PART 5

8. The following theorem is proved in [4] (Theorem 12, pages 95 and 96):

THEOREM 8.1 Suppose that g and h are simple automorphic functions on a group G, such that g has precisely α poles in the fundamental region of G and h has precisely β poles in the fundamental region of G. Then there is a polynomial in u and v, $P(u, v)$, such that $P(g, h) = 0$ and $\deg_u P = \beta$, $\deg_v P = \alpha$.

In our application of this theorem, q is prime and $G = \Gamma_o(q^2)$, where the subgroup $\Gamma_o(n)$ (n a non-zero integer) of the modular group is defined as the group of transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}, \text{ a, b, c, d integral, } ad - bc = 1,\ c \equiv 0 \text{ (mod. n)}.$$

Also we choose

$$g = g(\tau) = \{\eta(q\tau)/\eta(\tau)\}^s, \quad h = h(\tau) = \eta(q^2\tau)/\eta(\tau),$$

where $\eta(\tau)$, the Dedekind modular form, is defined by

$$\eta(\tau) = \exp(\pi i\tau/12).f(x), \quad x = \exp(2\pi i\tau), \quad \text{im}\,\tau > 0,$$

and $s = s(q)$ is the least positive even integer such that

$$\delta = s(q - 1)/24$$

is integral. Clearly

$$g = x^{\delta}\ f^s(y)/f^s(x), \qquad h = x^{\Delta}f(y^q)/f(x),$$

where

$$\Delta = (q^2 - 1)/24$$

{and is integral since $(q, 6) = 1$}. Now, it is shown by Newman in [9] (g and h are precisely as in this paper) that g is an entire modular function* on $\Gamma_0(q)$ {and so on $\Gamma_0(q^2)$}, h is an entire modular function on $\Gamma_0(q^2)$. Furthermore (see [9]) g has a pole of order δ {in the uniformising variable $z_q = \exp(-2\pi i/q\tau)$} at the parabolic vertex $\tau = 0$ and is regular elsewhere throughout the fundamental region of $\Gamma_0(q)$, h has a pole of order Δ at $\tau = 0$ and is regular elsewhere throughout the fundamental region of $\Gamma_0(q^2)$. Since $\Gamma_0(q^2)$ is of index q in $\Gamma_0(q)$, it follows that g has precisely $q\delta$ poles in the fundamental region of $\Gamma_0(q^2)$. Thus by Theorem 8.1 there is a polynomial in u and v, $P(u, v)$, such that $P(g, h) = 0$, $\deg_u P = \Delta$, $\deg_v P = q\delta$.†

From this point onwards q has the value 13. Then $s = 2$, $\delta = 1$, $\Delta = 7$, and we have shown that there is a relation

$$(8.1) \qquad \sum_{\ell=0}^{7} \sum_{m=0}^{13} c(\ell, m) g^{\ell} h^{m} = 0,$$

with coefficients $c(\ell, m)$, not all zero. Replacing g and h by the variables

$$A = A(\tau) = g/h^2 = \{\eta(13\tau)/\eta(169\tau)\}^2 = y^{-1} f^2(y)/f^2(y^{13}),$$

$$b = b(\tau) = 1/h = \eta(\tau)/\eta(169\tau) = x^{-7} f(x)/f(y^{13})$$

---

* The term "entire modular function" is not used in [9]; it is defined by Newman in [10] (page 352).

† This result was communicated to us, with the proof, by Dr. Newman.

for convenience, we have $g = A/b^2$, $h = 1/b$, and (8.1) becomes

$$(8.2) \qquad \sum_{\ell=0}^{7} \sum_{m=0}^{13} c(\ell, m) A^{\ell} b^{-2\ell-m} = 0.$$

We now examine (for a reason which will appear shortly) the effect of the transformation $\tau \to -1/169\tau$ on equation (8.2). As a special case of the transformation formula (1.4) of [9] we have

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}} \eta(\tau).$$

Whence

$$A(-1/169\tau) = \{\eta(-1/13\tau)/\eta(-1/\tau)\}^2 = 13\{\eta(13\tau)/\eta(\tau)\}^2 = 13A/b^2,$$

$$b(-1/169\tau) = \eta(-1/169\eta)/\eta(-1/\tau) = 13\eta(169\tau)/\eta(\tau) = 13/b,$$

and so, replacing τ by $-1/169\tau$ in (8.2), we obtain

$$(8.3) \qquad \sum_{\ell=0}^{7} \sum_{m=0}^{13} 13^{-\ell-m} c(\ell, m) A^{\ell} b^{m} = 0.^{*}$$

