PART 3

$q = 19$ throughout this Part.

6. We write

$$\alpha_1 = -x^{-8}P(2)/P(1),\ \alpha_2 = x^{-13}P(4)/P(2),\ \alpha_3 = x^{-14}P(8)/P(4),$$
$$\alpha_4 = x^{20}P(3)/P(8),\ \alpha_5 = -x^{-15}P(6)/P(3),\ \alpha_6 = x^{-3}P(7)/P(6),$$
$$\alpha_7 = -x^{7}P(5)/P(7),\ \alpha_8 = -x^{-10}P(9)/P(5),\ \alpha_9 = -x^{36}P(1)/P(9);$$

then by (ASD), Lemma 6 (with $q = 19$) we have

$$(6.1)\quad -x^{-15}f(x)/f(y^{19}) = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+1.$$

In (6.1) we replace x by $w_r x$ where $w_r (r = 1 \text{ to } 19)$ are the nineteenth roots of unity, and multiply together the nineteen resulting equations, obtaining

$$(6.2)\quad -y^{-15}f^{20}(y)/f^{20}(y^{19}) = \prod_{r=1}^{19}(\alpha_1 w_r^{-8}+\alpha_2 w_r^{-13}+\alpha_3 w_r^{-14}+\alpha_4 w_r^{20}+$$
$$+\alpha_5 w_r^{-15}+\alpha_6 w_r^{-3}+\alpha_7 w_r^{7}+\alpha_8 w_r^{-10}+\alpha_9 w_r^{36}+1).$$

Now as w_r runs through the nineteenth roots of unity so does w_r^5, so that the product on the right-hand side of (6.2) is equal to

$$\prod_{r=1}^{19}(\alpha_1 w_r^{36}+\alpha_2 w_r^{-8}+\alpha_3 w_r^{-13}+\alpha_4 w_r^{-14}+\alpha_5 w_r^{20}+\alpha_6 w_r^{-15}+\alpha_7 w_r^{-3}+\alpha_8 w_r^{7}+\alpha_9 w_r^{-10}+1),$$

and is thus unchanged if $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$, and α_9, are interchanged cyclically. The product is thus a linear combination of terms $[\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_5^{i_5}\alpha_6^{i_6}\alpha_7^{i_7}\alpha_8^{i_8}\alpha_9^{i_9}]$ where i_1 to i_9 are non-negative integers, and considering the left-hand side of (6.2) such terms as occur can only involve x in terms of $y = x^{19}$. Thus if $\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8} \alpha_9^{i_9}$ (or any other term of $[\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8} \alpha_9^{i_9}]$) occurs we must have

$$(6.3) \quad -8i_1-13i_2-14i_3+20i_4-15i_5-3i_6+7i_7-10i_8+36i_9 \equiv 0 \pmod{19}$$

(interchanging $i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8$, and i_9, cyclically gives the same congruence).

Now, writing

$a_1=y^{-1}P(6)P(7)/P(2)P(9)$, $\quad a_2=y^{-2}P(7)P(5)/P(4)P(1)$,

$a_3=-y^{-1}P(5)P(9)/P(8)P(2)$, $\quad a_4=-P(9)P(1)/P(3)P(4)$,

$a_5=y^{3}P(1)P(2)/P(6)P(8)$, $\quad a_6=-yP(2)P(4)/P(7)P(3)$,

$a_7=-P(4)P(8)/P(5)P(6)$, $\quad a_8=yP(8)P(3)/P(9)P(7)$,

$a_9=-y^{-1}P(3)P(6)/P(1)P(5)$,

it is easily verified that

$$\alpha_1^{19}=a_2^{16}a_3a_4^2a_5^7a_6^{13}a_7^5a_8^3a_9^{12}, \quad \alpha_2^{19}=a_3^{16}a_4a_5^2a_6^7a_7^{13}a_8^5a_9^3a_1^{12},$$

$$\alpha_3^{19}=a_4^{16}a_5a_6^2a_7^7a_8^{13}a_9^5a_1^3a_2^{12}, \quad \alpha_4^{19}=a_5^{16}a_6a_7^2a_8^7a_9^{13}a_1^5a_2^3a_3^{12},$$

