PART 2

q = 17 throughout this Part

5. We write

$\alpha_1 = -x^{-7}P(2)/P(1)$, $\alpha_2 = -x^{-12}P(6)/P(3)$, $\alpha_3 = x^{28}P(1)/P(8)$,
$\alpha_4 = -x^{14}P(3)/P(7)$, $\alpha_5 = x^{-10}P(8)/P(4)$, $\alpha_6 = -x^{-5}P(7)/P(5)$,
$\alpha_7 = x^{-11}P(4)/P(2)$, $\alpha_8 = x^{3}P(5)/P(6)$;

then by (ASD), Lemma 6 (with q = 17) we have

(5.1) $-x^{-12}f(x)/f(y^{17}) = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+1.$

In (5.1) we replace x by $w_r x$ where w_r (r = 1 to 17) are the seventeenth roots of unity, and multiply together the seventeen resulting equations, obtaining

(5.2) $$-y^{-12}f^{18}(y)/f^{18}(y^{17})=\prod_{r=1}^{17}(\alpha_1 w_r^{-7}+\alpha_2 w_r^{-12}+\alpha_3 w_r^{28}+\alpha_4 w_r^{14}+\alpha_5 w_r^{-10}+\alpha_6 w_r^{-5}+\alpha_7 w_r^{-11}+\alpha_8 w_r^{3}+1).$$

Now as w_r runs through the seventeenth roots of unity so does w_r^2, so that the product on the right-hand side of (5.2) is equal to

$$\prod_{r=1}^{17}(\alpha_1 w_r^{3}+\alpha_2 w_r^{-7}+\alpha_3 w_r^{-12}+\alpha_4 w_r^{28}+\alpha_5 w_r^{14}+\alpha_6 w_r^{-10}+\alpha_7 w_r^{-5}+\alpha_8 w_r^{-11}+1),$$

and is thus unchanged if α_1, α_2, α_3, α_4, α_5, α_6, α_7, and α_8, are interchanged cyclically. The product is thus a linear combination of terms $[\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_5^{i_5}\alpha_6^{i_6}\alpha_7^{i_7}\alpha_8^{i_8}]$ where i_1 to i_8 are non-negative integers, and considering the left-hand side of (5.2) such terms as occur can only involve x in terms of $y = x^{17}$. Thus if $\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8}$ (or any other term of $[\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8}]$) occurs we must have

$$(5.3) \quad -7i_1 - 12i_2 + 28i_3 + 14i_4 - 10i_5 - 5i_6 - 11i_7 + 3i_8 \equiv 0 \pmod{17}$$

(interchanging i_1, i_2, i_3, i_4, i_5, i_6, i_7, and i_8, cyclically gives the same congruence).

Now, writing

$a_1 = P(1)P(6)/P(2)P(4)$, $a_2 = -y^2P(3)P(1)/P(6)P(5)$,

$a_3 = y^{-2}P(8)P(3)/P(1)P(2)$, $a_4 = -y^{-1}P(7)P(8)/P(3)P(6)$,

$a_5 = y^{-1}P(4)P(7)/P(8)P(1)$, $a_6 = P(5)P(4)/P(7)P(3)$,

$a_7 = -yP(2)P(5)/P(4)P(8)$, $a_8 = yP(6)P(2)/P(5)P(7)$,

it is easily verified that

$$(5.4) \quad \begin{aligned} \alpha_1^{17} &= a_2^4 a_3^{12} a_4^{11} a_5^9 a_6^5 a_7^{14} a_8^{15}, & \alpha_5^{17} &= a_6^4 a_7^{12} a_8^{11} a_1^9 a_2^5 a_3^{14} a_4^{15}, \\ \alpha_2^{17} &= a_3^4 a_4^{12} a_5^{11} a_6^9 a_7^5 a_8^{14} a_1^{15}, & \alpha_6^{17} &= a_7^4 a_8^{12} a_1^{11} a_2^9 a_3^5 a_4^{14} a_5^{15}, \\ \alpha_3^{17} &= a_4^4 a_5^{12} a_6^{11} a_7^9 a_8^5 a_1^{14} a_2^{15}, & \alpha_7^{17} &= a_8^4 a_1^{12} a_2^{11} a_3^9 a_4^5 a_5^{14} a_6^{15}, \\ \alpha_4^{17} &= a_5^4 a_6^{12} a_7^{11} a_8^9 a_1^5 a_2^{14} a_3^{15}, & \alpha_8^{17} &= a_1^4 a_2^{12} a_3^{11} a_4^9 a_5^5 a_6^{14} a_7^{15}. \end{aligned}$$