Furthermore, this relation must be irreducible. We prove this in an elementary manner as follows. Consider the more general result

$$(8.4) \qquad \sum_{\ell=0}^{\lambda} \sum_{m=0}^{\mu} d(\ell, m) A^{\ell} b^{m} = 0,$$

as a relation in x. We observe that $A^{\ell} b^{m}$ begins $x^{-13\ell-7m} + \ldots$ and denote by $-t$ the overall lowest power of x in the expansions of those terms $d(\ell, m) A^{\ell} b^{m}$ which actually occur, i.e. for which $d(\ell, m) \neq 0$. Then, since the left-hand side of (8.4) is

---

* We may note that $A(\tau) = 13g(-1/169\tau)$ and $b(\tau) = 13h(-1/169\tau)$.

identically zero, x^{-t} must be the initial power of x in the expansions of at least two such terms. In other words there exist distinct integer pairs (ℓ_1, m_1) and (ℓ_2, m_2) such that

$$t = 13\ell_1 + 7m_1 = 13\ell_2 + 7m_2, \quad d(\ell_1, m_1), d(\ell_2, m_2) \neq 0, \tag{8.5}$$
$$0 \leq \ell_1 \leq \lambda, 0 \leq \ell_2 \leq \lambda, 0 \leq m_1 \leq \mu, 0 \leq m_2 \leq \mu.$$

Now $\ell_1 \neq \ell_2$ (otherwise $m_1 = m_2$ also), so that without loss of generality we may take $\ell_1 > \ell_2 (\geq 0)$ (giving $0 \leq m_1 < m_2$). But from (8.5) $\ell_1 \equiv \ell_2$ (mod. 7). Hence $\ell_1 \geq 7$. Similarly $m_2 \geq 13$. Thus, since $d(\ell_1, m_1), d(\ell_2, m_2) \neq 0$, the degrees in A and b of any relation of the form (8.4) must be at least 7 and at least 13 respectively. It follows that (8.3) is irreducible, of degrees 7 and 13 in A and b. Further, taking $\lambda = 7$, $\mu = 13$, so that $\ell_1 \leq 7$, $m_2 \leq 13$, and remembering that, whatever the values of λ and μ, $\ell_1 \geq 7$, $m_2 \geq 13$, we see that in the case of (8.3) $\ell_1 = 7$ and $m_2 = 13$; since $\ell_1 > \ell_2 \geq 0$ and $\ell_1 \equiv \ell_2$ (mod. 7), this means that $\ell_2 = 0$, and similarly $m_1 = 0$, so that $t = 91$. Thus

$$c(7, 0), c(0, 13) \neq 0$$

and $c(\ell, m) = 0$ if $13\ell + 7m > 91$, i.e. $m > 13 - 13\ell/7$, i.e. if

$$m > 13 - 2\ell (0 \leq \ell < 7), \quad m > 0 \ (\ell = 7).$$

It follows that we may rewrite (8.2) and (8.3) respectively as

$$(8.6)\qquad c(7,\ 0)A^7b^{-14} + \sum_{\ell=0}^{6}\sum_{m=0}^{13-2\ell} c(\ell,\ m)A^{\ell}b^{-2\ell-m} = 0,$$

$$(8.7)\qquad 13^{-7}c(7,\ 0)A^7 + \sum_{\ell=0}^{6}\sum_{m=0}^{13-2\ell} 13^{-\ell-m}\, c(\ell,\ m)A^{\ell}b^{m} = 0.$$

Multiplying (8.6) by $13^{-7}b^{14}$ and writing m for $14-2\ell - m$ in the summation we obtain

$$(8.8)\qquad 13^{-7}c(7,\ 0)A^7 + \sum_{\ell=0}^{6}\sum_{m=1}^{14-2\ell} 13^{-7}c(\ell, 14 - 2\ell - m)A^{\ell}b^{m} = 0.$$

Now in each of equations (8.7) and (8.8) the highest power of A occurring is 7 $\{$since $c(7, 0) \neq 0\}$ and A^7 is present in and only in the initial term. Also, these initial terms are the same and (8.7) is irreducible. It follows, since there can be only one irreducible relation between A and b, that the left-hand sides of the equations must be identical. Hence, equating coefficients of $A^{\ell}b^{m}$, we have

$$c(\ell,\ 14 - 2\ell - m) = 13^{7-\ell-m}\, c(\ell,\ m),$$

and the overlapping of the m-summation ranges means that either side of this equation must be zero whenever $m = 0$, so that in (8.7) $\{$or (8.8)$\}$ we may take $1 \leq m \leq 13 - 2\ell$. Thus, taking $c(0,\ 7) = -\ 13^7$ (without loss of generality) and writing $d(\ell,m)$ for $13^{-\ell-m}\, c(\ell,\ m)$ in (8.7), we arrive at the following.