(6.4)

$$\alpha_5^{19}=a_6^{16}a_7a_8^2a_9^7a_1^{13}a_2^5a_3^3a_4^{12}, \quad \alpha_6^{19}=a_7^{16}a_8a_9^2a_1^7a_2^{13}a_3^5a_4^3a_5^{12},$$

$$\alpha_7^{19}=a_8^{16}a_9a_1^2a_2^7a_3^{13}a_4^5a_5^3a_6^{12}, \quad \alpha_8^{19}=a_9^{16}a_1a_2^2a_3^7a_4^{13}a_5^5a_6^3a_7^{12},$$

$$\alpha_9^{19}=a_1^{16}a_2a_3^2a_4^7a_5^{13}a_6^5a_7^3a_8^{12}.$$

It will be noticed that all of the equations (6.4) may be obtained from any one of them by interchanging

$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9$, cyclically. By (6.4), since

$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 = -1$,

$$(\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8} \alpha_9^{i_9})^{19} =$$

$$= (a_1a_2a_3a_4a_5a_6a_7a_8a_9)^{\sigma} a_1^{\sigma_1} a_2^{\sigma_2} a_3^{\sigma_3} a_4^{\sigma_4} a_5^{\sigma_5} a_6^{\sigma_6} a_7^{\sigma_7} a_8^{\sigma_8} a_9^{\sigma_9}$$

where $\sigma = 30i_1+32i_2+2i_3+34i_4+10i_5+28i_6+24i_7+8i_8+20i_9$, an even integer, and

$$\sigma_1 = 12i_2+3i_3+5i_4+13i_5+7i_6+2i_7+i_8+16i_9,$$
$$\sigma_2 = 12i_3+3i_4+5i_5+13i_6+7i_7+2i_8+i_9+16i_1,$$
$$\sigma_3 = 12i_4+3i_5+5i_6+13i_7+7i_8+2i_9+i_1+16i_2,$$
$$\sigma_4 = 12i_5+3i_6+5i_7+13i_8+7i_9+2i_1+i_2+16i_3,$$
$$\sigma_5 = 12i_6+3i_7+5i_8+13i_9+7i_1+2i_2+i_3+16i_4,$$
$$\sigma_6 = 12i_7+3i_8+5i_9+13i_1+7i_2+2i_3+i_4+16i_5,$$
$$\sigma_7 = 12i_8+3i_9+5i_1+13i_2+7i_3+2i_4+i_5+16i_6,$$
$$\sigma_8 = 12i_9+3i_1+5i_2+13i_3+7i_4+2i_5+i_6+16i_7,$$
$$\sigma_9 = 12i_1+3i_2+5i_3+13i_4+7i_5+2i_6+i_7+16i_8;$$

moreover $\sigma + \sigma_1$ to $\sigma + \sigma_9$ are multiples of 19 by (6.3), hence any expression of the form $\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8} \alpha_9^{i_9}$ for which (6.3) holds is of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$ where j_1 to j_9 are non-negative integers. Thus every term occurring in the right-hand side of (6.2) is of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$, and such terms occur in cylically symmetrical sets of nine terms each.

Further, $\Phi(4)$ is the coefficient of x^4 in $1/f(x)$ regarded as a polynomial of degree 18 in x with coefficients involving x in terms of $y = x^{19}$, so that $y^{-14}f^{20}(y)\Phi(4)/f^{19}(y^{19})$ is the coefficient of x^o in $y^{-15}f^{20}(y)/\{f^{20}(y^{19})(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+1)\}$. This is a cyclically symmetric polynomial of degree 18 in $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$, and α_9; and the terms which give the coefficient of x^o occur only in symmetrical sets of nine expressible as $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$, as before. (This is not true for the coefficient of any power of x other than 0; the nine terms of $[a_1]$, for example, do not appertain to the same power of x.)