It will be noticed that all of the equations (5.4) may be obtained from any one of them by interchanging a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, and α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, cyclically. By (5.4), since $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 = -1$,

$$(\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_5^{i_5}\alpha_6^{i_6}\alpha_7^{i_7}\alpha_8^{i_8})^{17} =$$

$$= (a_1a_2a_3a_4a_5a_6a_7a_8)^{\sigma}\, a_1^{\sigma_1}a_2^{\sigma_2}a_3^{\sigma_3}a_4^{\sigma_4}a_5^{\sigma_5}a_6^{\sigma_6}a_7^{\sigma_7}a_8^{\sigma_8}$$

where $\sigma = 10i_1 + 24i_2 + 14i_3 + 26i_4 + 32i_5 + 18i_6 + 28i_7 + 16i_8$, an even integer, and

$$\sigma_1 = 15i_2 + 14i_3 + 5i_4 + 9i_5 + 11i_6 + 12i_7 + 4i_8,$$
$$\sigma_2 = 15i_3 + 14i_4 + 5i_5 + 9i_6 + 11i_7 + 12i_8 + 4i_1,$$
$$\sigma_3 = 15i_4 + 14i_5 + 5i_6 + 9i_7 + 11i_8 + 12i_1 + 4i_2,$$
$$\sigma_4 = 15i_5 + 14i_6 + 5i_7 + 9i_8 + 11i_1 + 12i_2 + 4i_3,$$
$$\sigma_5 = 15i_6 + 14i_7 + 5i_8 + 9i_1 + 11i_2 + 12i_3 + 4i_4,$$
$$\sigma_6 = 15i_7 + 14i_8 + 5i_1 + 9i_2 + 11i_3 + 12i_4 + 4i_5,$$
$$\sigma_7 = 15i_8 + 14i_1 + 5i_2 + 9i_3 + 11i_4 + 12i_5 + 4i_6,$$
$$\sigma_8 = 15i_1 + 14i_2 + 5i_3 + 9i_4 + 11i_5 + 12i_6 + 4i_7;$$

moreover $\sigma + \sigma_1$ to $\sigma + \sigma_8$ are multiples of 17 by (5.3), hence any expression of the form $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_5^{i_5}\alpha_6^{i_6}\alpha_7^{i_7}\alpha_8^{i_8}$ for which (5.3) holds is of the form

$a_1^{j_1}a_2^{j_2}a_3^{j_3}a_4^{j_4}a_5^{j_5}a_6^{j_6}a_7^{j_7}a_8^{j_8}$ where j_1 to j_8 are non-negative integers. Thus every term occurring in the right-hand side of (5.2) is of the form $a_1^{j_1}a_2^{j_2}a_3^{j_3}a_4^{j_4}a_5^{j_5}a_6^{j_6}a_7^{j_7}a_8^{j_8}$, and such such terms occur in cyclically symmetrical sets of eight terms each.

Further, $\Phi(5)$ is the coefficient of x^5 in $1/f(x)$ regarded as a polynomial of degree 16 in x with coefficients involving x in terms of $y = x^{17}$, so that $y^{-11}f^{18}(y)\Phi(5)/f^{17}(y^{17})$

is the coefficient of x^o in
$y^{-12}f^{18}(y)/\{f^{18}(y^{17})(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+1)\}$. This is a cyclically symmetric polynomial of degree 16 in $\alpha_1, a_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7,$ and α_8; and the terms which give the coefficient of x^o occur only in symmetrical sets of eight expressible as $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$, as before. (This is not true for the coefficient of any power of x other than 0; the eight terms of $[a_1]$, for example, do not appertain to the same power of x.)

