# Automatic Proof of Theta-Function Identities

Jie Frye and Frank Garvan

**Abstract**
This is a tutorial for using two new MAPLE packages, `thetaids` and `ramarobinsids`. The `thetaids` package is designed to prove generalized etaproduct identities using the valence formula for modular functions. We show how this thetaids package can be used to find theta-function identities as well as prove them. As an application, we show how to find and prove Ramanujan's 40 identities for his so called Rogers-Ramanujan functions G(q) and H(q). In his thesis Robins found similar identities for higher level generalized eta-products. Our `ramarobinsids` package is for finding and proving identities for generalizations of Ramanujan's G(q) and H(q) and Robin's extensions. These generalizations are associated with certain real Dirichlet characters. We find a total of over 300 identities.

## 1 Introduction

The Rogers-Ramanujan functions are $G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

The ratio of these two functions is the famous Rogers-Ramanujan continued fraction

$$\frac{G(q)}{H(q)} = \prod_{n=0}^{\infty} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$= 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cfrac{q^4}{1 + \ldots}}}}$$

Ramanujan also found

$$H(q)\, G(q)^{11} - q^2 G(q)\, H(q)^{11} = 1 + 11\, G(q)^6 H(q)^6$$

and

$$H(q) G(q^{11}) - q^2 G(q) H(q^{11}) = 1, \quad (1.3)$$

and remarked that "each of these formulae is the simplest of a large class." Here we have used the standard $q$-notation

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \text{ and } (a;q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

In 1974 B. J. Birch published a description of some manuscripts of Ramanujan including a list of forty identities for the Rogers-Ramanjan functions. Biagioli [5] show how the theory of modular forms could prove identities of this type efficiently. See [2] and [4] for recent work. It is instructive to write the Rogers-Ramanujan functions in terms of generalized eta-products.

The Dedekind eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

where $\tau \in H := \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ and $q := e^{2\pi i \tau}$, and the generalized Dedekind eta function is defined to be

$$\eta_{\delta, g}(\tau) = q^{\frac{\delta}{2} P_2\left(\frac{g}{\delta}\right)} \prod_{m \equiv \pm g (\mathbf{mod}\ \delta)} (1-q^m), \quad (1.4)$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial, $\{t\} = t - [t]$ is the fractional part of $t$, and $g, \delta, m \in \mathbb{Z}+$ and $0 < g < \delta$. The function $\eta_{\delta, g}(\tau)$ is a modular function (modular form of weight 0) on $SL_2(\mathbb{Z})$ with a multiplier system.

Ramanujan's identities (1.2) and (1.3) can be rewritten as
Ramanujan's identity (1.3) can be rewritten as

$$\frac{1}{\eta_{5.2}(\tau)\eta_{5.1}(\tau)^{11}} - \frac{1}{\eta_{5.1}(\tau)\eta_{5.2}(\tau)^{11}} = 1 + 11 \frac{\eta(5\tau)^6}{\eta(\tau)^6}$$

and

$$\frac{1}{\eta_{5.2}(\tau)\eta_{5.1}(11\tau)} - \frac{1}{\eta_{5.1}(\tau)\eta_{5.2}(11\tau)} = 1.$$

It is natural to consider higher level analogues of Ramanujan's identities (1.2) and (1.3). The following are nice level 13 analogues:

$$\frac{1}{\eta_{13;2,5,6}(\tau)\eta_{13;1,3,4}(\tau)^3} - \frac{1}{\eta_{13;1,3,4}(\tau)\eta_{13;2,5,6}(\tau)^3} = 1 + 3 \frac{\eta(13\tau)^2}{\eta(\tau)^2} \quad (1.7)$$

and

$$\frac{1}{\eta_{13;2,5,6}(\tau)\eta_{13;1,3,4}(3\tau)} - \frac{1}{\eta_{13;1,3,4}(\tau)\eta_{13;2,5,6}(3\tau)} = 1. \quad (1.8)$$

Here we have used the notation

$$\eta_{\delta; g_1, g_2, \ldots, g_k}(\tau) = \eta_{\delta; g_1}(\tau)\eta_{\delta; g_2}(\tau) \ldots \eta_{\delta; g_k}(\tau).$$

Equation (1.7) was found by Ramanujan [3, Eq.(8.4),p.373], and equation (1.8) is due to Robins [20], who considered more general identities. The following is level 17 analogue of (1.8) and appears to be new.

$$\frac{1}{\eta_{17;3,5,6,7}(\tau)\eta_{17;1,2,4,8}(2\tau)} - \frac{1}{\eta_{17;1,2,4,8}(\tau)\eta_{17;3,5,6,7}(2\tau)} = 1.$$

Motivated by these examples and other work of Robins [20] one is led naturally to consider

$$G(n, N, \chi) = G(n) := \prod_{\chi(g)=1,\, 0<g<\frac{N}{2}} \eta_{N;g}(n\tau), \quad H(n, N, \chi) = H(n) := \prod_{\chi(g)=-1,\, 0<g<\frac{N}{2}} \eta_{N;g}(n\tau),$$

where $\chi$ is a non-principal real Dirichlet character mod N satisfying $\chi(-1) = 1$.

Ratios of functions of this type were studied by Huber and Schultz [13]. They found the following level 17 identity:

$$(r^2+8r-1)^2-2r(r^2+1)s+r^2=0,$$

where

$$r=\frac{H\left(1,17,\left(\frac{\cdot}{17}\right)\right)}{G\left(1,17,\left(\frac{\cdot}{17}\right)\right)},\quad s=\frac{\eta(17\tau)^2}{\eta(\tau)^3}.$$

The main goal of the thetaids MAPLE package is to automatically prove identities for generalized eta-products using the thepry of modualr functions.

In Sections 3-4 we describe the ramarobinsids package, which uses the thetaids package to search for and prove theta-function identities for general functions $G(n,N,\chi)$ and $H(n,N,\chi)$ that are like the theta-function identities considered by Ramanujan [5] and Robins [20].

We note that Liangjie [20] gave an algorithm for proving relations for certain theta-functions and their derivatives using a different method. We also note that Lovejoy and Osburn [13], [15], [14], [16], have used an earlier version of the thetaids package to prove theta-functions identities that were needed to establish an number of results for mock-theta functions.

### *1.1 Installation Instructions*

First install the qseries package from
http://qseries.org/fgarvan/qmaple/qseries
and follow the directions on that page. Before proceeding it is advisable to become familiar with the functions in the qseries package. See [9] for a tutorial. Then go to
http://qseries.org/fgarvan/qmaple/thetaids
to install the thetaids package. In Section 3 you will need to install the ramarobinsids package from
http://qseries.org/fgarvan/qmaple/ramarobinsids

## 2 Proving theta-function identities

To prove a given theta-function identity one needs to basically do the following.

(i) Rewrite the identity in terms of generalized eta-functions.
(ii) Check that each term in the identity is a modular function on some group $\Gamma_1(N)$.
(iii) Determine the order at each cusp of $\Gamma_1(N)$ of each term in the identity.
(iv) Use the valence formula to determine up to which power of $q$ is needed to verify the identity.
(v) Finally prove the identity by carrying out the verification.

In this section we explain how to carry out each of these steps in MAPLE. Then we show how the whole process of proof can be automated.