Thus writing

$$F = y^{-3}f^4(y)/f^4(y^{19})$$

we have the following:

LEMMA 6.1 F^5 and $yf(y^{19})F^5\,\Phi(4)$ are each equal to a linear combination of terms $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$.

We now write

(6.5) to (6.7) $b_1 = a_1a_4a_7$, $b_2 = a_2a_5a_8$, $b_3 = a_3a_6a_9$;

(6.8) to (6.10) $c_1 = a_1a_4+a_4a_7+a_7a_1$, $c_2 = a_2a_5+a_5a_8+a_8a_2$,

$c_3 = a_3a_6+a_6a_9+a_9a_3$;

(6.11)to(6.13) $d_1 = a_1+a_4+a_7$, $d_2 = a_2+a_5+a_8$, $d_3 = a_3+a_6+a_9$;

so that

(6.14) $$b_1 b_2 b_3 + 1 = 0.$$

<9, 6, 5, 3>, <9, 7, 6, 1>, <7, 5, 2, 1>, <9, 5, 4, 2>, <9, 8, 4, 1>, <8, 3, 2, 1>, <6, 4, 3, 2>, <8, 7, 6, 4>, and <8, 7, 5, 3>, give, respectively,

(6.15) to (6.17) $a_1a_4 = a_3+1, \quad a_2a_5 = a_4+1, \quad a_3a_6 = a_5+1,$

(6.18) to (6.20) $a_4a_7 = a_6+1, \quad a_5a_8 = a_7+1, \quad a_6a_9 = a_8+1,$

(6.21) to (6.23) $a_7a_1 = a_9+1, \quad a_8a_2 = a_1+1, \quad a_9a_3 = a_2+1.$

It will be observed that each of the equations (6.5) to (6.23) remains valid when b_1, b_2, b_3, and c_1, c_2, c_3, and d_1, d_2, d_3, and a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, are interchanged cyclically. We are now in a position to prove

LEMMA 6.2 Any expression of the form $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ is equal to a linear combination of terms $[b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}]$, where the square bracket in this case denotes a summation of the <u>three</u> different terms obtained by interchanging b_1, b_2, b_3, and c_1, c_2, c_3, and d_1, d_2, d_3, <u>separately</u>, and k_1 to k_9 are non-negative integers.

By eliminating a_3 and a_9 from equations (6.15), (6.21), and (6.23), we obtain

(6.24) $$a_2 = a_1^2+(b_1 - d_1)a_1,$$

and clearly this equation remains valid when a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, and b_1, b_2, b_3, and d_1, d_2, d_3, are interchanged cyclically. Thus, by means of (6.24) and the eight similar equations, each of the a_1 to a_9 can be expressed as a polynomial in b_1, b_2, b_3, d_1, d_2, d_3, and a_1, with integral coefficients; and hence any expression of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$ is equal to such a polynomial. (We could of course have used any other of the a_1 to a_9 here instead of a_1.) But (in view of the definitions of b_1, c_1, and d_1) a_1 (and a_4 and a_7) satisfies a cubic equation with coefficients in terms of b_1, c_1, and d_1. Hence any $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$ may be expressed in the form

$$Pa_1^2 + Qa_1 + R,$$

where P, Q, and R, are polynomials in b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, and d_3, with integral coefficients.

Now in $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ the terms

$a_4^{j_1} a_5^{j_2} a_6^{j_3} a_7^{j_4} a_8^{j_5} a_9^{j_6} a_1^{j_7} a_2^{j_8} a_3^{j_9}$ and

$a_7^{j_1} a_8^{j_2} a_9^{j_3} a_1^{j_4} a_2^{j_5} a_3^{j_6} a_4^{j_7} a_5^{j_8} a_6^{j_9}$, obtained under the cyclic interchanges (a_1, a_4, a_7), (a_2, a_5, a_8), and (a_3, a_6, a_9), also occur. Further b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, and d_3, are not affected by these interchanges; so that the sum of the three terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ under discussion is equal to an expression of the form

$$P(a_1^2 + a_4^2 + a_7^2)+Q(a_1 + a_4 + a_7)+3R,$$

using the cyclic properties of our relations. Since from the definitions of c_1 and d_1

$$a_1 + a_4 + a_7 = d_1,$$
$$a_1^2 + a_4^2 + a_7^2 = d_1^2 - 2c_1,$$

this expression is equal to a linear combination of terms $b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}$. Hence Lemma 6.2 follows, since clearly (again using the cyclic properties of our relations) the other two triplets of terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ correspond to the other two terms of each $[b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}]$.

