma(4,0)}{P(0)}\Big\}+g(4)+P^2(0)\Big\{x^4y\,\frac{P(1)P(6)}{P(2)P(3)P(4)} - x^7\,\frac{P(3)P(5)}{P(1)P^2(4)} -$$

$$-x^8\,\frac{y}{P(6)} + x^9\,\frac{P(3)}{P(1)P(4)} + x^{10}y^2\,\frac{P(1)P(2)}{P(3)P(4)P(5)}\Big\}.$$

Now, as with $q = 11$, it is clearly convenient to avoid the terms involving $\Sigma(m, 0)$ which occur in (4.4). For example, from (4.3) and (4.4) N_{01} contains a term

$$-1 + 3\Big\{y^2\,\frac{\Sigma(2,0)}{P(0)} + 1\Big\} - x^4y^4\,\frac{\Sigma(6,0)}{P(0)},$$

i.e., in view of (4.2), $r_{01}(0)$ contains a term $3y^2\,\frac{\Sigma(2,0)}{P(0)} + 2$, and $r_{01}(4)$ contains a term $-y^4\,\frac{\Sigma(6,0)}{P(0)}$. Also, the forms of the $r_{bc}(d)$ for $q = 5, 7$, given in (ASD), and for $q=11$, together with the congruences for the $\bar{\Phi}_{13}(b)$ given in Theorem 3.1, suggest that the values of the $r_{bc}(0)$, for example, will involve either a factor $P(6)/P(3)$ or a factor $y/P(5)$; it is found to be preferable to consider the factors of the former type. We accordingly (following the case of $q = 11$) define $R_{bc}(d)(0 \leqslant d \leqslant 12)$, the "normalised" form of $r_{bc}(d)$, for $q=13$ as shown; clearly, from the definition of $r_b(d)$ and the relation $N(m, q, n) = N(q - m, q, n)$ given in (ASD), we may consider b and c to lie between 0 and 6 inclusive.

$$R_{01}(0) = P(3)\{r_{01}(0)-3y^2\Sigma(2,0)/P(0)-2\}/P(6),$$
$$R_{12}(0) = P(3)\{r_{12}(0)+y^2\Sigma(2,0)/P(0)+1\}/P(6),$$
$$R_{34}(1) = P(1)\{r_{34}(1)-y^4\Sigma(5,0)/P(0)\}/P(2),$$
$$R_{45}(1) = P(1)\{r_{45}(1)+2y^4\Sigma(5,0)/P(0)\}/P(2),$$
$$R_{56}(1) = P(1)\{r_{56}(1)-y^4\Sigma(5,0)/P(0)\}/P(2),$$
$$R_{01}(4) = -P(4)\{r_{01}(4)+y^4\Sigma(6,0)/P(0)\}/P(5),$$
$$R_{12}(4) = -P(4)\{r_{12}(4)-2y^4\Sigma(6,0)/P(0)\}/P(5),$$
$$R_{23}(4) = -P(4)\{r_{23}(4)+y^4\Sigma(6,0)/P(0)\}/P(5),$$
$$R_{45}(8) = -y^{-1}P(6)\{r_{45}(8)-y^3\Sigma(4,0)/P(0)\}/P(1),$$
$$R_{56}(8) = -y^{-1}P(6)\{r_{56}(8)+2y^3\Sigma(4,0)/P(0)\}/P(1),$$
$$R_{23}(11) = P(5)\{r_{23}(11)-\Sigma(1,0)/P(0)-y^{-1}\}/P(3),$$
$$R_{34}(11) = P(5)\{r_{34}(11)+2\Sigma(1,0)/P(0)+2y^{-1}\}/P(3),$$
$$R_{45}(11) = P(5)\{r_{45}(11)-\Sigma(1,0)/P(0)-y^{-1}\}/P(3),$$
$$R_{12}(12) = -yP(2)\{r_{12}(12)+y^2\Sigma(3,0)/P(0)\}/P(4),$$
$$R_{23}(12) = -yP(2)\{r_{23}(12)-2y^2\Sigma(3,0)/P(0)\}/P(4),$$
$$R_{34}(12) = -yP(2)\{r_{34}(12)+y^2\Sigma(3,0)/P(0)\}/P(4),$$

and, for all other values of b and c with c = b + 1,

$$R_{bc}(0) = P(3)r_{bc}(0)/P(6),$$
$$R_{bc}(1) = P(1)r_{bc}(1)/P(2),$$
$$R_{bc}(2) = -P(2)r_{bc}(2)/P(3),$$
$$R_{bc}(3) = P(4)r_{bc}(3)/P(6),$$
$$R_{bc}(4) = -P(4)r_{bc}(4)/P(5),$$
$$R_{bc}(5) = -yP(1)r_{bc}(5)/P(5),$$
$$R_{bc}(6) = r_{bc}(6),$$

$$R_{bc}(7) = y^{-1}P(5)r_{bc}(7)/P(1),$$
$$R_{bc}(8) = -y^{-1}P(6)r_{bc}(8)/P(1),$$
$$R_{bc}(9) = -P(6)r_{bc}(9)/P(4),$$
$$R_{bc}(10) = P(3)r_{bc}(10)/P(2),$$
$$R_{bc}(11) = P(5)r_{bc}(11)/P(3),$$
$$R_{bc}(12) = -yP(2)r_{bc}(12)/P(4),$$

and, for all remaining values of b and c, we use the relations

$$R_{bc}(d)+R_{ce}(d) = R_{be}(d),$$
$$R_{cb}(d) = -R_{bc}(d).$$

It will be noticed that in the above definitions the coefficient of any $r_{bc}(d)$ is precisely the coefficient of $\Phi(d)$ in the definition of $\phi(d)$, given in § 2.