### *2.1 Encoding theta-functions, eta-functions and generalized*

## *eta-functions*

We recall Jacobi's triple product for theta-functions:

$$\prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^{n})(1 - q^{n}) = \sum_{n=-\infty}^{\infty} (-1)^{n} z^{n} q^{\frac{n(n-1)}{2}}, \quad (2.1)$$

so that

$$\prod_{n=1}^{\infty} (1 - q^{\delta n - g})(1 - q^{\delta n + g - \delta})(1 - q^{\delta n}) = \sum_{n=-\infty}^{\infty} (-1)^{n} q^{\frac{n(\delta n - \delta + 2g)}{2}}. \quad (2.2)$$

In the `qseries` MAPLE package the function on the left side of (2.2) is encoded symbollically as `JAC(g,d,infinity)`. This is the building block of the functions in our package. In the `qseries` package `JAC(0,d,infinity)` corresponds symbollically to

$$\prod_{n=1}^{\infty} (1 - q^{\delta n}) = \sum_{n=-\infty}^{\infty} (-1)^{n} q^{\frac{\delta n(3n+1)}{2}},$$

which is Euler’s Pentagonal Number Theorem.

| Function | Symbolic MAPLE form |
|---|---|
| $\prod_{n=1}^{\infty} (1 - q^{\delta n - g})(1 - q^{\delta n + g - \delta})(1 - q^{\delta n})$ | `JAC(g,δ,infinity)` |
| $\prod_{n=1}^{\infty} (1 - q^{\delta n})$ | `JAC(0,δ,infinity)` |
| $\eta_{\delta; g}(\tau)$ | `GETA(δ,g)` |
| $\eta(\delta\tau)$ | `EETA(δ)` |

We will also consider generalized eta-products. Let $N$ be a fixed positive integer. A generalized Dedekind eta-product of level $N$ has the form

$$f(\tau) = \prod_{\delta | N,\, 0 < g < \delta} \eta_{\delta; g}(\tau)^{r_{\delta; g}} \quad (2.4)$$

where

$$r_{\delta; g} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \textbf{if } g = \frac{\delta}{2} \\ \mathbb{Z} & \textit{otherwise} \end{cases}$$

In MAPLE we represent the generalized eta-product

$$\eta_{N; g_1}(\tau)^{r_1} \eta_{N; g_2}(\tau)^{r_2} \ldots \eta_{N; g_m}(\tau)^{r_m}$$

symbollically by the list

`[[N, g_1, r_1], [N, g_2, r_2], ..., [N, g_m, r_m]]`

We call such a list a *geta-list*.

### *2.2 Symbolic product conversion*

jac2eprod—Converts a quotient of theta-functions in JAC notation to a product of generalized eta-functions in EETA and GETA notation.

**EXAMPLE:**

```
> currentdir();
```

"C:\cygwin64\home\Owner\math\mypapers\auto-theta\tutorial\maple" **(1)**

```
> with(qseries):
  with(thetaids):
  G:=q->add(q^(n^2)/aqprod(q,q,n),n=0..10):
  H:=q->add(q^(n^2+n)/aqprod(q,q,n),n=0..10):
  JG:=jacprodmake(G(q),q,50);
```

$$JG := \frac{JAC(0, 5, \infty)}{JAC(1, 5, \infty)} \quad (2)$$

```
> HG:=jacprodmake(H(q),q,50);
```

$$HG := \frac{JAC(0, 5, \infty)}{JAC(2, 5, \infty)} \quad (3)$$

```
> JP:=jacprodmake(H(q)*G(q)^(11),q,80);
```

$$JP := \frac{JAC(0, 5, \infty)^{12}}{JAC(1, 5, \infty)^{11}\, JAC(2, 5, \infty)} \quad (4)$$

```
> GP:=jac2eprod(JP);
```

$$GP := \frac{1}{GETA(5, 1)^{11}\, GETA(5, 2)} \quad (5)$$

jac2getaprod — Converts a quotient of theta-function in JAC notation to a product of generalized eta-functions in standard notation.

```
>  jac2getaprod(JP);
```

$$\frac{1}{\eta_{5,\,1}(\tau)^{11}\, \eta_{5,\,2}(\tau)} \quad (6)$$

GETAP2getalist — Converts a product of generalized eta-functions into a list as described above.

```
> GETAP2getalist(GP);
```

$$[[5, 1, -11], [5, 2, -1]] \quad (7)$$

### *2.3 Processing theta-functions*

There are two main functions in the thetaids package for processing combinations of theta-functions.

mixedjac2jac — Converts a sum of quotients of theta-functions written in terms of JAC(a,b,infinity) to a sum with the same base b. The functions jac2series and jacprodmake from the qseries package are used.

**EXAMPLE:**

```
> Y1:=1+jacprodmake(G(q),q,100)*jacprodmake(H(q^2),q,100);
```

$$Y1 := 1 + \frac{JAC(0, 5, \infty)\, JAC(0, 10, \infty)}{JAC(1, 5, \infty)\, JAC(4, 10, \infty)} \quad (8)$$

```
> Y2:=mixedjac2jac(Y1);
```

$$Y2 := 1 + \frac{JAC(0, 10, \infty)^3}{JAC(1, 10, \infty)\, JAC(4, 10, \infty)^2} \quad \textbf{(9)}$$

processjacid — Processes a theta-function identity written as a rational function of JAC-functions using mixedjac2jac and renormalizing by dividing by the term with the lowest power of $q$.
As an example, we consider the well-known identity

$\theta_3(q)^4 = \theta_2(q)^4 + \theta_4(q)^4$ (2.6)

```
> with(qseries):
  with(thetaids):
  F1:=theta2(q,100)^4:
  F2:=theta3(q,100)^4:
  F3:=theta4(q,100)^4:
  findhom([F1,F2,F3],q,1,0);
```

$$\{X_1 - X_2 + X_3\} \quad \textbf{(10)}$$

```
> JACID0:=qs2jaccombo(F1-F2+F3,q,100);
```

$$JACID0 := \frac{16\, q\, JAC(0, 4, \infty)^6}{JAC(2, 4, \infty)^2} - \frac{JAC(0, 4, \infty)^6\, JAC(2, 4, \infty)^6}{JAC(1, 4, \infty)^8} + JAC(1, 2, \infty)^4 \quad \textbf{(11)}$$

```
> JACID1:=processjacid(JACID0);
```

$$JACID1 := -\frac{16\, q\, JAC(1, 4, \infty)^8}{JAC(2, 4, \infty)^8} + 1 - \frac{JAC(1, 4, \infty)^{16}}{JAC(0, 4, \infty)^{12}\, JAC(2, 4, \infty)^4} \quad \textbf{(12)}$$

```
> expand(jac2getaprod(JACID1));
```

$$-\frac{\eta_{4,1}(\tau)^{16}}{\eta_{4,2}(\tau)^4} + 1 - \frac{16\,\eta_{4,1}(\tau)^8}{\eta_{4,2}(\tau)^8} \quad \textbf{(13)}$$

We see that (2.6) is equivalent to the identity

$$\frac{\eta_{4;1}(\tau)^{16}}{\eta_{4;2}(\tau)^4} + \frac{16\,\eta_{4;1}(\tau)^8}{\eta_{4;2}(\tau)^8} = 1.$$

## *2.4 Checking modularity*

Robins [21] has found sufficient conditions under which a generalized etaproduct is a modular function on $\Gamma_1(N)$.