We now prove

LEMMA 6.3 Any expression of the form $[b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}]$ is equal to a linear combination of terms $[b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}]$, where k_1' to k_3' are non-negative integers.

Clearly it will be sufficient to show that c_1, c_2, c_3, d_1, d_2, and d_3, can all be expressed as polynomials in b_1, b_2, and b_3, with integral coefficients. For then any $b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}$ may be expressed as a linear combination of terms $b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}$, and Lemma

6.3 follows from cyclic considerations.

We have

(6.25) to (6.27) $c_1 = d_3+3, \quad c_2 = d_1+3, \quad c_3 = d_2+3,$

the first of which is (6.15) + (6.18) + (6.21), in the obvious notation, and

(6.28) to (6.30) $b_1^2=b_3+c_3+d_3+1, \quad b_2^2=b_1+c_1+d_1+1,$
$b_3^2=b_2+c_2+d_2+1,$

the first of which is (6.15).(6.18).(6.21). Substituting for c_1, c_2, and c_3, in (6.28) to (6.30) from (6.25) to (6.27), and solving the resulting equations for d_1, d_2, and d_3, we obtain

(6.31) $$2d_1=-b_1^2+b_2^2+b_3^2-b_1-b_2+b_3-4,$$

(6.32) $$2d_2=-b_2^2+b_3^2+b_1^2-b_2-b_3+b_1-4,$$

(6.33) $$2d_3=-b_3^2+b_1^2+b_2^2-b_3-b_1+b_2-4.$$

We now show that

(6.34) $$b_1+b_2+b_3+2 = 0.$$

Then the right-hand side of (6.31) is equal to

$$(b_1+b_2+b_3)^2-2b_1-2(b_1b_2+b_2b_3+b_3b_1)-(b_1+b_2+b_3)+2b_3-4$$

$$= -2b_1^2-2(b_1b_2+b_2b_3+b_3b_1)+2b_3+2,$$

and since the latter expression has a factor 2 we have d_1, and hence c_2 by (6.26), as a polynomial in b_1, b_2, and b_3, with integral coefficients; clearly from cyclic considerations the same is true of d_2, d_3, c_1, and c_3, and we have the Lemma.

(6.34) is proved as follows. We have

$$b_1d_1 = c_3+2d_3+3,$$

which is (6.15).(6.18) + (6.18).(6.21) + (6.21).(6.15). Substituting for c_3 from (6.27) and then for d_1, d_2, and d_3, from (6.31) to (6.33), the resulting equation simplifies to

$$b_1^3-b_1b_2^2-b_3^2b_1+4b_1^2+b_2^2-b_3^2+b_1b_2-b_3b_1+3b_1+b_2-3b_3 = 0,$$

and of course we may interchange b_1, b_2, and b_3, cyclically in this equation to obtain two other similar relations. Adding all three equations we arrive at

$$[b_1^3]-[b_1b_2^2]-[b_1^2b_2]+4[b_1^2]+[b_1] = 0.$$

But it is easily verified that the left-hand side of this equation is equal to

$$([b_1]+2)([b_1^2]-2[b_1b_2]+2[b_1]-3),$$

using (6.14); and the second of these two factors, expanded as a power series in y, begins $4y^{-2}+ \ldots$ and is therefore non-zero. Thus we arrive at the relation (6.34), and complete the proof of Lemma 6.3.