We might now proceed as for $q = 11$, and use (4.3) and (4.4), together with the congruent form of $1/f(x)$ given by Theorem 3.1, to obtain congruent forms of all the $R_{bc}(d)$, as a first step in the attempt to obtain identical forms. Indeed, it would be possible to find identical forms directly, by using the identical form of $1/f(x)$ given by Theorem 2.1 or Theorem 2.2. However, either of these methods would be extremely tedious, and instead we proceed as follows.

Using (2.13) of (ASD) we determine* each of N_{01} to N_{56}, as a power series in x, as far as x^{142}. In view of (4.2) this

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*The divisions by $f(x)$ were carried out by means of a single-length programme on Durham University's Ferranti "Pegasus" computer; further details are given at the end of the thesis (page 90).

gives us every $r_{bc}(d)$, as a power series in y, as far as y^{10}, and it is a simple matter to find the corresponding terminated power series for the $R_{bc}(d)$.

We now seek congruences for the $R_{bc}(d)$, in the following manner. The factor $P(0)/f^2(y)$ occurring in the congruences for the $\Phi(b)$ given in Theorem 3.1, together with the factor $P^2(0)$ occurring in the expressions for the $S(b)$ given in (4.4), suggest that each $R_{bc}(d)$-congruence will involve a factor $P^3(0)/f^2(y)$. Also, the form of the $R_{bc}(d)$-congruences for $q = 11$, given in [6], and the fact that in (4.4) the terms in the brackets prefixed by $P^2(0)$ are of degree -1 in the $P(a)$, suggest that each $R_{bc}(d)$-congruence will involve a linear combination of l, m', n, l', m, n', and a further variable, the further variable being different only for different values of d and being a multiplicative combination of these quantities, of degree 1. It is obvious that we may consider this further variable to be linearly independent of l, m', n, l', m, and n'.

We find, by comparing coefficients of powers of y in the expansions of the appropriate quantities (the coefficients are of course all integral), that in fact, each $R_{bc}(d)$ <u>appears</u> to be congruent to the product of $P^3(0)/f^2(y)$ and a linear combination of l, m', n, l', m, n', and up to <u>two</u> further variables; the further variables found to suffice are given in the following table.

d	0	1	2	3	4	5	6	7	8	9	10	11	12
	Kl	Kn	n'/K	Kl	Km	Km	-	m'/K	m'/K	l'/K	Kn	n'/K	l'/K
	Kn	Km	-	-	Kl	-	-	-	l'/K	-	-	m'/K	n'/K

Table 4.1

We draw up a list of apparent congruences for all the $R_{bc}(d)$ with c = b + 1. The number of terms found in the expansion of each $R_{bc}(d)$ is sufficient to determine and check the 8 (or less) coefficients involved in each such congruence. Inspection of this list reveals no sets of congruent relations between the $R_{bc}(d)$ for different values of d such as are given for q = 11 in (9.1) to (9.14) of (AH), so that we cannot hope to find identities for the $R_{bc}(d)$ in the way used for q = 11. Instead we adopt the following method.

The form of the identities for the $\Phi(b)$ given in Theorem 3.2 suggests that each $R_{bc}(d)$ may be <u>equal</u> to the sum of <u>two</u> linear combinations of the type already indicated, multiplied by $P^3(0)/f^2(y)$ <u>and</u> $13yP^5(0)/f^4(y)$ respectively. A difficulty now arises: we have not found a sufficient number of terms of any $R_{bc}(d)$ to enable us to determine the 16 (or less) coefficients involved in such an identity. We circumvent this difficulty in a manner sufficiently well illustrated by the following example.

Writing

$$U = P^3(0)/f^2(y), \qquad V = yP^5(0)/f^4(y),$$

so that

(4.5) $$U = FV,$$

and noting that for $q = 11$ the numerical values of the coefficients involved in the $R_{bc}(d)$-identities are small, we assume that there is an identity for $R_{01}(0)$ of the form

$$R_{01}(0) = U(-5l-3m-3n-2l'-2m'+3Kn)+$$
$$+13V(f_1 l+f_2 m+f_3 n+f_4 l'+f_5 m'+f_6 n'+f_7 Kl+f_8 Kn),$$

where the U-term on the right-hand side is our congruent form of $R_{01}(0)$ written so that its coefficients all lie between ± 6 inclusive, and f_1 to f_8 are integers. The numbers of terms found in the expansion of $R_{01}(0)$ is sufficient to determine f_1 to f_8 and check the resulting identity.

In obtaining apparent identities for all the $R_{bc}(d)$ we occasionally find that in the U-bracket a 4, for example, should be a -9; this presents no serious difficulty. Also, we should note that for any particular $R_{bc}(d)$ a certain amount of transfer between U- and V- brackets is possible. For example, in the case of $R_{01}(0)$ we have the relations

(4.6) and (4.7) $U(13l) = 13V(-3l+l'-Kl)$, $U(13n)=13V(-3n+n'-Kn)$,

found by multiplying (1.17) through by l and n respectively and using (4.5).

We state the result, a complete set of conjectural values of the $R_{bc}(d)$ for $q = 13$, in the form of a theorem, and then prove that the values are in fact correct.