**Theorem 2.1 ( [21](Theorem 3)).** The function $f(\tau)$, defined in (2.4), is a modular function on $\Gamma_1(N)$ if

(i) $\sum_{\delta | N, g} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta; g} \equiv 0 \ \mathbf{mod}\ 2$

and

(ii) $\sum_{\delta | N, g} \frac{N}{\delta} P_2(0)\, r_{\delta; g} \equiv 0 \ \mathbf{mod}\ 2$

The functions on the left side of (i), (ii) above are computed using the MAPLE functions `vinf` and `v0` respectively. Suppose $f(\tau)$ is given as in (2.4) and this generalized eta-product is encoded as the geta-list `L`. Recall that each item in the list `L` has the form $[\delta, g, r_{\delta; g}]$. The syntax is `vinf(L,N)` and `v0(L,N)`. As an example we consider the two generalized eta-products in (2.7).

**EXAMPLE:**

```
> L1:=[[4,1,16],[4,2,-4]];
```

$$L1 := [[4, 1, 16], [4, 2, -4]] \quad (14)$$

```
> vinf(L1,4),v0(L1,4);
```

$$0, 2 \quad (15)$$

```
> L2:=[[4,1,8],[4,2,-8]];
```

$$L2 := [[4, 1, 8], [4, 2, -8]] \quad (16)$$

```
> vinf(L2,4),v0(L2,4);
```

$$2, 0 \quad (17)$$

The numbers 0, 2 are even and we see that both generalized eta-products in (2.7) are modular functions on $\Gamma_1(4)$ by Theorem 2.1.

`Gamma1ModFunc(L,N)` — Checks whether a given generalized eta-product is a modular function on $\Gamma_1(N)$. Here the generalized eta-product is encoded as the geta-list `L`. The function first checks whether each $\delta$ is a divisor of N and checks whether both `vinf(L,N)` and `v0(L,N)` are even. It returns 1 if it is a modular function on $\Gamma_1(N)$ otherwise it returns 0. If the global variable `xprint` is set to *true* then more detailed information is printed. Thus here and throughout `xprint` can be used for debugging purposes.

**EXAMPLE:**

```
> Gamma1ModFunc(L1,4);
```

$$1 \quad (18)$$

```
> xprint:=true;
```

$$xprint := true \quad (19)$$

```
> Gamma1ModFunc(L1,4);
* starting Gamma1ModFunc with L=[[4, 1, 16], [4, 2, -4]] and N=4

All n are divisors of 4
val0=2
which is even.
valinf=0
which is even.
It IS a modfunc on Gamma1(4)
```

$$1 \quad (20)$$

## *2.5 Cusps*

Cho, Koo and Park [8] have found a set of inequivalent cusps for $\Gamma_1(N) \cap \Gamma_0(mN)$. The group $\Gamma_1(N)$ corresponds to the case $m = 1$.

**Theorem 2.2 ( [8](Corollary 4, p.930)).** Let $a, c, a', c' \in \mathbb{Z}$ with $(a, c) = (a', c') = 1$.

(ii) The cusps

$\frac{a}{c}$ and $\frac{a'}{c}$ are equivalent mod $\Gamma_1(N)$ if and only if

$$\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \pm \begin{pmatrix} a \\ c \end{pmatrix} \textbf{ mod } N$$

(ii) The following is a complete set of inequivalent cusps mod $\Gamma_1(N)$.

$$S = \{ \frac{y_{c,j}}{x_{c,i}} : 0 < c \mid N, 0 < s_{c,i}, a_{c,j} \le N, (s_{c,i}, N) = (a_{c,j}, N) = 1,$$

$$s_{c,i} = s_{c,i'} \quad \Leftrightarrow \quad s_{c,i} \equiv \pm s_{c,i'} \textbf{ mod } \frac{N}{c},$$

$$a_{c,j} = a_{c,j'} \quad \Leftrightarrow \begin{cases} a_{c,j} \equiv \mp a_{c,j'} \textbf{ mod } c & \textbf{if } c = \frac{N}{2} \textbf{ or } N, \\ a_{c,j} \equiv a_{c,j'} \textbf{ mod } c & \textit{otherwise}, \end{cases}$$

$$x_{c,i}, y_{c,j} \in \mathbb{Z} \text{ chosen s.th. } x_{c,i} \equiv c\ s_{c,i},\ \ y_{c,j} \equiv a_{c,j} \textbf{ mod } N,\ \ (x_{c,i}, y_{c,j}) = 1 \},$$

(iii) and the fan width of the cusp $\frac{a}{c}$ is given by

$$\kappa\left(\frac{a}{c}, \Gamma_1(N)\right) = \begin{cases} 1, & \textbf{if } N = 4 \textbf{ or } (c, 4) = 2, \\ \frac{N}{(c, N)}, & \textit{otherwise}. \end{cases}$$

In this theorem, it is understood as usual that the fraction $\pm\frac{1}{0}$ corresponds to $\infty$.

`cuspequiv1(`$a_1$, $c_1$, $a_2$, $c_2$, $N$`)` —determines whether the cusps $\frac{a_1}{c_1}$ and $\frac{a_2}{c_2}$ are $\Gamma_1(N)$-equivalent using Theorem 2.2(i).

**EXAMPLE:**

```
> cuspequiv1(1,3,1,9,40);
```

*false* **(21)**

```
> cuspequiv1(1,9,2,9,40);
```

*true* **(22)**

We see that modulo $\Gamma_1(40)$ the cusps $\frac{1}{3}$ and $\frac{1}{9}$ are inequivalent and the cusps $\frac{1}{9}$ and $\frac{2}{9}$ are equivalent.

`Acmake(c,N)` — returns the set $\{a_{c,j}\}$ where $c$ is a positive divisor of $N$.

`Scmake(c,N)` — returns the set $\{s_{c,i}\}$ where $c$ is a positive divisor of $N$.

`newxy(x,y,N)` — returns $[x_1, y_1]$ for given $(x, y, N) = 1$ such that $x_1 \equiv x$ **mod** $N$ and $y_1 \equiv y$ **mod** $N$

`cuspmake1(N)` —returns a set of inequivalent cusps for $\Gamma_1(N)$ using Theorem 2.2. Each cusp $\frac{a}{c}$ in the list is represented by `[a,c]`, so that $\infty$ is represented by `[1,0]`.

This MAPLE procedure uses the functions Acmake, Scmake and newxy.

**EXAMPLE:**

```
> C10:=cuspmake1(10);
```

$$C10 := \{[0,1],[1,0],[1,2],[1,3],[1,4],[1,5],[2,5],[3,10]\} \quad \textbf{(23)}$$

```
> for L in C10 do lprint(L,cuspwid1(L[1],L[2],10));od;
[0, 1], 10
[1, 0], 1
[1, 2], 5
[1, 3], 10
[1, 4], 5
[1, 5], 2
[2, 5], 2
[3, 10], 1
```

We have the following table of cusps for $\Gamma_1(10)$.

| Cusp | Cusp-width |
|---|---|
| 0 | 10 |
| $\infty$ | 1 |
| $\frac{1}{2}$ | 5 |
| $\frac{1}{3}$ | 10 |
| $\frac{1}{4}$ | 5 |
| $\frac{1}{5}$ | 2 |
| $\frac{2}{5}$ | 2 |
| $\frac{3}{10}$ | 1 |

CUSPSANDWIDMAKE(N) — returns a set of inequivalent cusps for $\Gamma_1(N)$, and corresponding widths. Output has the form [CUSPLIST,WIDTHLIST].