We further write

$$\lambda = [b_1b_2],$$

$$\mu = [b_1^2b_2],$$

and prove the following:

LEMMA 6.4 Any expression of the form $[b_1^{k'_1} b_2^{k'_2} b_3^{k'_3}]$ is equal to

$$S(\lambda) + \mu T(\lambda),$$

where $S(\lambda)$ and $T(\lambda)$ are polynomials in λ with integral coefficients.

By (6.34) any expression of the form $b_1^{k'_1} b_2^{k'_2} b_3^{k'_3}$ can be expressed as a linear combination of terms $b_1^{l_1} b_2^{l_2}$ where l_1 and l_2 are non-negative integers. Clearly then, performing a cyclic summation, any $[b_1^{k'_1} b_2^{k'_2} b_3^{k'_3}]$ is equal to a linear combination of terms $[b_1^{l_1} b_2^{l_2}]$, and we need only consider the latter expression, rather than the former.

Now, by (6.14), (6.34), and the definition of λ, b_1 to b_3 are the roots of the cubic equation

$$z^3 + 2z^2 + \lambda z + 1 = 0,$$

so that we have

$$b_1^3 = -2b_1^2-\lambda b_1-1, \tag{6.35}$$

$$b_2^3 = -2b_2^2-\lambda b_2-1. \tag{6.36}$$

In view of (6.35) and (6.36) any $b_1^{l_1} b_2^{l_2}$ may be expressed in the form

$$G+Hb_1+Ib_2+Jb_1^2+Kb_2^2+Lb_1b_2+Mb_1^2b_2+Nb_1b_2^2+Pb_1^2b_2^2,$$

where G, H, I, J, K, L, M, N, and P, are polynomials in λ with integral coefficients. Then, since λ is not affected when b_1, b_2, and b_3, are interchanged cyclically, we have

(6.37) $[b_1^{l_1} b_2^{l_2}] = 3G + (H+I)[b_1] + (J+K)[b_1^2] + L[b_1 b_2] + M[b_1^2 b_2] +$

$+ N[b_1 b_2^2] + P[b_1^2 b_2^2].$

But we have (6.34) and the definitions of λ and μ

(6.38) to (6.40) $[b_1] = -2$, $[b_1 b_2] = \lambda$, $[b_1^2 b_2] = \mu$,

and

(6.41) $[b_1^2] = [b_1]^2 - 2[b_1 b_2] = 4 - 2\lambda$,

(6.42) $[b_1 b_2^2] = [b_1][b_1 b_2] - [b_1^2 b_2] - 3 b_1 b_2 b_3 = -2\lambda - \mu + 3$,

(6.43) $[b_1^2 b_2^2] = [b_1 b_2]^2 - 2[b_1^2 b_2 b_3] = [b_1 b_2]^2 + 2[b_1] = \lambda^2 - 4$,

using (6.14). Hence (6.37) becomes

$[b_1^{l_1} b_2^{l_2}] = \{3G - 2(H+I) + (4-2\lambda)(J+K) + \lambda L + (-2\lambda+3)N + (\lambda^2-4)P\} + \mu\{M-N\}$,

and since both curly brackets on the right-hand side of this equation are polynomials in λ with integral coefficients, Lemma 6.4 follows.

We have the following relation between λ and μ:

(6.44) $\mu^2 + (2\lambda-3)\mu + \lambda^3 - 12\lambda + 17 = 0.$

Since μ^2 is certainly of the form $[b_1^{k'_1} b_2^{k'_2} b_3^{k'_3}]$ we know by Lemma 6.4 that a relation of the above form exists, and the coefficients in the equation are found by comparing coefficients of powers of y in the expansions of the appropriate quantities as power series in y; {cf. the proof of (AH), equation (8.13).} We give a direct proof also: we have

$$\mu^2+(2\lambda-3)\mu = -[b_1^2b_2][b_1b_2^2]$$

by (6.40) and (6.42),

$$= -[b_1^3b_2^3] + [b_1^3] - 3$$

using (6.14). But

$$[b_1^3b_2^3] = [b_1b_2][b_1^2b_2^2] + [b_1^2b_2] + [b_1b_2^2]$$

using (6.14),

$$= \lambda^3 - 6\lambda + 3$$

by (6.39), (6.40), (6.42), and (6.43); and

$$[b_1^3] = [b_1][b_1^2] - [b_1^2b_2] - [b_1b_2^2],$$

$$= 6\lambda - 11$$

by (6.38), (6.40), (6.41), and (6.42). Equation (6.44) follows.