Thus writing

$$F = y^{-2}f^3(y)/f^3(y^{17})$$

we have the following:

LEMMA 5.1 F^6 and $yf(y^{17})F^6\,\Phi(5)$ are each equal to a linear combination of terms $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$.

We now write

(5.5) to (5.8) $b_1 = a_1a_5, \quad b_2 = a_2a_6, \quad b_3 = a_3a_7, \quad b_4 = a_4a_8,$

so that

(5.9) $$b_1b_2b_3b_4 + 1 = 0.$$

<7, 6, 5, 3> and <8, 4, 2, 1> give, respectively,

(5.10) $$b_1 + b_3 + 1 = 0,$$

(5.11) $$b_2 + b_4 + 1 = 0,$$

while <8, 5, 4, 3>, <8, 7, 5, 2>, <7, 6, 4, 2>, <6, 5, 4, 1>, <5, 3, 2, 1>, <8, 6, 3, 2>, <8, 7, 6, 1>, and <7, 4, 3, 1>, give, respectively, (5.12) to (5.15) $a_1 = b_1a_2 + 1, \quad a_2 = b_2a_3 + 1,$

$a_3 = b_3a_4 + 1, \quad a_4 = b_4a_5 + 1,$

(5.16) to (5.19) $a_5 = b_1a_6 + 1, \quad a_6 = b_2a_7 + 1,$

$a_7 = b_3a_8 + 1, \quad a_8 = b_4a_1 + 1.$

It will be observed that each of the equations (5.5) to (5.19) remains valid when b_1, b_2, b_3, b_4, and a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, are interchanged cyclically. We are now in a position to prove

LEMMA 5.2 Any expression of the form $[a_1^{j_1}\ a_2^{j_2}\ a_3^{j_3}\ a_4^{j_4}\ a_5^{j_5}\ a_6^{j_6}\ a_7^{j_7}\ a_8^{j_8}]$ is equal to a linear combination of terms $[b_1^{k_1}\ b_2^{k_2}\ b_3^{k_3}\ b_4^{k_4}]$, where k_1 to k_4 are non-negative integers.

Eliminating a_2, a_3, and a_4, from equations (5.12) to (5.15), and using (5.9), we have

(5.20) $a_1 + a_5 = b_1\ b_2\ b_3 + b_1b_2 + b_1 + 1.$

Multiplying this equation through by a_1, and substituting for $a_1\ a_5$ from (5.5), we have

(5.21) $a_1^2 = (b_1b_2b_3+b_1b_2+b_1+1)a_1 - b_1.$

Now, by means of (5.13) to (5.19), each of the a_1 to a_8 can be expressed in the form

(5.22) $Pa_1 + Q,$

where P and Q are polynomials in b_1 to b_4 with integral coefficients. (We could of course have used any other of the

a_1 to a_8 here instead of a_1.) It follows that any expression of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}$ may be expressed as a polynomial in a_1, the coefficients being polynomials in b_1 to b_4 (with integral coefficients). In view of (5.21) this means that any $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}$ is equal to an expression of the form (5.22).

Now in $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ the term $a_5^{j_1} a_6^{j_2} a_7^{j_3} a_8^{j_4} a_1^{j_5} a_2^{j_6} a_3^{j_7} a_4^{j_8}$, obtained under the interchanges (a_1,a_5), (a_2,a_6), (a_3,a_7), and (a_4, a_8), also occurs. Further b_1 to b_4 are not affected by these interchanges; so that the sum of the two terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ under discussion is equal to an expression of the form

$$P(a_1 + a_5) + 2Q,$$

using the cyclic properties of our relations. But by (5.20) this expression is equal to a linear combination of terms $b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}$. Hence Lemma 5.2 follows, since clearly (again using the cyclic properties of our relations) the other three pairs of terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ correspond to the other three terms of each $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$.