THEOREM 4.1 We have the following; for each $R_{bc}(d)$ given, both brackets on the right-hand side involve l, m', n, l', m, n', and the quantities indicated in Table 4.1, only.

$$R_{01}(0) = U(-5l-3m-3n-2l'-2m'+3Kn)+13V(-2l-2m-2n+m'+n'-Kl),$$

$$R_{01}(1) = U(-8l+6m+n+l'+m'-2n'-8Kn)+13V(-l+2m+n+l'-m'-n'-Km-2Kn),$$

$$R_{01}(2) = U(7m-6l'+4m'+4n'+3n'/K)+13V(3m-2l'+m'+n'+n'/K),$$

$$R_{01}(3) = U(6l-9m+3n+m'+7n'-Kl)+13V(l-m+2n-l'+m'+n'+Kl),$$

$$R_{01}(4) = U(3l-m+7n+l'+n'-Kl+6Km)+13V(3n+Kl+2Km),$$

$$R_{01}(5) = U(5l-3m+3n+4l'+n'-5Km)+13V(2l+m+n+n'-2Km),$$

$$R_{01}(6) = U(-l+5m-6n+3l'-m'+2n')+13V(l+m-2n+2n'),$$

$$R_{01}(7) = U(-l-3n+6m'-6n'+2m'/K)+13V(-2n+3m'-n'-m'/K),$$

$$R_{01}(8) = U(-2m-n+3l'-5m'-n'+m'/K)+13V(-2m-n+l'),$$

$$R_{01}(9) = U(3m-10n-l'-2m'+l'/K)+13V(l-3n-l'-m'+n'+l'/K),$$

$$R_{01}(10)= U(8l-8m-2n-m'+6Kn)+13V(2l-4m-n+m'+2Kn),$$

$$R_{01}(11)= U(m+4n+4l'-3m'-4n'-4n'/K)+13V(m+n+l'-2m'-2n'-n'/K),$$

$$R_{01}(12)= U(m-n-6l'+3m'+4n'-3l'/K)+13V(m-n-3l'+m'+n');$$

$$R_{12}(0) = U(4l-m-2n-l'+m'+n'-2Kl-Kn)+13V(l+m-l'-Kn),$$

$$R_{12}(1) = U(7l+m-2n-n'+7Kn)+13V(2l-m-n+Km+2Kn),$$

$$R_{12}(2) = U(-l-4m-5l'+m'+n'+2n'/K)+13V(-l-m-l'+m'+n'),$$

$$R_{12}(3) = U(-4l+6m-4n-m'+n'+2Kl)+13V(-l+m-2n+n'),$$

$$R_{12}(4) = U(6l+m-5n-m'-n'+2Kl-3Km)+13V(l-2n+n'-Km),$$

$$R_{12}(5) = U(-l-3m-7n+4l'+n'+Km)+13V(-m-n+l'+n'+Km),$$

$$R_{12}(6) = U(-l-3m+5n+l'+m'+n')+13V(-l+2n+l'),$$

$$R_{12}(7) = U(-2l+3n-l'+6m'-n'-m'/K)+13V(-l+m+n-n'),$$

$$R_{12}(8) = U(m+n+4l'+2m'+n'-m'/K)+13V(m+n+l'-m'),$$

$$R_{12}(9) = U(-m+9n-3l'-2m'+2n'+2l'/K)+13V(-l+3n-m'+l'/K),$$

$$R_{12}(10) = U(-5l+7m-n+l'+2m'-6Kn)+13V(-l+3m+n+l'-m'-2Kn),$$

$$R_{12}(11) = U(-3n+l'-2m'-6n'-n'/K)+13V(-m-2n-n'),$$

$$R_{12}(12) = U(l-m+n-l'+3m'-3n'-2l'/K+n'/K)+13V(n+m'-n'-l'/K);$$

$$R_{23}(0) = U(5l-m+4n+l'-n'-Kl)+13V(2l-m+n-n'+Kn),$$

$$R_{23}(1) = U(-6l+3m-n+3n'-6Kn)+13V(-2l+2m+n+n'-2Kn),$$

$$R_{23}(2) = U(-2m+6l'-4m'-6n'-4n'/K)+13V(-m+2l'-2m'-2n'-n'/K),$$

$$R_{23}(3) = U(-4l+3m+n+m'-7n'+Kl)+13V(-l+m+l'-m'-2n'),$$

$$R_{23}(4) = U(-3l-5m+m'-Kl-Km)+13V(-m-n'-Km),$$

$$R_{23}(5) = U(-2l+10n-3l'-2n'+3Km)+13V(-l-m+2n-2n'),$$

$$R_{23}(6) = U(-l-4n-l'-4n')+13V(-m-2n-n'),$$

$$R_{23}(7) = U(-2l-2n+l'-m'+5n'+m'/K)+13V(-n+m'+2n'),$$

$$R_{23}(8) = U(-l+m-n-2l'-4m')+13V(-n-l'+n'),$$

$$R_{23}(9) = U(-m-9n+4l'-m'+n')+13V(l-m-2n+l'+m'+n'-l'/K),$$

$$R_{23}(10) = U(7l-5m+3n-m'+5Kn)+13V(2l-2m-l'+m'+2Kn),$$

$$R_{23}(11) = U(-m+3n-6l'-3m'-5n'-m'/K+5n'/K)+13V(-l+n-l'-m'-n'+m'/K+2n'/K),$$