Now, by Lemmas 6.1, 6.2, 6.3, and 6.4, F^5 and $yf(y^{19})F^5\,\Phi(4)$ are each equal to an expression of the form $S(\lambda) + \mu T(\lambda)$. Since the lowest powers of y in the expansions of F^5, λ, and μ, as power series in y, are -15, -2, and -3, respectively, we assume a form for F^5 with $S(\lambda)$ of degree 7 and $T(\lambda)$ of degree 6. We find the 15 coefficients involved in these two polynomials by comparing coefficients of y^{-15}, y^{-14}, ..., y^{-2}, and y^0, (they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}. The resulting expression for F^5 is found, using (6.44), to be a perfect fifth power, and in fact we have

(6.45) $$F = \mu + 5\lambda + 9,$$

since F, λ, and μ, are real for real y. Similarly, in the case of $yf(y^{19})F^5\Phi(4)$, $S(\lambda)$ and $T(\lambda)$ are of degrees 7 and 5 respectively, and we find the 14 coefficients involved by comparing coefficients of y^{-14}, y^{-13}, ..., y^{-2}, and y^{0}, (again they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}; we obtain

(6.46) $$yf(y^{19})F^5\Phi(4)=-5\lambda^7+27734\lambda^6-1018027\lambda^5+4089364\lambda^4+10082120\lambda^3-61692429\lambda^2+67638607\lambda-319561+\mu(-1155\lambda^5+259455\lambda^4-3809331\lambda^3+10287942\lambda^2+2093087\lambda-16560108).$$

The equations (6.44), (6.45), and (6.46), for $q = 19$, are of course analogous to (AH), equations (8.13), (11.7), and (11.9), for $q = 11$.

We now write

$$m_1 = yP(1)P(7)P(8), \qquad m_2 = -y^2P(2)P(3)P(5),$$
$$m_3 = P(4)P(6)P(9).$$

Then

(6.47) $$m_1m_2m_3 = -y^3f(y)/f(y^{19});$$

(6.48) to (6.50) $m_1/m_2 = -b_1$, $m_2/m_3 = -b_2$, $m_3/m_1 = -b_3$.

Also, in terms of these new functions (6.34) becomes

(6.51) $$m_1m_2^2+m_2m_3^2+m_3m_1^2 = -2y^3f(y)/f(y^{19}),$$

by (6.47) to (6.50). We now prove the following relation

$$(6.52) \qquad m_1m_2+m_2m_3+m_3m_1 = yf^2(y)/f^2(y^{19}).$$

Denoting the left-hand side of this equation by X we have

$$X/m_1m_2 = 1-b_3+b_3b_1,$$

$$X/m_2m_3 = 1-b_1+b_1b_2,$$

$$X/m_3m_1 = 1-b_2+b_2b_3,$$

by (6.48) to (6.50). Multiplying together these three equations we obtain

$$X^3/m_1^2m_2^2m_3^2 = -[b_1b_2^2] + 3[b_1b_2] -3[b_1]+6,$$

using (6.14). But by (6.38), (6.39), and (6.42), the right-hand side of this equation is equal to $\mu + 5\lambda + 9$, or by (6.45) to F. Hence

$$X^3 = m_1^2m_2^2m_3^2y^{-3}f^4(y)/f^4(y^{19}) = y^3f^6(y)/f^6(y^{19})$$

using (6.47), and (6.52) follows, since X and $f(y)$ are real for real y. Next we show that