**EXAMPLE:**

```
> CUSPSANDWIDMAKE1(10);
```

$$\left[\left[oo, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{10}\right], [1, 10, 5, 10, 5, 2, 2, 1]\right] \quad \textbf{(24)}$$

## *2.6 Orders at cusps*

We will use Biagioli's [6] results for theta-functions to calculate orders at cusps of generalized eta-products. We define the theta-function

$$\theta_{\delta;g}(\tau) = q^{\frac{(\delta-2g)^2}{8\delta}} \prod_{m=1}^{\infty} (1-q^{m\delta-g})(1-q^{m\delta-(\delta-g)})(1-q^{m\delta}), \quad (2.8)$$

for $0 < g < \delta$. This corresponds to Biagioli's function $f_{\delta,g}$ [6, p.277]. The classical Dedekind eta-function can be written as

$$\eta(\tau) = \theta_{3;1}(\tau), \quad (2.9)$$

and the generalized Dedekind eta-function can be written as

$$\eta_{\delta;g}(\tau) = \frac{\theta_{\delta;g}(\tau)}{\eta(\delta\tau)} = \frac{\theta_{\delta;g}(\tau)}{\theta_{3\delta;\delta}(\tau)}. \quad (2.10)$$

Biagioli [6] has calculated the invariant order of $\theta_{\delta;g}(\tau)$ at any cusp. Using (2.10) this gives a method for calculating the invariant order at any cusp of a generalized eta-product.

**Theorem 2.3 ( [6](Lemma 3.2, p.285)).** The order at the cusp $s = \frac{b}{c}$ (assuming $(b,c)=1$) of the theta function $\theta_{\delta;g}(\tau)$ (defined above and assuming $\delta \nmid g$) is

$$ord(\theta_{\delta;g}(\tau), s) = \frac{e^2}{2\delta}\left(\frac{bg}{e} - \left[\frac{bg}{e}\right] - \frac{1}{2}\right)^2, \quad (2.11)$$

where $e = (\delta, c)$ and [ ] is the greatest integer function.

`Bord(δ,g,a,c)`—returns the order of $\theta_{\delta;g}(\tau)$ at the cusp $\frac{a}{c}$, assuming $(a,c)=1$ and $\delta \nmid g$.

`getacuspord(δ,g,a,c)`—returns the order of the generalized eta-function $\eta_{\delta;g}(\tau)$ at the cusp $\frac{a}{c}$, assuming $(a,c)=1$ and $\delta \nmid g$.

**EXAMPLE:**

```
> getacuspord(50,1,4,29);
```

$$\frac{1}{600} \quad \textbf{(25)}$$

We see that

$$ord\left(\eta_{50;1}(\tau), \frac{4}{29}\right) = \frac{1}{600}.$$

Let $G$ be a generalized eta-product corresponding to the getalist `L`. The following MAPLE procedure calculates the invariant order $ord(G,\zeta)$ for any cusp $\zeta$.

`getaprodcuspord(L,cusp)`—returns of the generalized eta-product corresponding to the geta-list `L` at the given `cusp`. The cusp is either a rational or `oo` (infinity).

**EXAMPLE:**

```
> GL:=[[4,1,16],[4,2,-4]];
```

$$GL := [[4,1,16],[4,2,-4]] \quad \textbf{(26)}$$

```
> getaprodcuspord(GL,1/2);
```

$$-1 \quad \textbf{(27)}$$

We see that

$$ord\left(\frac{\eta_{4;1}(\tau)^{16}}{\eta_{4;2}(\tau)^{4}}, \frac{1}{2}\right) = -1.$$

Following [6, p.275], [19, p.91] we consider the order of a function $f$ with respect to a congruence subgroup $\Gamma$ at the cusp $\zeta \in \mathbb{Q} \cup \{\infty\}$ and denote this by

$$\mathrm{ORD}(f, \zeta, \Gamma) = \kappa(\zeta, \Gamma)\ \mathrm{ord}(f, \zeta), \qquad (2.12)$$

getaprodcuspORDS(L,S,W) — returns a list of orders $\mathrm{ORD}(g, \zeta, \Gamma_1(N))$ where $G$ is the generalized eta-product corresponding to the getalist L, $\zeta \in S$ (list of inequivalent cusps of $\Gamma_1(N)$) and W is a list of corresponding fan-widths.

**EXAMPLE:**

```
> CW4:=CUSPSANDWIDMAKE1(4);
```

$$CW4 := \left[\left[oo, 0, \frac{1}{2}\right], [1, 4, 1]\right] \qquad \textbf{(28)}$$

```
> GL:=[[4,1,16],[4,2,-4]];
```

$$GL := [[4, 1, 16], [4, 2, -4]] \qquad \textbf{(29)}$$

```
> getaprodcuspORDS(GL,CW4[1],CW4[2]);
```

$$[0, 1, -1] \qquad \textbf{(30)}$$

We know that the generalized eta-product

$$f\cdot(\tau) = \frac{\eta_{4;1}(\tau)^{16}}{\eta_{4;2}(\tau)^{4}}$$

is a modular function on $\Gamma_1(4)$ . We calculated $\mathrm{ORD}(f, \zeta, \Gamma_1(4))$ at each cusp $\zeta$ of $\Gamma_1(4)$.

| $\zeta$ | $\mathrm{ORD}(f, \zeta, \Gamma_1(4))$ |
|---|---|
| $\infty$ | 0 |
| 0 | 1 |
| $\frac{1}{2}$ | -1 |

Observe that the total order of $f$ with respect to $\Gamma_1(4)$ is 0:

$$\mathrm{ORD}(f, \Gamma_1(4)) = \sum_{\zeta \in S} \mathrm{ORD}(f, \zeta, \Gamma_1(4)) = 0 + 1 - 1 = 0,$$

in agreement with the valence formula. See Theorem 2.4 below. Here $S$ is the set of inequivalent cusps of $\Gamma_1(4)$.

## *2.7 Proving theta-function identities*

Our method for proving theta-function or generalized eta-product identities depends on **Theorem 2.4 (The Valence Formula [19](p.98)).** Let $f \neq 0$ be a modular form of weight $k$ with respect to a subgroup $\Gamma$ of finite index in $\Gamma(1) = SL_2(\mathbb{Z})$.

Then

$$\mathrm{ORD}\left(f,\Gamma\right) = \frac{1}{12}\,\mu\,k, \quad (2.13)$$

where $\mu$ is the index of $\hat{\Gamma}$ in $\widehat{\Gamma(1)}$,

$$\mathrm{ORD}\left(f,\Gamma\right) = \sum_{\zeta\in R^*} \mathrm{ORD}\left(f,\zeta,\Gamma\right),$$

$R^*$ is a fundamental region for $\Gamma$, and $\mathrm{ORD}\left(f,\zeta,\Gamma\right)$ is given in equation (2.12).

*Remark 2.1.* For $z \in \mathfrak{h}$, $\mathrm{ORD}\left(f,\zeta,\Gamma\right)$ is defined in terms of the invariant order $\mathrm{ord}(f,\zeta)$ which is interpreted in the usual sense. See [19, p.91] for details of this and the notation used.