We further write

$$\lambda = b_1 b_3 + b_2 b_4,$$
$$\mu = b_1^2 b_2 b_3 + b_2^2 b_3 b_4 + b_3^2 b_4 b_1 + b_4^2 b_1 b_2,$$

and prove the following:

LEMMA 5.3 Any expression of the form $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$ is equal to

$$S(\lambda) + \mu T(\lambda),$$

where $S(\lambda)$ and $T(\lambda)$ are polynomials in λ with integral coefficients.

By (5.10) and (5.11) any expression of the form $b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}$ can be expressed as a linear combination of terms $b_1^{l_1} b_2^{l_2}$ where l_1 and l_2 are non-negative integers. Clearly then, performing a cyclic summation, any $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$ is equal to a linear combination of terms $[b_1^{l_1} b_2^{l_2}]$, and we need only consider the latter expression, rather than the former.

Writing

$$c_1 = b_1 b_3, \quad c_2 = b_2 b_4,$$

we have by multiplying (5.10) and (5.11) through by b_1 and b_2 respectively

$$b_1^2 = - b_1 - c_1, \tag{5.23}$$

$$b_2^2 = - b_2 - c_2. \tag{5.24}$$

In view of (5.23) and (5.24) any $b_1^{l_1} b_2^{l_2}$ may be expressed in the form

$$A + Bb_1 + Cb_2 + Db_1 b_2,$$

where A, B, C, and D, are polynomials in c_1 and c_2 with integral coefficients. Then, since c_1 and c_2 are not affected by the interchanges (b_1, b_3) and (b_2, b_4), $b_3^{l_1} b_4^{l_2}$ is equal to

$$A + Bb_3 + Cb_4 + Db_3b_4.$$

Hence, using (5.10) and (5.11), we have

$$b_1^{l_1} b_2^{l_2} + b_3^{l_3} b_4^{l_4} = E + D(b_1b_2 + b_3b_4), \tag{5.25}$$

where $E = 2A - B - C$.

Now, using the definitions of c_1 and c_2, the definition of λ, and (5.9), may be written as

$$c_1 + c_2 = \lambda, \tag{5.26}$$

$$c_1c_2 = -1, \tag{5.27}$$

respectively. From these two equations we derive

$$c_1^2 = \lambda c_1 + 1, \tag{5.28}$$

$$c_2^2 = \lambda c_2 + 1. \tag{5.29}$$

In view of (5.27), (5.28), and (5.29), any polynomial in c_1 and c_2, with integral coefficients, may be expressed in the form

$$G + Hc_1 + Ic_2,$$

where G, H, and I, are polynomials in λ with integral coefficients. Hence we may write (5.25) in the form

$$b_1^{l_1} b_2^{l_2} + b_3^{l_1} b_4^{l_2} = (G + Hc_1 + Ic_2)+(G' + H'c_1 + I'c_2)(b_1b_2+b_3b_4),$$

where G', H', and I', are also polynomials in λ with integral coefficients. Further since interchanging b_1, b_2, b_3, and b_4, cyclically corresponds to interchanging c_1 and c_2, and leaving λ unchanged, we also have

$$b_2^{l_1} b_3^{l_2} + b_4^{l_1} b_1^{l_2} = (G + Hc_2 + Ic_1)+(G' + H'c_2 + I'c_1)(b_2b_3+b_4b_1).$$

Thus, adding the last two equations, and using (5.26) and the definitions of c_1 and c_2, we obtain

(5.30) $[b_1^{l_1}\ b_2^{l_2}] = 2G + H\lambda + I\lambda + G'[b_1b_2]+H'[b_1^2b_2b_3]+I'[b_1b_2b_3^2]$.

But

$$[b_1b_2] = (b_1 + b_3)(b_2 + b_4) = 1$$

by (5.10) and (5.11), and

(5.31) $\mu + [b_1b_2b_3^2] = [b_1^2b_2b_3]+[b_1b_2b_3^2] = (b_1b_3+b_2b_4)[b_1b_2]=\lambda.1$.