$$(6.53) \qquad y^{-2}f(y)(m_1+m_2+m_3)/f(y^{19}) = -\lambda-5.$$

It would be possible to prove this relation by a method similar to that used for (6.52), however the following proof is simpler. Using (6.47) we write (6.52) in the form

$$1/m_1+1/m_2+1/m_3 = -y^{-2}f(y)/f(y^{19}).$$

Then, in view of this relation, the left-hand side of (6.53) is equal to

$$-(m_1+m_2+m_3)(1/m_1+1/m_2+1/m_3),$$
$$= -(m_1/m_2+m_2/m_3+m_3/m_1)-(m_1/m_3+m_2/m_1+m_3/m_2)-3,$$
$$= [b_1]-[b_1b_2]-3$$

by (6.48) to (6.50), and hence is equal to $-\lambda-5$ by (6.38) and (6.39); thus (6.53) is proved. Now, if we write

(6.54) $\delta = y^{-2}f(y)(m_1+m_2+m_3)/f(y^{19})$,

then instead of λ and μ we may take δ and F, as new variables, in view of (6.45) and (6.53). In fact from these two relations we have

(6.55) $\lambda = -\delta - 5$,

(6.56) $\mu = F + 5\delta + 16$.

Substituting for λ and μ from (6.55) and (6.56) in (6.44) we obtain the following relation between δ and F:

(6.57) $\delta^3 = F(F + 8\delta + 19)$.

Also, substituting for λ and μ in (6.46) we obtain $yf(y^{19})F^5\Phi(4)$ as a polynomial in δ and F. Further since (6.57) is a cubic in δ, this polynomial is equal to another polynomial in δ and F of degree 2 in δ; in fact we have

$$yf(y^{19})F^5\Phi(4)=\delta^2(65.19F^3+1137.19^3F^2+363.19^5F+7.19^7)+$$
(6.58) $$+\delta(5F^4+2504.19^2F^3+3016.19^4F^2+232.19^6F+19^8)+$$
$$+(2276.19F^4+5431.19^3F^3+717.19^5F^2+24.19^7F+19^8).$$

Making a slight change in notation for convenience, we now re-state (6.47), (6.52), (6.51), (6.54), (6.58), and (6.57), in order, as follows.

THEOREM 6.1 If we write

$$M_1 = y^2 f^3(y^{19})P(1)P(7)P(8)/f^3(y), \quad M_2 = -y^3 f^3(y^{19})P(2)P(3)P(5)/f^3(y),$$

$$M_3 = y f^3(y^{19})P(4)P(6)P(9)/f^3(y),$$

then we have

$$M_1M_2M_3 = -1/F^2,$$

$$M_1M_2+M_2M_3+M_3M_1 = 1/F,$$

$$M_1M_2^2+M_2M_3^2+M_3M_1^2 = -2/F^2,$$

where $F = y^{-3}f^4(y)/f^4(y^{19})$; and if we further write

$$\varepsilon = M_1 + M_2 + M_3,$$

then we have

$$y f(y^{19})\Phi(4) = \varepsilon^2(65.19+1137.19^3/F+363.19^5/F^2+7.19^7/F^3)+$$
$$+\varepsilon(5+2504.19^2/F+3016.19^4/F^2+232.19^6/F^3+19^8/F^4)+$$
$$+(2276.19/F+5431.19^3/F^2+717.19^5/F^3+24.19^7/F^4+19^8/F^5),$$

where, from the last four equations but one, there is the following relation between ε and F

$$\varepsilon^3 = (8\varepsilon + 1)/F + 19/F^2.$$

We conclude this Part by observing that in the last equation but one the only term on the right-hand side without a factor 19 is 5ε, so that, in view of the definitions of ε and M_1 to M_3, we have the following simple congruence, modulo 19,

(6.59) $\Phi(4) \equiv 5f(y^{19})f^{16}(y)\{P(4)P(6)P(9)+yP(1)P(7)P(8)-y^2P(2)P(3)P(5)\}$,

since $f^{19}(y) \equiv f(y^{19})$ (mod. 19).