Since any generalized eta-product has weight $k = 0$ and has no zeros and no poles on the upper-half plane we have

**Corollary 2.5** Let $f_1(\tau), f_2(\tau), \ldots f_n(\tau)$ be generalized eta-products that are modular functions on $\Gamma_1(N)$. Let $\mathcal{S}_N$ be a set of inequivalent cusps for $\Gamma_1(N)$. Define the constant $s \in \mathcal{S}_N$

$$B = \sum_{s\in\mathcal{S}_N,\, s\neq\infty} \min\left(\left\{\mathrm{ORD}\left(f_j, s, \Gamma_1(N)\right) : 1 \le j \le n\right\} \cup \{0\}\right), \quad (2.14)$$

and consider

$$g(\tau) := \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \ldots + \alpha_n f_n(\tau) + 1, \quad (2.15)$$

where each $\alpha_j \in \mathbb{C}$. Then

$$g(\tau) \equiv 0$$

if and only if

$$\mathrm{ORD}\left(g(\tau), \infty, \Gamma_1(N)\right) > -B. \quad (2.16)$$

To prove an alleged theta-function identity, we first rewrite it in the form

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \ldots + \alpha_n f_n(\tau) + 1 = 0, \quad (2.17)$$

where each $\alpha_j \in \mathbb{C}$ and each $f_i(\tau)$ is a generalized eta-product of level $N$. We use the following algorithhm:

*STEP 1*. Use Theorem 2.1 to check that $f_j(\tau)$ is a generalized eta-product on $\Gamma_1(N)$ for each $1 \le j \le n$.

*STEP 2*. Use Theorem 2.2 to find a set $\mathcal{S}_N$ of inequivalent cusps for $\Gamma_1(N)$ and the fan width of each cusp.

*STEP 3*. Use Theorem 2.3 to calculate the invariant order of each generalized etaproduct $f_j(\tau)$ at each cusp of $\Gamma_1(N)$.

*STEP 4*. Calculate

$$B = \sum_{s \in \mathcal{S}_N, s \neq \infty} \min\left( \left\{ \mathrm{ORD}\left(f_j, s, \Gamma_1(N)\right) : 1 \le j \le n \right\} \cup \{0\} \right).$$

*STEP 5*. Show that

$$\mathrm{ORD}\left(g(\tau), \infty, \Gamma_1(N)\right) > -B$$

where

$$g(\tau) = \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \ldots + \alpha_n f_n(\tau) + 1.$$

Corollary 2.5 then imples that $g(\tau) \equiv 0$ and hence the theta-function identity (2.17).

To calculate the constant $B$ in (2.14) and STEP 4 we use
`mintotORDS(L,n)` — returns the constant `B` in equation (2.14) where `L` the array of ORDS:

$$L := \left[\mathrm{ORD}(f_1), \mathrm{ORD}(f_2), \ldots, \mathrm{ORD}(f_n)\right],$$

where

$$\mathrm{ORD}(f) = \left[\, \mathrm{ORD}\left(f, \zeta_1, \Gamma_1(N)\right), \mathrm{ORD}\left(f, \zeta_2, \Gamma_1(N)\right), \ldots, \left(f, \zeta_m, \Gamma_1(N)\right)\right]$$

and $\zeta_1, \zeta_2, \ldots, \zeta_m$ are the inequivalent cusps of $\Gamma_1(N)$. Each $\mathrm{ORD}(f)$ is computed using `getprodcuspORDS`.

**EXAMPLE:**

As an example we prove Ramanujan's well-known identity

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{25n})} = R(q^5) - q - \frac{q^2}{R(q^5)}, \quad (2.18)$$

where

$$R(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-2})(1-q^{5n-3})}{(1-q^{5n-1})(1-q^{5n-4})}.$$

We rewrite this identity as

$$\frac{\eta(\tau)}{\eta(25\tau)} = \frac{\eta_{25;10}(\tau)}{\eta_{25;5}(\tau)} - 1 - \frac{\eta_{25;5}(\tau)}{\eta_{25;10}(\tau)}. \quad (2.19)$$

Let

$$g(\tau) = f_1(\tau) - f_2(\tau) + f_3(\tau) + 1, \quad (2.20)$$

where

$$f_1(\tau) = \frac{\eta(\tau)}{\eta(25\tau)} = \prod_{j=1}^{12} \eta_{25;j}(\tau), \quad f_2(\tau) = \frac{\eta_{25;10}(\tau)}{\eta_{25;5}(\tau)},$$

$$f_3(\tau) = \frac{1}{f_2(\tau)} = \frac{\eta_{25;5}(\tau)}{\eta_{25;10}(\tau)}.$$

*STEP 1*. We check that each function is a modular function on $\Gamma_1(25)$.

```
> f1:=mul(GETA(25,j), j=1..12):
  f2:=GETA(25,10)/GETA(25,5):
  f3:=1/f2:
```

```
GP1:=GETAP2getalist(f1):
GP2:=GETAP2getalist(f2):
GP3:=GETAP2getalist(f3):
Gamma1ModFunc(GP1,25),Gamma1ModFunc(GP2,25),Gamma1ModFunc(GP3,25)
;
```

$$1, 1, 1 \quad (31)$$

*STEP 2*. We find a set of inequivalent cusps for $\Gamma_1(25)$ and their fan widths

```
> CW25:=CUSPSANDWIDMAKE1(25):
> cusps25:=CW25[1];
```

$$cusps25 := \left[oo, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{2}{5}, \frac{2}{25}, \frac{3}{5}, \frac{3}{10}, \frac{3}{25}, \frac{4}{5}, \frac{4}{25}, \frac{6}{25}, \frac{7}{10}, \frac{7}{25}, \frac{8}{25}, \frac{9}{10}, \frac{9}{25}, \frac{11}{25}, \frac{12}{25}\right] \quad (32)$$

```
> widths25:=CW25[2];
```

$$widths25 := [1, 25, 25, 25, 25, 5, 25, 25, 25, 25, 5, 25, 25, 5, 1, 5, 5, 1, 5, 1, 1, 5, 1, 1, 5, 1, 1, 1] \quad (33)$$

STEP 3. We compute $\mathrm{ORD}\left(f_j, \zeta, \Gamma_1(25)\right)$ for each $j$ and cusp $\zeta$ of $\Gamma_1(25)$.

```
> ORDS1:=getaprodcuspORDS(GP1,cusps25,widths25);
```

$$ORDS1 := [-1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, -1, 0, 0, -1, 0, -1, -1, 0, -1, -1, 0, -1, -1, -1] \quad (34)$$

```
> ORDS2:=getaprodcuspORDS(GP2,cusps25,widths25);
```

$$ORDS2 := [-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 1] \quad (35)$$

```
> ORDS3:=getaprodcuspORDS(GP3,cusps25,widths25);
```

$$ORDS3 := [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, -1] \quad (36)$$

*STEP 4*. We calculate the constant $B$ in (2.14).

```
> mintotORDS([ORDS1,ORDS2,ORDS3],3);
```

$$-9 \quad (37)$$

*STEP 5*. To prove the identity (2.18) we need to verify that

$$\mathrm{ORD}\left(g(\tau), \infty, \Gamma_1(25)\right) > 9.$$

```
> JACL:=map(getalist2jacprod,[GP1,GP2,GP3]):
> JACID:=JACL[1]-JACL[2]+JACL[3]+1:
> QJ:=jac2series(JACID,100):
> series(QJ,q,100);
```

$$\mathrm{O}(q^{99}) \quad (38)$$

This completes the proof of the identity (2.18). We only had to show that the coefficient of $q^j$ was zero in the q-expansion of $g(\tau)$ for $j \le 10$. We actually did it for $j \le 98$ as a check.

STEPS 1–5 may be automated using
`provemodfuncid(JACID,N)` — returns the constant $B$ in equation (2.14)
and prints details of the verification and proof of the identity corresponding to
`JACID`, which is a linear combination of symbolic `JAC`-functions, and `N` is the
level. If `xprint=true` then more details of the verification are printed. When this function is called there is a query asking whether to verify the identity. Enter `yes` to carry out the verification.