Hence (5.30) becomes

$$[b_1^{l_1}\ b_2^{l_2}] = (2G+H\lambda+I\lambda+G'+I'\lambda) + \mu(H'-I'),$$

and since both brackets on the right-hand side of this equation are polynomials in λ with integral coefficients, Lemma 5.3 follows.

We have the following relation between λ and μ:

(5.32) $\mu^2 - \lambda\mu + \lambda^3 + 4\lambda^2 + 4\lambda + 15 = 0.$

Since μ^2 is certainly of the form $[b_1^{k_1}\ b_2^{k_2}\ b_3^{k_3}\ b_4^{k_4}]$ we know by Lemma 5.3 that a relation of the above form exists, and the coefficients in the equation are found by comparing coefficients of powers of y in the expansions of the appropriate quantities as power series in y;{cf. the proof of (AH), equation (8.13).} We give a direct proof also: we have

$$\mu^2 - \lambda u = -[b_1^2b_2b_3][b_1b_2b_3^2],$$

using (5.31),

$$= -\{c_1(b_1b_2+b_3b_4)+c_2(b_2b_3+b_4b_1)\}\{c_2(b_1b_2+b_3b_4)+c_1(b_2b_3+b_4b_1)\}$$

$$= -c_1c_2([b_1^2\ b_2^2]+4b_1b_2b_3b_4)-(c_1^2 + c_2^2)[b_1b_2^2\ b_3],$$

$$= [b_1^2\ b_2^2] - 4 - (\lambda^2 + 2)[b_1b_2^2\ b_3]$$

by (5.9), (5.26) and (5.27). But

(5.33) $[b_1^2\, b_2^2] = (b_1^2 + b_3^2)(b_2^2 + b_4^2),$

$$= (1 - 2b_1b_3)(1 - 2b_2b_4)$$

using (5.10) and (5.11),

$$= -2\lambda - 3$$

using (5.9); and

(5.34) $[b_1 b_2^2\, b_3] = b_2b_4(b_1^2 + b_3^2) + b_1b_3(b_2^2 + b_4^2)$

$$= b_2b_4(1 - 2b_1b_3) + b_1b_3(1 - 2b_2b_4)$$

$$= \lambda + 4.$$

Equation (5.32) follows.

Now, by Lemmas 5.1, 5.2, and 5.3, F^6 and $yf(y^{17})F^6\Phi(5)$ are each equal to an expression of the form $S(\lambda) + \mu T(\lambda)$. Since the lowest powers of y in the expansions of F^6, λ, and μ, as power series in y, are -12, -2, and -3, respectively, we assume a form for F^6 with $S(\lambda)$ of degree 6 and $G(\lambda)$ of degree 4. We find the 12 coefficients involved in these two polynomials by comparing coefficients of $y^{-12}, y^{-11}, \ldots, y^{-2}$, and y^{o}, (they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}. The resulting expression for F^6 is found, using (5.32), to be a perfect cube, and in fact we have

(5.35) $$F^2 = \lambda^2 - 20\lambda - 56 + 8\mu,$$

since F, λ, and μ, are real for real y. Similarly, in the case of $yf(y^{17})F^6\,\Phi(5)$, $S(\lambda)$ and $T(\lambda)$ are of degrees 5 and 4 respectively, and we find the 11 coefficients involved by comparing coefficients of y^{-11}, y^{-10}, ..., y^{-2}, and y^{0}, (again they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}; we obtain

$$yf(y^{17})F^6\,\Phi(5)=-834\lambda^5+31236\lambda^4-34498\lambda^3+126757\lambda^2-14022\lambda-$$

(5.36) $$-112984+\mu(-7\lambda^4+9756\lambda^3-69280\lambda^2+162020\lambda-164885).$$

The equations (5.32), (5.35), and (5.36), for q = 17, are of course analagous to (AH), equations (8.13), (11.7), and (11.9), for q = 11.

