## PART 4

q = 11 throughout this Part

The notation is that of (AH).

7. The following relations, not given in (AH), are needed. They are of a type which has no analogue in the cases

$q = 5$, 7, and 13.

(7.1) $[rt] = yf^2(y)/f^2(y^{11})$,

(7.2) $y^{-4}f^3(y)[rsu]/f^3(y^{11}) = \lambda + 13$,

(7.3) $y^{-2}f^5(y)[r]/f^5(y^{11}) = -\mu + 6\lambda + 16$,

(7.4) $y^{-7}f^8(y)[rstu]/f^8(y^{11}) = -\lambda^2 - 11\mu + 40\lambda + 7$.

We prove (7.1) as follows. Denoting the left-hand side of the equation by X we have, using the definitions of

$\alpha$, $\beta$, $\gamma$, $\delta$, and $\epsilon$,

$$X/rt = \delta\epsilon\alpha\beta + \delta\epsilon\beta + \epsilon\beta + \epsilon + 1,$$

together with the other four equations obtained on interchanging r, s, t, u, v, and $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, cyclically. Multiplying together these five equations we see that $X^5/(rstuv)^2$, i.e. $y^{-10}f^2(y)X^5/f^2(y^{11})$, is equal to a cyclically symmetric polynomial in $\alpha$, $\beta$, $\gamma$, $\delta$, and $\epsilon$, with integral

---

We may note that in view of the relation (7.1) the factor D in the expressions for the $r_{bc}(d)$ for $q = 11$ is equal to $y^{-1}f^2(y^{11})/f^2(y)$.

coefficients, i.e. to a linear combination of terms $[\alpha^{\ell}\ \beta^{m}\ \gamma^{n}\ \delta^{p}\ \epsilon^{q}]$, each of which is equal to an expression of the form $Q_1(\lambda) + \mu Q_2(\lambda)$ by (AH), Lemma 9. Using the method employed in (AH) to find the relations (11.7), (11.8), and (11.9), that of comparing coefficients of powers of y in power series expansions, we obtain

$$y^{-10}f^{2}(y)X^{5}/f^{2}(y^{11}) = \lambda\mu - 17\lambda^{2} - 108\mu + 346\lambda - 131.$$

But the right-hand side of this equation is the same as the right-hand side of (AH), equation (11.7). Thus, taking fifth roots, (7.1) follows, since X and f(y) are real for real y.

(7.2), (7.3), and (7.4), may be proved in a manner similar to that used for (7.1), and we omit the details, although it should be pointed out that we now need (AH), equation (8.13) as well as (AH), equation (11.7).

From (7.2), (7.3), and (7.4), together with (AH), equation (11.9), we have the following result:

(7.5) $$\Phi(6) = -11y^{-3}f^{3}(y^{11})[rstu]/f^{4}(y) + 2.11^{2}yf^{6}(y^{11})[r]/f^{7}(y) - 11^{3}f^{8}(y^{11})[rsu]/f^{9}(y) + 11^{4}y^{4}f^{11}(y^{11})/f^{12}(y).$$

We now give conjectural expressions for the other ten $\Phi(s)$ as follows. We write

$$\phi(0) = y^{-1}P(4)\,\Phi(0)/P(2), \qquad \phi(5) = y^{-1}P(5)\Phi(5)/P(1),$$

$$\phi(4) = P(3)\Phi(4)/P(4), \qquad \phi(2) = P(1)\Phi(2)/P(2),$$

$$\phi(9) = -\,P(5)\Phi(9)/P(3), \qquad \phi(1) = -\,P(2)\Phi(1)/P(4),$$

$$\phi(7) = -\,yP(1)\Phi(7)/P(5), \qquad \phi(8) = -\,P(4)\Phi(8)/P(3),$$

$$\phi(10) = -\,P(2)\Phi(10)/P(1), \qquad \phi(3) = P(3)\Phi(3)/P(5).$$

Then

THEOREM 7.1 We have

$$\begin{aligned}\phi(0) = {} & (tv)y^{-2}f(y^{11})/f^2(y) + \\ & +(-5rstu-53stuv+41tuvr-uvrs+29vrst)y^{-2}f^3(y^{11})/f^4(y)+ \\ & +11(-45r+6s+63t-48u+2v)yf^6(y^{11})/f^7(y)+ \\ & +11^2(-6rsu-32stv+20tur+4uvs+25vrt)f^8(y^{11})/f^9(y)+ \\ & +11^3(-r/s-3s/t+2t/u-5u/v-4v/r)y^4f^{11}(y^{11})/f^{12}(y),\end{aligned}$$

$$\begin{aligned}\phi(5) = {} & (7tv)y^{-2}f(y^{11})/f^2(y) + \\ & +(-2rstu-217stuv-10tuvr+15uvrs-6vrst)y^{-2}f^3(y^{11})/f^4(y)+ \\ & +11(-9r-12s+171t+30u-15v)yf^6(y^{11})/f^7(y)+ \\ & +11^2(6rsu-67stv-20tur+18uvs+19vrt)f^8(y^{11})/f^9(y)+ \\ & +11^3(6r/s+7s/t+10t/u+8u/v+2v/r)y^4f^{11}(y^{11})/f^{12}(y),\end{aligned}$$

---

It is of interest to note that (7.5) is essentially the "right" form for $\Phi(6)$, being equivalent to the equation

$$yf(y^{11})\Phi(6) = 11g_2 + 2.11^2g_3 + 11^3g_4 + 11^4g_5$$

given by Atkin [ 1 ] in his proof of the Ramanujan congruence for $11^2$.

and these equations still hold if $\phi(0)$, $\phi(4)$, $\phi(9)$, $\phi(7)$, and $\phi(10)$, or $\phi(5)$, $\phi(2)$, $\phi(1)$, $\phi(8)$, and $\phi(3)$, are interchanged cyclically, so long as r, s, t, u, and v, are also interchanged cyclically.

We hope to prove the above theorem at a later date. The following considerations put its validity beyond any reasonable doubt.

Firstly the definitions of the $\phi(s)$ and the general form of the theorem are analogous for $q = 11$ to the case of $q = 13$ (§.2). Secondly, noting (7.1) and the following relation

$$r/s + s/t + t/u + u/v + v/r = 1$$

which is given in (AH) (page 186) as $[\alpha] = 1$, we point out the correspondence between the expressions for the $\phi(s)$ given in the theorem and the expression for $\Phi(6)$ given by (7.5).

Lastly, in finding the theorem, we assumed that the $\phi(s)$ could be expressed in such a form, and then found <u>and checked</u> the values of the coefficients involved by comparing coefficients of powers of y in power series expansions, in a manner similar to that used for $q = 13$. In fact we made five distinct checks in the case of each of our two sets of coefficients. The powers of 11 which appear in the coefficients serve as an additional check.