**EXAMPLE:**

```
> provemodfuncid(JACID,25);
"TERM ", 1, "of ", 4, " *****************"
"TERM ", 2, "of ", 4, " *****************"
```

```
"TERM ", 3, "of ", 4, " *****************"
"TERM ", 4, "of ", 4, " *****************"
"mintotord = ", -9
"TO PROVE the identity we need to show that v[oo](ID) > ", 9
*** There were NO errors.
*** o Each term was modular function on
      Gamma1(25).
*** o We also checked that the total order of
      each term was zero.
*** o We also checked that the power of q was correct in
      each term.
"*** WARNING: some terms were constants. ***"
"See array CONTERMS."
To prove the identity we will need to verify if up to
q^(9).
Do you want to prove the identity? (yes/no)
You entered yes.
We verify the identity to O(q^(59)).
RESULT: The identity holds to O(q^(59)).
CONCLUSION: This proves the identity since we had only
            to show that v[oo](ID) > 9.
```

$$9 \tag{39}$$

provemodfuncidBATCH(JACID,N)—is a version of provemodfuncid that prints less detail and does not query.

**EXAMPLE:**

```
> provemodfuncidBATCH(JACID,25);
*** There were NO errors.  Each term was modular function on
    Gamma1(25). Also -mintotord=9. To prove the identity
    we need to  check up to O(q^(11)).
    To be on the safe side we check up to O(q^(59)).
*** The identity below is PROVED!
```

$$1 \tag{40}$$

printJACIDORDStable() —prints an ORDs table for the $f_j$ and lower bound for $g$ after provemodfuncid is run. Formatted output from our example is given below in Table 1. By summing the last column we see that $B$ =-9, which confirms an earlier calculation using mintotORDS.

```
> printJACIDORDStable();
-------------------------------------------------------------
printJACIDORDStable()
   Print a table of ORDS for each term in a jacprod-identity
   using global data produced by the function provemodfuncid.
   Table is stored in the matrix bigmat which is returned.
-------------------------------------------------------------
ORDS Table for the jacprod identity
_G =
```

$$\_F_1 - \_F_2 + \_F_3 + \_F_4 = 0$$

where
_F[1] =

$GETA(25,1)\, GETA(25,2)\, GETA(25,3)\, GETA(25,4)\, GETA(25,5)\, GETA(25,6)\, GETA(25,7)\, GETA(25,8)\, GETA(25,9)\, GETA(25,10)\, GETA(25,11)\, GETA(25,12)$

_F[2] =

$$\frac{GETA(25, 10)}{GETA(25, 5)}$$

_F[3] =

$$\frac{GETA(25, 5)}{GETA(25, 10)}$$

_F[4] =

$$1$$

| *cusp* | $ORD(\_F_1)$ | $ORD(\_F_2)$ | $ORD(\_F_3)$ | $ORD(\_F_4)$ | $ORD(\_G)$ |
|---|---|---|---|---|---|
| *oo* | $-1$ | $-1$ | 1 | 0 | $-1$ |
| 0 | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{2}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{3}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{4}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{5}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{1}{6}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{7}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{8}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{9}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{10}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{1}{11}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{1}{12}$ | 1 | 0 | 0 | 0 | 0 |
| $\frac{2}{5}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{2}{25}$ | $-1$ | 1 | $-1$ | 0 | $-1$ |
| $\frac{3}{5}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{3}{10}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{3}{25}$ | $-1$ | 1 | $-1$ | 0 | $-1$ |
| $\frac{4}{5}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{4}{25}$ | $-1$ | $-1$ | 1 | 0 | $-1$ |

```
The last column of the table gives a lower bound for
ORDS of _G. By summing this last column (except for oo)
we see that the identity can be proved by showing that
the coefficients of
q^0, q^1, ... q^10 are all zero.
This confirms the calculation done by provemodfuncid.
```

*bigmat* **(41)**

# 3 Generalized Ramanujan-Robins identities

As an application of our `thetaids` package we show how to find and prove generalized eta-product identities due to Ramanujan and Robins, and some natural extensions. In Section 1 we defined the functions $G(n, N, \chi)$ and $H(n, N, \chi)$, where $\chi$ is a non-principal real Dirichlet character mod $N$ satisfying $\chi(-1)=1$. Robins [20] proved the following striking analogue of Ramanujan's identity (1.3) (or (1.6)):

$$G(3)H(1) - G(1)H(3) = 1, \quad (3.1)$$

where

$$G(n) = \frac{1}{\eta_{13;\,1,\,3,\,4}(n\tau)}, \qquad H(n) = \frac{1}{\eta_{13;\,2,\,5,\,6}(n\tau)}.$$

Equation (3.1) is a restatement of (1.8). In this case $N = 13$ and $\chi = \left(\frac{\cdot}{13}\right)$ is the Legendre symbol.

We will also consider

$$G^{*}(n, N, \chi) = G^{*}(n) := \prod_{\chi(g)=1,\, 0<g<\frac{N}{2}} \eta^{*}_{N;g}(n\tau), \; H^{*}(n, N, \chi) = H^{*}(n) := \prod_{\chi(g)=-1,\, 0<g<\frac{N}{2}} \eta^{*}_{N;g}(n\tau),$$

$$\eta^{*}_{\delta;g}(\tau) = q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)} \prod_{m \equiv \pm g \,(\mathbf{mod}\ \delta)} \left(1 - (-q)^{m}\right). \quad (3.3)$$

We note that

$$\eta^{*}_{\delta;g}(\tau) = \omega_{\delta;g}\, \eta_{\delta;g}(\tau + \pi i)$$

where $\omega_{\delta;g}$ is a root of unity. Using the notation (1.10) (with N = 5 and $\chi = \left(\frac{\cdot}{15}\right)$) we may rewrite Ramanujan's identities (1.2), (1.3) as

$$G(1)^{11} H(1) - G(1)\, H(1)^{11} = 1 + 11\, G(1)^{6} H(1)^{6},$$
$$H(1)\, G(11) - G(1)\, H(11) = 1,$$

respectively.

We have written a number of specialised functions for the purpose of finding and proving identities for these more general *G*- and *H*-functions. We have collected these functions into the new `ramarobinsids` package. Go to

http://qseries.org/fgarvan/qmaple/ramarobinsids

and follow the directions on that page. This package requires both the `qseries` and `thetaids` packages.

## *3.1 Some MAPLE functions*

`Geta(g,d,n)` — returns the generalized eta-function $\eta_{d;g}(n\tau)$ in symbolic `JAC`-form.

`GetaB(g,d,n)` — returns `Geta(g,d,n)` without the $q^{\frac{\delta}{2}P_2\left(\frac{g}{\delta}\right)}$ factor.

`GetaL(L,d,n)` — returns the generalized eta-product corresponding to the geta-list in `JAC`-form with $\tau$ replaced by $n\tau$.

`GetaBL(L,d,n)` — returns the generalized eta-product `GetaL(g,d,n)` without the $q$-factor.

`GetaEXP(g,d,n)` — returns lowest power of $q$ in $\eta_{d;g}(n\tau)$.

`GetaLEXP(L,d,n)` — returns lowest power of `q` for the generalized eta-product corresponding to `GetaL(L,d,n)`.