We now write

$$\delta = b_1b_2 - b_2b_3 + b_3b_4 - b_4b_1.$$

Then

$$\delta^2 = [b_1^2\,b_2^2] - 2[b_1b_2^2b_3]+4b_1b_2b_3b_4,$$

(5.37) $$= -4\lambda - 15$$

by (5.9), (5.33), and (5.34). Also, by (5.35) and (5.37),

$$F^2\delta^2 = (-4\lambda - 15)(\lambda^2 - 20\lambda - 56 + 8\mu),$$

and, using (5.32), it is easily verified that the right-hand side of this equation is equal to

$$(-2\mu + 9\lambda + 30)^2;$$

hence we have

(5.38) $$F\delta = -2\mu + 9\lambda + 30,$$

where the sign of the coefficient of the lowest power of y in the expansion of each side of this equation is examined to determine the appropriate root. Thus, instead of λ and μ, we may take δ and F, as new variables; in fact from (5.37) and (5.38) we have

$$\lambda = -(\delta^2 + 15)/4, \tag{5.39}$$

$$\mu = -(4F\delta + 9\delta^2 + 15)/8. \tag{5.40}$$

Substituting for λ and μ from (5.39) and (5.40) in (5.35) we obtain the following relation between δ and F:

$$(\delta^2 - 17)^2 = 16F(F + 4\delta). \tag{5.41}$$

Also, substituting for λ and μ in (5.36) we obtain $yf(y^{17})F^6\,\Phi(5)$ as a polynomial in δ and F. Further since (5.41) is a quartic in δ, this polynomial is equal to another polynomial in δ and F of degree 3 in δ; in fact we have

$$\begin{aligned} 8yf(y^{17})F^6\,\Phi(5) = {} & \delta^3(84.17^2F^3 + 20.17^5F) + \\ & +\delta^2(115.17F^4+316.17^4F^2+17^7) + \\ & +\delta(28F^5+2476.17^3F^3+32.17^6F) + \\ & +(6677.17^2F^4 + 124.17^5F^2 - 9.17^7) \end{aligned} \tag{5.42}$$

{it is of course obvious from the form of (5.39), (5.40), and (5.41), that the right-hand side of this equation must be a function of δ^2, $F\delta$, and F^2, only}.

We further write

$m_1 = -\,yP(2)P(8)P(3)P(5) - yP(1)P(4)P(6)P(7)$, $n_1 = -y^2P(1)P(4)P(2)P(8)$,

$m_2 = P(6)P(7)P(2)P(8) - y^2P(3)P(5)P(1)P(4)$, $n_2 = P(3)P(5)P(6)P(7)$.

Then

(5.43) and (5.44) $\quad m_1/n_1 = b_1 - b_3,\ m_2/n_2 = b_2 - b_4;$

(5.45) $\quad n_1 n_2 = -y^2 f(y)/f(y^{17});$

(5.46) and (5.47) $\quad n_2/n_1 = b_1 b_3, \quad n_1/n_2 = -b_2 b_4.$

Also,

$$m_1^2/n_1^2 = (b_1-b_3)^2 = (b_1+b_3)^2 - 4b_1 b_3 = 1 - 4n_2/n_1,$$

using (5.10), (5.43), and (5.46), i.e.

(5.48) $\quad m_1^2 = n_1^2 - 4n_1 n_2,$

and correspondingly we may obtain

(5.49) $\quad m_2^2 = n_2^2 + 4n_1 n_2.$

In terms of these new functions we have

(5.50) $\quad \delta = (b_1 - b_3)(b_2 - b_4) = -y^{-2} f(y^{17}) m_1 m_2 / f(y)$

by (5.43), (5.44), and (5.45), and also

(5.51) $\quad \delta^2 = -4\lambda - 15 = -4(b_1 b_3 + b_2 b_4) - 15 = -4(n_2/n_1 - n_1/n_2) - 15,$