$$R_{23}(12) = U(l+m-n-4l'-3m'-n'+l'/K-2n'/K)+13V(-m-n-l'-m'-n'+l'/K);$$

$$R_{34}(0) = U(-3l-6n+l')+13V(-l-n+l'+n'-Kn),$$

$$R_{34}(1) = U(6l+m+6n-n'-Km+5Kn)+13V(2l-l'-n'+2Kn),$$

$$R_{34}(2) = U(8m+3l'+m'-2n'+n'/K)+13V(l+2m-n+n'/K),$$

$$R_{34}(3) = U(-l-7m-n+3n'-Kl)+13V(-2m+m'+n'),$$

$$R_{34}(4) = U(-5l+3m+5n-l'-m'+5Km)+13V(-2l+n+2Km),$$

$$R_{34}(5) = U(5l+5m-11n-3l'+n'-5Km)+13V(2l+2m-3n-2l'+n'-Km),$$

$$R_{34}(6) = U(3m+4n-2l'+m'+2n')+13V(2m+n-l'),$$

$$R_{34}(7) = U(l+n+l'+3m'-3n')+13V(n+l'-n'),$$

$$R_{34}(8) = U(l-3m+n+6l'+m'+n'-m'/K)+13V(-m+n+2l'-n'-l'/K),$$

$$R_{34}(9) = U(2m+8n+6l'+4m'-4n'-4l'/K)+13V(2m+n-2n'-l'/K),$$

$$R_{34}(10) = U(-4l+2m-4n-2l'+m'-3Kn)+13V(-2l+m-n-2Kn),$$

$$R_{34}(11) = U(-l+m-3n+4l'+5m'+n'+2m'/K-2n'/K)+13V(l+m-n+l'+2m'+n'-m'/K-n'/K),$$

$$R_{34}(12) = U(-l+n-2m'+4n'+3l'/K+n'/K)+13V(m+n-m'+2n'+l'/K);$$

$$R_{45}(0) = U(-5l+2m+6n+n'+2Kl)+13V(-l+m+n+Kn),$$

$$R_{45}(1) = U(-5l-m+4n-l'-2n'+2Km-3Kn)+13V(-2l-m+n+l'-n'-Kn),$$

$$R_{45}(2) = U(l-10m-5l'+3m'+3n'/K)+13V(-l-2m+n-l'+2m'+n'/K),$$

$$R_{45}(3) = U(-6l+6m+2n+5n'-Kl)+13V(-l+2m+n+n'-Kl),$$

$$R_{45}(4) = U(-3l-2m-7n+l'+n'-7Km)+13V(m-n+n'-Kl-2Km),$$

$$R_{45}(5) = U(-6l+10n+n'+5Km)+13V(-3l+3n+l'+2Km),$$

$$R_{45}(6) = U(2l-5m-3n-m'+3n')+13V(-2m+m'+n'),$$

$$R_{45}(7) = U(l-n+l'-m'-5n'+m'/K)+13V(l-n-n'),$$

$$R_{45}(8) = U(4m+5l'-2m'-n'-l'/K)+13V(l+m-n-m'),$$

$$R_{45}(9) = U(-2m-5n-2l'-2m'+2l'/K)+13V(-2m-n+l'),$$

$$R_{45}(10) = U(-2n+l'+m'+2Kn)+13V(l-n+Kn),$$

$$R_{45}(11) = U(l-m+2n+3l'+2n'+m'/k-4n'/k)+13V(-m+n+l'-m'-n'-n'/K),$$

$$R_{45}(12) = U(-m-n+6l'+5n')+13V(-m+2l'+m'+n'-l'/K);$$

$$R_{56}(0) = U(-6l+m+2n-2l'-n'+Kl)+13V(-2l-n'),$$

$$R_{56}(1) = U(2l+2m+2n+l'+2n'-Km+Kn)+13V(l+m+n+n'+Kn),$$

$$R_{56}(2) = U(7m+l'-4m'-5n'-4n'/k)+13V(l+m-n-2m'-2n'-n'/K),$$

$$R_{56}(3) = U(5l-6m+n-6n'-Kl)+13V(l-2m-2n'),$$

$$R_{56}(4) = U(-3l-2m+5n-n'+5Km)+13V(-l-2m+n+l'-n'+Km),$$

$$R_{56}(5) = U(3l+6m-6n+5l'-2n'-3Km)+13V(2l+m-2n+l'-m'-Km),$$

$$R_{56}(6) = U(4m+n+3l'+m'-4n')+13V(l+m+l'-m'-n'),$$

$$R_{56}(7) = U(3l+n-2l'+4m'+7n'-m'/K)+13V(n-l'+m'+n'),$$

$$R_{56}(8) = U(-3m-n-l'-3m'+2n'+2l'/K)+13V(-m+n+l'+n'),$$

$R_{56}(9) = U(m+2n+2l'-3m'+4n'+2l'/K)+13V(m+n-m'+2n'+l'/K),$

$R_{56}(10)= U(-3l+4n+l'-2Kn)+13V(-l+2n+l'),$

$R_{56}(11)= U(m-n-5l'+5m'+6n'+5n'/K)+13V(m-2l'+2m'+2n'+n'/K),$

$R_{56}(12)= U(-l+m+n+3l'+4m'+n'-4l'/K)+13V(m-l'/K-n'/K).$

The following relations will be required in the proof of this theorem for systematic simplification of expressions involving l, m', n, l', m, and n'.