`MGeta(g,d,n)` — $\eta^*$ analogue of `Geta(g,d,n)`

`MGetaL(g,d,n)` — $\eta^*$ analogue of `GetaL(g,d,n)`

`Eeta(n)` — returns Dedekind eta-function $\eta(n\tau)$ in `JAC`-form

**EXAMPLE:**

```
> with(ramarobinsids):
> Geta(1,5,2);
```

$$\frac{q^{1/30}\,JAC(2, 10, \infty)}{JAC(0, 10, \infty)} \tag{42}$$

```
> GetaB(1,5,2);
```

$$\frac{JAC(2, 10, \infty)}{JAC(0, 10, \infty)} \tag{43}$$

```
> GetaEXP(1,5,2);
```

$$\frac{1}{30} \tag{44}$$

```
> GetaL([1,3,4],13,1);
```

$$\frac{q^{1/4}\,JAC(1, 13, \infty)\,JAC(3, 13, \infty)\,JAC(4, 13, \infty)}{JAC(0, 13, \infty)^3} \tag{45}$$

```
> GetaLB([1,3,4],13,1);
```

$$\frac{JAC(1, 13, \infty)\,JAC(3, 13, \infty)\,JAC(4, 13, \infty)}{JAC(0, 13, \infty)^3} \tag{46}$$

```
> GetaLEXP([1,3,4],13,1);
```

$$\frac{1}{4} \tag{47}$$

```
> MGeta(1,5,2);
```

$$\frac{q^{1/30}\,JAC(2, 10, \infty)\,JAC(4, 40, \infty)\,JAC(0, 20, \infty)^2}{JAC(0, 10, \infty)\,JAC(0, 40, \infty)\,JAC(2, 20, \infty)^2} \tag{48}$$

```
> MGetaL([1,3,4],13,1);
```

$$\left(q^{1/4}\,JAC(1, 13, \infty)\,JAC(2, 52, \infty)\,JAC(0, 26, \infty)\,JAC(3, 13, \infty)\,JAC(6, 52, \infty)\,JAC(4, 26, \infty)^2\,JAC(8, 26, \infty)\right) \Big/ \left(JAC(0, 13, \infty)\,JAC(0, 52, \infty)\,JAC(1, 26, \infty)^2\,JAC(3, 26, \infty)^2\,JAC(4, 13, \infty)\,JAC(8, 52, \infty)\right) \tag{49}$$

```
> Eeta(3);
```

$$q^{1/8}\,JAC(0, 3, \infty) \quad \textbf{(50)}$$

`CHECKRAMIDF(SYMF,ACC,T)`—checks whether a certain symbolic expression of *G*- and *H*-functions is an eta-product. This assumes that `G(n)`, `H(n)`, `GM(n)`, `HM(n)` have already been defined. `GM` and `HM` are the $\eta^*$ analogues of `G`, `H`. The `SYMF` symbolic form is written in terms of `_G`, `_H`, `_GM`, `_HM`. `ACC` is an upperbound on the absolute value of exponents allowed in the formal product, `T` is highest power of $q$ considered. This procedure returns a list of exponents in the formal product if it is a likely eta-product otherwise it returns `NULL`. A number of global variables are also assigned. The main ones are

- `_JFUNC`: `JAC`-expression of `SYMF`.
- `LQD`: lowest power of $q$.
- `RID`: the conjectured eta-product.
- `ebase`: base of the conjectured eta-product.
- `SYMID`: symbolic form of the identity.

**EXAMPLE:**

```
> with(qseries):
> with(thetaids):
> with(ramarobinsids):
> G:=j->1/GetaL([1,3,4],13,j):
> H:=j->1/GetaL([2,5,6],13,j):
> GM:=j->1/MGetaL([1,3,4],13,j):
> HM:=j->1/MGetaL([2,5,6],13,j):
> GE:=j->-GetaLEXP([1,3,4],13,j):
> HE:=j->-GetaLEXP([2,5,6],13,j):
> GHID:=(_G(1)*_G(2)+_H(1)*_H(2))/(_G(2)*_H(1)-_G(1)*_H(2));
```

$$GHID := \frac{\_G(1)\,\_G(2) + \_H(1)\,\_H(2)}{\_G(2)\,\_H(1) - \_G(1)\,\_H(2)} \quad \textbf{(51)}$$

```
> CHECKRAMIDF(GHID,10,50);
```

$$[-2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, 0, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, -2, 0, 0] \quad \textbf{(52)}$$

```
> ebase;
```

$$26 \quad \textbf{(53)}$$

```
> _JFUNC;
```

$$\big(q^3\,JAC(1, 13, \infty)\,JAC(3, 13, \infty)\,JAC(4, 13, \infty)\,JAC(2, 26, \infty)\,JAC(6, 26, \infty)\,JAC(8, 26, \infty) + JAC(2, 13, \infty)\,JAC(5, 13, \infty)\,JAC(6, 13, \infty)\,JAC(4, 26, \infty)\,JAC(10, 26, \infty)\,JAC(12, 26, \infty)\big) \big/ \big(q\,(-q\,JAC(2, 26, \infty)\,JAC(6, 26, \infty)\,JAC(8, 26, \infty)\,JAC(2, 13, \infty)\,JAC(5, 13, \infty)\,JAC(6, 13, \infty) + JAC(1, 13, \infty)\,JAC(3, 13, \infty)\,JAC(4, 13, \infty)\,JAC(4, 26, \infty)\,JAC(10, 26, \infty)\,JAC(12, 26, \infty))\big) \quad \textbf{(54)}$$

```
> LDQ;
```

$$-1 \quad \textbf{(55)}$$

```
> RID;
```

$$\frac{\eta(13\,\tau)^2\,\eta(2\,\tau)^2}{\eta(26\,\tau)^2\,\eta(\tau)^2} \quad \textbf{(56)}$$

```
> SYMID;
```

$$\frac{\_G(1)\ \_G(2) + \_H(1)\ \_H(2)}{\_G(2)\ \_H(1) - \_G(1)\ \_H(2)} = \frac{\eta(13\,\tau)^2\,\eta(2\,\tau)^2}{\eta(26\,\tau)^2\,\eta(\tau)^2} \quad (57)$$

```
> etamake(jac2series(_JFUNC,1001),q,1001);
```

$$\frac{\eta(13\,\tau)^2\,\eta(2\,\tau)^2}{\eta(26\,\tau)^2\,\eta(\tau)^2} \quad (58)$$

It seems that

$$\frac{G(1)\ G(2) + H(1)\ H(2)}{G(2)\ H(1) - G(1)\ H(2)} = \frac{\eta(13\,\tau)^2\,\eta(2\,\tau)^2}{\eta(26\,\tau)^2\,\eta(\tau)^2} \qquad (3.4)$$

when $N = 13$ and $\chi = \left(\frac{\cdot}{13}\right)$, at least up to $q^{1000}$.

**EXAMPLE:**

```
> RRID1:=_JFUNC-Eeta(13)^2*Eeta(2)^2/Eeta(26)^2/Eeta(1)^2:
> JRID1:=processjacid(RRID1):
> jmxperiod;
```

$$26 \quad (59)$$

```
> provemodfuncidBATCH(JRID1,26);
*** There were NO errors.  Each term was modular function on
    Gamma1(26). Also -mintotord=18. To prove the identity
    we need to  check up to O(q^(20)).
    To be on the safe side we check up to O(q^(70)).
*** The identity below is PROVED!
```

$$1 \quad (60)$$

Thus identity (3.4) is proved.
The search for and proof of such identities may be automated.

### *3.2 Ten types of identities for Ramanujan's functions G(q) and H(q)*

We consider ten types of identities.We write a MAPLE function to search for and prove identities of each type. Here we assume $N = 5$ and $\chi = \left(\frac{\cdot}{5}\right)$. We continue to use the notation (1.10).