by (5.37), (5.46) and (5.47). Now (5.41) may be written in the form

$$16(F + 2\delta)^2 = \delta^4 + 30\delta^2 + 289,$$

but by (5.51) the right-hand side of this equation is equal to

$$16(n_1/n_2 + n_2/n_1)^2,$$

hence we have

$$F + 2\delta = -(n_1/n_2 + n_2/n_1),$$

where the sign of the coefficient of the lowest power of y on each side of this equation is examined to determine the appropriate root. Now the right-hand side of this equation is equal to

$$y^{-2}f(y^{17})(m_1^2 + m_2^2)/f(y)$$

by (5.45), (5.48), and (5.49). Thus using (5.50) we have

(5.52) $y^2 f(y)F/f(y^{17}) = f^4(y)/f^4(y^{17}) = (m_1 + m_2)^2$,

whence

(5.53) $$f^2(y)/f^2(y^{17}) = m_1 + m_2,$$

where again care is taken to select the appropriate root. Further, in view of (5,50) and (5.52) the right-hand side of (5.41) is equal to

$$16y^{-4}f^2(y)(m_1-m_2)^2/f^2(y^{17}),$$

whence, taking the appropriate square root of this expression,

(5.54) $$\delta^2-17=4y^{-2}f(y)(-m_1+m_2)/f(y^{17}).$$

We note that elimination of δ from equations (5.50) and (5.54) gives

(5.55) $m_1^2\, m_2^2 + 4y^2f^3(y)(m_1-m_2)/f^3(y^{17})-17y^4f^2(y)/f^2(y^{17})=0.$

Making a slight change in notation for convenience, we now re-state (5.53), (5.55), (5.50), (5.42), and (5.41), in order, as follows.

THEOREM 5.1 If we write

$$M_1 = f^2(y^{17})\{-yP(2)P(8)P(3)P(5)-yP(1)P(4)P(6)P(7)\}/f^2(y),$$
$$M_2 = f^2(y^{17})\{P(6)P(7)P(2)P(8)-y^2P(3)P(5)P(1)P(4)\}/f^2(y),$$

then we have

$$M_1 + M_2 = 1,$$
$$M_1^2\, M_2^2 + 4(M_1 - M_2)/F - 17/F^2 = 0,$$

where $F = y^{-2}f^3(y)/f^3(y^{17})$; and if we further write

$$\varepsilon = -M_1M_2,$$

then we have

$$8yf(y^{17})\,\bar{\Phi}(5) = \varepsilon^3(84.17^2 + 20.17^5/F^2) + $$
$$+\varepsilon^2(115.17.+316.17^4/F^2+17^7/F^4)+$$
$$+\varepsilon(28+2476.17^3/F^2+32.17^6/F^4) +$$
$$+(6677.17^2/F^2+124.17^5/F^4-9.17^7/F^6),$$

where, from the last three equations but one, there is the following relation between ε and F

$$(\varepsilon^2 - 17/F^2)^2 = 16(4\varepsilon + 1)/F^2.$$

§ We conclude this Part by deriving the following simple congruence

(5.56) $\bar{\Phi}(5) \equiv f^2(y^{17})f^5(y)\{7P(3)P(5)P(6)P(7)+6y^2P(1)P(2)P(4)P(8)\}$ (mod. 17).

Since the only term on the right-hand side of (5.42) without a factor 17 is $28\delta F^5$, we have

(5.57) $yf(y^{17})F\,\bar{\Phi}(5) \equiv -5\delta$ (mod. 17).

But from (5.51)

$$\delta^2 \equiv -4(n_2 + 4n_1)^2/n_1n_2 \quad \text{(mod. 17)},$$

and using (5.45)

$$-1/n_1n_2 = y^{-2}f(y^{17})/f(y) \equiv y^{-2}f^{16}(y) \quad \text{(mod. 17)}$$

since $f^{17}(y) \equiv f(y^{17})$ (mod. 17), so that, taking the appropriate square root,

(5.58) $\delta \equiv 2y^{-1}f^8(y)(n_2+4n_1)$ (mod. 17).

(5.56) follows immediately, from (5.57), (5.58), and the definitions of n_1, n_2, and F.