(4.8) to (4.10) $lm/n = -l-m, \quad mn/l = -m-n, \quad nl/m = -n-l;$

(4.11) to (4.13) $l^2/m = -Fl-3l+m-n, \quad m^2/n=-Fm-3m+n-l,$

$n^2/l = -Fn-3n+l-m;$

(4.14) to (4.16) $l^2/n = Fl+2l-m+n, \quad m^2/l = Fm+2m-n+l,$

$n^2/m = Fn+2n-l+m;$

(4.17) to (4.19) $Kl = -Fl-3l+l', \quad Km = -Fm-3m+m', \quad Kn = -Fn-3n+n',$

(4.20) to (4.22) $l'/K = Fl'+3l'+l, \quad m'/K = Fm'+3m'+m,$

$n'/K = Fn'+3n'+n;$

(4.23) to (4.25) $K^2l = F(3l-Kl)+10l-3l', \quad K^2m=F(3m-Km)+10m-3m',$

$K^2n = F(3n-Kn)+10n-3n',$

(4.26) to (4.28) $l'/K^2=F(3l'+l'/K)+10l'+3l,$

$m'/K^2=F(3m'+m'/K)+10m'+3m,$

$n'/K^2=F(3n'+n'/K)+10n'+3n.$

(4.8) to (4.16) follow from (2.16) and (2.17), (4.17) to (4.22) from (1.17), and (4.23) to (4.28) from (4.17) to (4.22) respectively; (4.8) and (4.11) have already been given as (2.18) and (2.19) respectively. We shall also need the relations

(4.29) to (4.31) $a = l/m-K$, $b = m/n-K$, $c = n/l-K$,

(4.32) to (4.34) $a' = l'/m' + 1/K$, $b' = m'/n'+1/K$, $c' = n'/l'+1/K$,

arising from (1.6) to (1.11) and (2.11) to (2.13). Of course all of the equations (4.8) to (4.34) remain valid when l, m', n, l', m, and n', are interchanged according to (2.14) or (2.15) and a, b', c, a', b, and c', are interchanged cyclically. Finally, the following will be required

(4.35)
$$\begin{aligned}
2g(1)-g(2)+1 &= -P^2(0)l'b = P^2(0)(l+l'+m'),\\
2g(2)-g(4)+1 &= P^2(0)mc' = P^2(0)(-m-n+m'),\\
2g(3)-g(6)+1 &= -P^2(0)m'c = P^2(0)(m+m'+n'),\\
2g(4)+g(5) &= P^2(0)n'a = P^2(0)(-n-l'-n'),\\
2g(5)+g(3) &= P^2(0)lb' = P^2(0)(-l-m+l'),\\
2g(6)+g(1) &= P^2(0)na' = P^2(0)(-l-n+n');
\end{aligned}$$

these relations arise from (ASD), Lemma 8 (with $q = 13$), and (4.29) to (4.34) above, using (4.8) to (4.10) (divided through by K if necessary).

The proof of Theorem 4.1 is similar to those of (ASD), Theorems 4 and 5, and (AH), Theorem 6. If we write

$$N'_{01} = N_{01}+\{-3y^2\Sigma(2,0)/P(0)-2\}+x^4\{y^4\Sigma(6,0)/P(0)\},$$

$$N'_{12} = N_{12}+\{y^2\Sigma(2,0)/P(0)+1\}+x^4\{-2y^4\Sigma(6,0)/P(0)\}+x^{12}\{y^2\Sigma(3,0)/P(0)\},$$

$$N'_{23} = N_{23}+x^4\{y^4\Sigma(6,0)/P(0)\}+x^{11}\{-\Sigma(1,0)/P(0)-y^{-1}\}+ +x^{12}\{-2y^2\Sigma(3,0)/P(0)\},$$

$$N'_{34} = N_{34}+x\{-y^4\Sigma(5,0)/P(0)\}+x^{11}\{2\Sigma(1,0)/P(0)+2y^{-1}\} + +x^{12}\{y^2\Sigma(3,0)/P(0)\},$$

$$N'_{45} = N_{45}+x\{2y^4\Sigma(5,0)/P(0)\}+x^8\{-y^3\Sigma(4,0)/P(0)\}+ +x^{11}\{-\Sigma(1,0)/P(0)-y^{-1}\},$$

$$N'_{56} = N_{56}+x\{-y^4\Sigma(5,0)/P(0)\}+x^8\{2y^3\Sigma(4,0)/P(0)\},$$

then in view of (4.2) and the definitions of the $R_{bc}(d)$ we have for any fixed values of b and c with c = b + 1

(4.36) $$N'=P(6)R_0/P(3)+xP(2)R_1/P(1)-x^2P(3)R_2/P(2)+x^3P(6)R_3/P(4)- -x^4P(5)R_4/P(4)-x^5y^{-1}P(5)R_5/P(1)+x^6R_6+x^7yP(1)R_7/P(5)- -x^8yP(1)R_8/P(6)-x^9P(4)R_9/P(6)+x^{10}P(2)R_{10}/P(3)+ +x^{11}P(3)R_{11}/P(5)-x^{12}y^{-1}P(4)R_{12}/P(2)$$

where for convenience the suffix bc is dropped, and R(d) is written as R_d. Thus writing

$$f(x)N'/P(0) = \sum_{d=0}^{12} t_d x^d$$

we can use (4.36) and the expression for $f(x)/P(0)$ given by (1.1) to find each t_d as a linear combination of R_d in which each R_d occurring is multiplied by some multiplicative combination of the P(a); for example we find that

$$t_2 = -P(2)P(6)(R_o+R_1)/P(1)P(3)-P(3)P(4)R_2/P^2(2)-$$
$$-yP(3)P(6)R_3/P(4)P(5)+y^2P(1)(R_6-R_8)/P(6)+$$
$$+yP(2)P(5)R_{10}/P(3)P(4).$$

If in this example we define T_2, the "normalised" form of t_2, by

$$T_2 = -y^{-2}P(6)t_2/P(1)$$

then we find that

$$T_2=-B(R_o+R_1)/K-BCbR_2-ABc'R_3-R_6+R_8-R_{10}/K,$$

and the coefficient of each R_d in this equation is equal to a simple expression in l, m', n, l', m, and n', as follows:

$-B/K = -m'/n$ by (2.12);

$-BCb = -m(m/n-K)/l$ by (2.12), (2.13), and (4.30),

$= -m(-l/m-n/l-1/k-1)/l$ by (1.17) and (2.17),

$= m'/l-n/l+1$ by (2.16);

$-ABc' = -l'/n-1$ by (2.11), (2.12), and (4.34).