THEOREM 8.2 Let

$$A = y^{-1}f^{2}(y)/f^{2}(y^{13}), \quad b = x^{-7}f(x)/f(y^{13}).$$

Then there is an irreducible polynomial relation

$$A^{7} = \sum_{\ell=0}^{6} \sum_{m=1}^{13-2\ell} d(\ell, m)A^{\ell}b^{m}$$

with integral coefficients $d(\ell, m)$ which satisfy

$$d(\ell, 14 - 2\ell - m) = 13^{\ell+m-7} d(\ell, m).$$

The last equation of course follows from the corresponding result for the $c(\ell, m)$. The word "integral" is valid as follows. We have seen that, in the polynomial relation of Theorem 8.2, if two or more of the quantities $A^{\ell}b^{m}$ have the same initial power of x, then this power must be -91, and that x^{-91} is the initial power of x in precisely two of these quantities one of which is A^{7}. In other words in the right-hand side no two $A^{\ell}b^{m}$ have the same initial power of x. Thus the $d(\ell, m)$, determined by equating the coefficients of powers of x in the expansions of each side, appear strictly seriatim. Since in the expansion of any $A^{\ell}b^{m}$, including A^{7}, the coefficient of the initial power of x is unity and that of any other power of x integral, it follows that every $d(\ell, m)$ must be integral.

In obtaining the values of the $d(\ell, m)$ only the 28 values such that $\ell + m \geq 7$ need to be calculated; the remainder can then be written down. These 28 values may be obtained by comparing the coefficients of x^{-91}, x^{-90}, ... as far as x^{-49}; 9 of these 43 powers (viz. -78, -71, -65, -64, -58, -57, -52, -51, -50) are not expressible in the form $-13\ell - 7m$ ($0 \le \ell \le 6$, $1 \le m \le 13 - 2\ell$), so that no new $d(\ell, m)$ is obtained, and 6 (viz. -72, -66, -60, -59, -54, -53) give, superfluously, $d(\ell, m)$ such that $\ell + m < 7$.*

We find that

$$\begin{aligned} A^7 = {} & A^6(11.13b) + \\ & +A^5(36.13b^3-204.13b^2+36.13^2b) + \\ & +A^4(38.13b^5-346.13b^4+126.13^2b^3-346.13^2b^2+38.13^3b) + \\ & +A^3(20.13b^7-222.13b^6+102.13^2b^5-422.13^2b^4+102.13^3b^3 - \\ & \qquad -222.13^3b^2 + 20.13^4b) + \\ & +A^2(6.13b^9-74.13b^8+38.13^2b^7-184.13^2b^6+56.13^3b^5-184.13^3b^4+ \\ & \qquad +38.13^4b^3-74.13^4b^2+6.13^5b) + \\ & +A(13b^{11}-13^2b^{10}+7.13^2b^9-37.13^2b^8+13^4b^7-51.13^3b^6+13^5b^5- \\ & \qquad -37.13^4b^4+7.13^5b^3-13^6b^2+13^6b) + \\ & +(b^{13}-13b^{12}+7.13b^{11}-3.13^2b^{10}+15.13^2b^9-5.13^3b^8+19.13^3b^7- \\ & \qquad -5.13^4b^6+15.13^4b^5-3.13^5b^4+7.13^5b^3-13^6b^2+13^6b). \end{aligned} \tag{8.9}$$

It turns out then that the $d(\ell, m)$ are all non-zero and that they contain powers of 13 which could not have been anticipated from Theorem 8.2.

---

* In actual fact we examined the coefficients of sufficient of x^{-91}, x^{-90}, ..., x^{-49}, and of x^{-48}, to enable us to find each of and to make 12 independent checks on the 28 values.

We observe, finally, that while the above result is new, the relation between A and b^2 obtainable by "squaring" (8.9) is given, in effect, by Lehner in [8] (pages 376 and 379).