By proceeding in the above manner for all the t_d, suitably normalising the t_d in each case, we arrive at the following:

$$T_0 = y^{-1}t_0 = m(R_0+R_{12})/l+l(R_1+R_{11})/n+n(R_4+R_8)/m+R_6,$$
$$T_1 = y^{-2}P(5)t_1/P(1) = -m'R_0/l + (-l/n-m'/n)(R_1+R_{12})+$$
$$+(-m/n+K)R_2+(-l/m-l/K)(R_5+R_9)+R_7,$$
$$T_2 = -y^{-2}P(6)t_2/P(1) = -m'(R_0+R_1)/n+(m'/l-n/l+1)R_2+$$
$$+(-l'/n-1)R_3-R_6+R_8-R_{10}/K,$$
$$T_3 = -y^{-1}P(6)t_3/P(4) = -l'R_1/n+(-n/m-l'/m)(R_4+R_{11})+$$
$$+(-l/m+K)R_7+(-n/l-1/K)(R_3+R_2)+R_9,$$

$$T_4 = y^{-1}P(3)t_4/P(2) = nR_8/m' + (-m'/l' + n/l')(R_{12}+R_4) + (-n'/l' - 1/K)R_3 + (-m'/n' + K)(R_2+R_5) + R_{10},$$

$$T_5 = y^{-1}P(5)t_5/P(3) = -n'(R_4+R_0)/l + (n'/m - l/m + 1)R_9 + (-m'/l - 1)R_5 - R_6 + R_{11} - R_3/K,$$

(4.37)
$$T_6 = -P(2)t_6/P(4) = -l'(R_1+R_4)/m + (l'/n - m/n + 1)R_7 + (-n'/m - 1)R_{10} - R_6 + R_{12} - R_5/K,$$

$$T_7 = P(3)t_7/P(6) = l(R_{11}+R_8)/m' + (-l/n' - m'/n' + 1)R_5 + (n/m' - 1)R_2 - R_6 + R_0 + KR_7,$$

$$T_8 = P(1)t_8/P(2) = n(R_8+R_{12})/l' + (-n/m' - l'/m' + 1)R_3 + (m/l' - 1)R_7 - R_6 + R_1 + KR_9,$$

$$T_9 = -P(2)t_9/P(3) = -n'R_4/m + (-m/l - n'/l)(R_0+R_8) + (-n/l + K)R_9 + (-m/n - 1/K)(R_{10}+R_7) + R_2,$$

$$T_{10} = P(4)t_{10}/P(6) = lR_{11}/n' + (-n'/m' + l/m')(R_8+R_1) + (-l'/m' - 1/K)R_5 + (-n'/l' + K)(R_9+R_{10}) + R_3,$$

$$T_{11} = -P(4)t_{11}/P(5) = m(R_{12}+R_{11})/n' + (-m/l' - n'/l' + 1)R_{10} + (l/n' - 1)R_9 - R_6 + R_4 + KR_2,$$

$$T_{12} = -yP(1)t_{12}/P(5) = mR_{12}/l' + (-l'/n' + m/n')(R_{11}+R_0) + (-m'/n' - 1/K)R_{10} + (-l'/m' + K)(R_7+R_3) + R_5.$$

We observe that, apart from T_0, the T_d fall naturally into two groups of six given by d = 1, 3, 4, 9, 10, 12, and d = 2, 5, 6, 7, 8, 11, respectively, and that with the normalising factors as chosen, interchanging either T_1, T_4, T_3, T_{12}, T_9, and T_{10}, or T_2, T_8, T_6, T_{11}, T_5, and T_7, cyclically corresponds to interchanging R_0, R_8, R_1, R_{12}, R_4, R_{11}, and R_2, R_3, R_7, R_{10}, R_9, R_5, cyclically (leaving R_6 unchanged) if we interchange l, m', n, l', m, and n', according to (2.14) or (2.15); the two groups of six R_d occur naturally in Table 4.1. T_o is invariant under these interchanges. We might have anticipated such a situation as an aid in finding the identities of (4.37) (c.f. the proofs of Theorems 2.1 and 2.2).

We now find alternative expressions for the T_d. This time each pair of values of b and c (with c = b + 1) is considered separately, so that we have 78 $T_{bc}(d)$ (in the obvious notation) to determine, viz. $T_{01}(d)$ to $T_{56}(d)$ for d = 0 to d = 12. These expressions are found as in the following examples.

$t_{01}(9)$ (again in the obvious notation) is by definition the coefficient of x^9 in $f(x)N'_{01}/P(0)$, thus we have

(4.38) $t_{01}(9)=P(0)\{-3yP(1)P(6)/P(2)P(4)P(5)-P(4)P(5)/P(2)P(3)P(6)\}$

from the definition of N'_{01}, the expression for $f(x)N_{01}$ given in (4.3), and the values of S(0) and S(1) given in (4.4); of course the terms involving $\Sigma(m,0)$ all disappear. Multiplying (4.38) by $-yP(2)/P(3)$ we obtain

$$yT_{01}(9) = P(0)(3m'a+n'c),$$
$$= P(0)(3l'-3m+nn'/l-n) \quad \text{by (4.29) and (4.31)},$$
$$= P(0)(-3m+4l'-m'-n'/K)$$

by (4.13) (divided through by K) and (1.17). The method of this example applies when $d \neq 0$. When $d = 0$ the procedure is slightly different.

$t_{01}(0)$ is the coefficient of x^0 in $f(x)N'_{01}/P(0)$, and proceeding as in the previous example we obtain

$$t_{01}(0) = \{-3g(2)+g(6)-2\}/P(0).$$

Since $T_{01}(0) = y^{-1}t_{01}(0)$ this equation becomes

$$yT_{01}(0) = P(0)(-1+m+2n+1'-2m'+n')$$

by means of relations (4.35).

A complete set of alternative values of $yT_{bc}(d)/P(0)$ is given in Table 4.2 at the end of this Part (page 46).

By equating our two expressions for each $T_{bc}(d)$ we now have, for any fixed values of b and c, a set of 13 simultaneous linear equations for $R_{bc}(d)(d=0 \text{ to } 12)$. Moreover these equations have a unique solution; this may be seen by proving that a determinant is non-zero, but it is easier to observe that the equations are in fact the necessary and sufficient conditions that $\sum_{d=0}^{12} R_{bc}(d)x^d$ be the quotient of two given power series. Accordingly to prove Theorem 4.1 all that remains is to show that for $(b, c) = (0, 1)$ to $(5, 6)$ respectively the values of the $R_{bc}(d)$ given in the theorem satisfy these equations. In other words we need to show that for each of the 78 $T_{bc}(d)$ the value found by substituting for the $R_{bc}(d)$ from the theorem in the appropriate equation of (4.37) agrees with the value given by Table 4.2. This is tedious but straightforward; we proceed as in the following example.

Consider $T_{01}(1)$ as given by substituting for the $R_{01}(d)$ from the theorem in the second equation of (4.37). Each $R_{01}(d)$ is expressed in the theorem as the sum of two brackets, one multiplied by U and the other by 13V. We write down and simplify {by means of (4.8) to (4.28)} the total contribution of the U-brackets and the total contribution of the V-brackets separately, and combine the resulting two expressions. The contribution of the V-brackets is

$$-m'(-2l-2m-2n+m'+n'-Kl)/l+(-1/n-m'/n)(-l+3m-2l'-Km-2Kn)+$$
$$+(-m/n+K)(3m-2l'+m'+n'+n'/K)+(-1/m-1/K)(3l+m-2n-l'-m'+2n'-2Km+$$
$$+l'/K)+(-2n+3m'-n'-m'/K)$$
$$= (-3l+6m-n-2l'+3m'+2n')+(4Kl+3Km+l'/K-m'/K-2n'/K)+(-l'/K^2)+$$
$$+(2/K^2+3/K-3+K)lm/n+(-1/K^2+2/K)mn/l+(-2/K+2)nl/m+$$
$$+(-1/K^2+1/K-3)l^2/m+(-4/K-2)m^2/n+(2/K+1)l^2/n+(-1/K^2+2/K)m^2/l$$

and this expression, on substituting for l'/K^2, lm/n, mn/l, nl/m, l^2/m, m^2/n, l^2/n, and m^2/l, from (4.8) to (4.16) and (4.23) to (4.28), reduces to

$$F(4l+2m-2l'+6m'-m'/K)+(8l+11m-n-6l'+13m'-3n')+$$
$$+(3Kl+2Km+l'/K-5m'/K+n'/K)$$

which expression, on substituting for each term in the third bracket from (4.17) to (4.22), reduces to

$$F(l-l'+m'+n'-m'/K), \qquad (4.39)$$

only terms containing a factor F remain. The contribution of the U-brackets is

$$-m'(-5l-3m-3n-2l'-2m'+3Kn)/l+(-l/n-m'/n)(-8l+7m-5l'+4m'+2n'-8Kn-3l'/K)+(-m/n+K)(7m-6l'+4m'+4n'+3n'/K)+(-l/m-1/K)(5l-7n+3l'-2m'+n'-5Km+l'/K)+(-l-3n+6m'-6n'+2m'/K)$$

$$=(-7l+17m+n-5l'+7m'+4n')+(13Kl+7Km-3l'/K+m'/K-n'/K)+(-l'/K^2)+ +(3/K^3+5/K^2+10/K-7)lm/n+(3/K-3)mn/l+(-1/K+7)nl/m+ +(-1/K^2-3/K-5)l^2/m+(-4/K^2-11/K-7)m^2/n+(3/K^2+5/K+8)l^2/n+ +(2/K^2+3/K)m^2/l$$

and this expression, on substituting for l'/K^2, lm/n, mn/l, nl/m, l^2/m, m^2/n, l^2/n, and m^2/l, reduces to

$$F(13l+7m+5l'+14m'+3l'/K+6m'/K)+(28l+35m+3n+9l'+25m'-4n')+ +(13Kl+7Km+7l'/K+8m'/K-3n'/K)+(-3l'/K^2-3m'/K^2)$$

which expression, on substituting for each term in the third and fourth brackets from (4.17) to (4.22) and (4.23) to (4.28) respectively, reduces to

$$F(3l'+13m'-3n'+3m'/K)+13(-l+m+l'+2m'-n'), \qquad (4.40)$$

only terms containing either a factor F or a factor 13 remain. Multiplying expressions (4.39) and (4.40) by 13V and U respectively, and adding, remembering that $FV = U$, we obtain the following expression for $T_{01}(1)$

$$FU(31'+13m'-3n'+3m'/K)+13U(m+3m'-m'/K),$$

and this expression, on substituting for m'/K in the second bracket from (4.21), reduces to

$$FU(31'-3n'+3m'/K).$$

Since $FU=y^{-1}P(O)$, this is the same as the value of $T_{\phi 1}(1)$ given by Table 4.2.

We perform the above verification for each of the 78 $T_{bc}(d)$; the working is always essentially the same as the above, and is therefore omitted. This completes the proof of Theorem 4.1.

As in the case of $q = 11$, there are certain linear congruence relations (but no identities) between the $r_{bc}(d)$ for a given value of d when $q = 13$; if we write

$$s_1(d) = r_{o1}(d)-6r_{56}(d),$$
$$s_2(d) = r_{12}(d)-5r_{56}(d),$$
$$s_3(d) = r_{23}(d)-4r_{56}(d),$$
$$s_4(d) = r_{34}(d)-3r_{56}(d),$$
$$s_5(d) = r_{45}(d)-2r_{56}(d),$$

we have, modulo 13,

$$
\begin{aligned}
s_3(0)-6s_4(0)+5s_5(0) &\equiv 0,\\
s_2(1)+3s_3(1)-5s_4(1)-5s_5(1) &\equiv 0,\\
s_4(2) &\equiv 0,\\
s_1(2)+s_2(2)-5s_3(2)+s_5(2) &\equiv 0,\\
s_1(3)-s_3(3) &\equiv 0,\\
s_2(3)+s_3(3)-3s_4(3)-6s_5(3) &\equiv 0,\\
s_1(4)-4s_2(4)+4s_3(4)-5s_4(4)-6s_5(4) &\equiv 0,\\
s_1(5) &\equiv 0,\\
s_2(5)-2s_3(5)-4s_4(5)-2s_5(5) &\equiv 0,\\
s_1(6)+2s_2(6)-5s_5(6) &\equiv 0,\\
s_2(6)+5s_3(6)+3s_4(6)+3s_5(6) &\equiv 0,\\
s_1(7)-3s_2(7)+6s_3(7) &\equiv 0,\\
s_2(7)-s_3(7)-3s_4(7)-s_5(7) &\equiv 0,\\
s_1(8)+6s_2(8)-5s_3(8)-5s_4(8)-3s_5(8) &\equiv 0,\\
s_2(9)-6s_4(9) &\equiv 0,\\
s_1(9)-4s_3(9)+2s_4(9)-6s_5(9) &\equiv 0,\\
s_1(10)+3s_2(10)-5s_5(10) &\equiv 0,\\
s_2(10)+6s_3(10)+5s_4(10)-s_5(10) &\equiv 0,\\
s_1(11)+5s_2(11)-3s_3(11)-3s_4(11)-3s_5(11) &\equiv 0,\\
s_1(12)+2s_2(12)+5s_3(12)-5s_4(12)+3s_5(12) &\equiv 0.
\end{aligned}
$$

The above congruences with each $r_{bc}(d)$ replaced by the corresponding $R_{bc}(d)$ follow immediately from Theorem 4.1, and for each value of d we simply divide through by the normalising factor contained in the $R_{bc}(d)$ (the coefficients of the $r_{bc}(d)$ in the congruences are such that the terms involving $\Sigma(m, 0)$ disappear).

We may note that since

$$\bar{\Phi}(d) = \sum_{n=0}^{\infty} p(13n + d) = \sum_{n=0}^{\infty} \sum_{b=0}^{12} N(b, 13, 13n + d)y^n$$

$$= \sum_{n=0}^{\infty} N(0, 13, 13n+d)y^n + 2\sum_{b=1}^{6} \sum_{n=0}^{\infty} N(b,13,13n+d)y^n$$

{using the relation $N(m, q, n) = N(q - m, q, n)$ given in (ASD)}

$$= r_0(d) + 2\sum_{b=1}^{6} r_b(d)$$

$$\equiv r_{01}(d)+3r_{12}(d)+5r_{23}(d)+7r_{34}(d)+9r_{45}(d)+11r_{56}(d) \pmod{13}$$

{using (6.8) and (6.9) of (ASD)}, Theorem 4.1 may be used in an alternative proof of Theorem 3.1.

Table 4.2

$yT_{bc}(d)/P(0)$

b,c / d	0,1	1,2	2,3	3,4	4,5	5,6
0	-l+m+2n+ +l'-2m'+n'	2l-m+m'- -2n'	-l+m-n+ +m'+2n'	-l-m+n- -2m'-n'	l+m+m'+n'	-n-l'-n'
1	3l'-3n'+ +3m'/K	-l'-m'/K	2n'	l-2n'	-2l+2n'	l-n'
2	3m'	-m'	0	-m+l'+m'	2m-2l'-2m'	-m+l'+m'
3	n-m'	-2n+2m'	n	-m'-n'-l'/K	-m'+2n'+ +2l'/K	m'-n'-l'/K
4	-l	3l+m'	-3l-2m'	l+m'	l-m+Kn	-2l+2m- -2Kn
5	3n-3m'-4n'	-n+m'+3n'	-n'	0	0	0
6	-l+l'+n'	2l-2l'-2n'	-l+l'+n'	-l'	2l'	-l'
7	0	0	-l	2l	n+l'	-2l-2n-2l'
8	0	-m-n-n'	2m+2n+2n'	-m-n-n'	-n	2n
9	-3m+4l'- -m'-n'/K	m-3l'+2m'+ +2n'/K	l'-m'-n'/K	0	l'	-2l'
10	0	0	m-n+Kl	-3m+2n- -2Kl	2m-n-n'+ +Kl	m+2n'
11	0	m	l-m+m'	-2l-m-2m'	l+m+m'	0
12	3n	l-2n-Km	-2l+n-l'+ +2Km	l+n+2l'- -Km	-n-l'	0