In this section

$$G(1) = G\left(1, 5, \left(\frac{\cdot}{5}\right)\right) = \frac{1}{\eta_{5;\,1}(\tau)} = \frac{q^{-\frac{1}{60}}}{(q, q^4; q^5)_\infty},$$

$$H(1) = H\left(1, 5, \left(\frac{\cdot}{5}\right)\right) = \frac{1}{\eta_{5;\,2}(\tau)} = \frac{q^{\frac{11}{60}}}{(q^2, q^3; q^5)_\infty}.$$

**EXAMPLE:**

```
> with(qseries):
> with(thetaids):
> with(ramarobinsids):
> G:=j->1/GetaL([1],5,j):
> H:=j->1/GetaL([2],5,j):
```

```
> GM:=j->1/MGetaL([1],5,j):
> HM:=j->1/MGetaL([2],5,j):
> GE:=j->-GetaLEXP([1],5,j):
> HE:=j->-GetaLEXP([2],5,j):
```

### 3.2.1 Type 1

We consider identities of the form

$$G(a)\,H(b)\,\pm\,G(b)\,H(a) = f(\tau),$$

where $f(\tau)$ is an eta-product and $a$, $b$ are positive relatively prime integers.

findtype1(T) — cycles through symbolic expressions

$$\_G(a)\ \_H(b)\ +c\ \_G(b)\_H(a)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $b < a$, $c \in \{-1, 1\}$, and

$$GE(a) + HE(b) - (GE(b) + HE(a)) = \frac{1}{5}(b-a) \in \mathbb{Z}, \quad (3.5)$$

using CHECKRAMIDF to check whether the expression corresponds to a likely eta-product, and if so uses provemodfuncidBATCH to prove it. Condition (3.5) eliminates the case of fractional powers of $q$, which in our case means $a \equiv b \mod 5$. The procedure also returns a list of [a,b,c] which give identities .

**EXAMPLE:**

```
> proveit:=true: noprint:=false:
> qthreshold:=6000:
> findtype1(11);
*** There were NO errors.  Each term was modular function on
    Gamma1(30). Also -mintotord=8. To prove the identity
    we need to  check up to O(q^(10)).
    To be on the safe side we check up to O(q^(68)).
*** The identity below is PROVED!
[6, 1, -1]
```

$$\_G(6)\ \_H(1) - \_G(1)\ \_H(6) = \frac{\eta(6\,\tau)\,\eta(\tau)}{\eta(3\,\tau)\,\eta(2\,\tau)}$$

```
"n=", 10
*** There were NO errors.  Each term was modular function on
    Gamma1(55). Also -mintotord=40. To prove the identity
    we need to  check up to O(q^(42)).
    To be on the safe side we check up to O(q^(150)).
*** The identity below is PROVED!
[11, 1, -1]
```

$$\_G(11)\ \_H(1) - \_G(1)\ \_H(11) = 1$$

$$[[6, 1, -1], [11, 1, -1]] \quad \mathbf{(61)}$$

```
> PROVEDFL1;
```

$$[[6, 1, -1, 30, -8], [11, 1, -1, 55, -40]] \quad \mathbf{(62)}$$

```
> myramtype1:=findtype1(36);
*** There were NO errors.  Each term was modular function on
    Gamma1(30). Also -mintotord=8. To prove the identity
    we need to  check up to O(q^(10)).
    To be on the safe side we check up to O(q^(68)).
*** The identity below is PROVED!
[6, 1, -1]
```

$$\_G(6)\ \_H(1) - \_G(1)\ \_H(6) = \frac{\eta(6\,\tau)\,\eta(\tau)}{\eta(3\,\tau)\,\eta(2\,\tau)}$$

```
"n=", 10
*** There were NO errors.  Each term was modular function on
    Gamma1(55). Also -mintotord=40. To prove the identity
    we need to  check up to O(q^(42)).
    To be on the safe side we check up to O(q^(150)).
*** The identity below is PROVED!
[11, 1, -1]
```

$$\_G(11)\ \_H(1) - \_G(1)\ \_H(11) = 1$$

```
*** There were NO errors.  Each term was modular function on
    Gamma1(70). Also -mintotord=48. To prove the identity
    we need to  check up to O(q^(50)).
    To be on the safe side we check up to O(q^(188)).
*** The identity below is PROVED!
[7, 2, -1]
```

$$\_G(7)\ \_H(2) - \_G(2)\ \_H(7) = \frac{\eta(14\,\tau)\ \eta(\tau)}{\eta(7\,\tau)\ \eta(2\,\tau)}$$

```
*** There were NO errors.  Each term was modular function on
    Gamma1(80). Also -mintotord=64. To prove the identity
    we need to  check up to O(q^(66)).
    To be on the safe side we check up to O(q^(224)).
*** The identity below is PROVED!
[16, 1, -1]
```

$$\_G(16)\ \_H(1) - \_G(1)\ \_H(16) = \frac{\eta(4\,\tau)^2}{\eta(8\,\tau)\ \eta(2\,\tau)}$$

```
"n=", 20
*** There were NO errors.  Each term was modular function on
    Gamma1(120). Also -mintotord=128. To prove the identity
    we need to  check up to O(q^(130)).
    To be on the safe side we check up to O(q^(368)).
*** The identity below is PROVED!
[8, 3, -1]
```

$$\_G(8)\ \_H(3) - \_G(3)\ \_H(8) = \frac{\eta(24\,\tau)\ \eta(6\,\tau)\ \eta(4\,\tau)\ \eta(\tau)}{\eta(12\,\tau)\ \eta(8\,\tau)\ \eta(3\,\tau)\ \eta(2\,\tau)}$$

```
"n=", 30
*** There were NO errors.  Each term was modular function on
    Gamma1(180). Also -mintotord=288. To prove the identity
    we need to  check up to O(q^(290)).
    To be on the safe side we check up to O(q^(648)).
*** The identity below is PROVED!
[9, 4, -1]
```

$$\_G(9)\ \_H(4) - \_G(4)\ \_H(9) = \frac{\eta(36\,\tau)\ \eta(6\,\tau)^2\ \eta(\tau)}{\eta(18\,\tau)\ \eta(12\,\tau)\ \eta(3\,\tau)\ \eta(2\,\tau)}$$

```
*** There were NO errors.  Each term was modular function on
    Gamma1(180). Also -mintotord=288. To prove the identity
    we need to  check up to O(q^(290)).
    To be on the safe side we check up to O(q^(648)).
*** The identity below is PROVED!
[36, 1, -1]
```

$$\_G(36)\ \_H(1) - \_G(1)\ \_H(36) = \frac{\eta(9\,\tau)\ \eta(6\,\tau)^2\ \eta(4\,\tau)}{\eta(18\,\tau)\ \eta(12\,\tau)\ \eta(3\,\tau)\ \eta(2\,\tau)}$$

(62)

$$myramtype1 := [[6, 1, -1], [11, 1, -1], [7, 2, -1], [16, 1, -1], [8, 3, -1], [9, 4, -1], [36, 1, -1]] \quad \textbf{(63)}$$

```
> read xprogs:
> PROVEDFL1;
```

$$[[6, 1, -1, 30, -8], [11, 1, -1, 55, -40], [7, 2, -1, 70, -48], [16, 1, -1, 80, -64], [8, 3, -1, 120, -128], [9, 4, -1, 180, -288], [36, 1, -1, 180, -288]] \quad \textbf{(64)}$$

```
> for j from 1 to nops(PROVEDFL1) do printtype1(PROVEDFL1[j],3,5+j)
  ;od;
```

$$G(6)\,H(1) - G(1)\,H(6) = \frac{\eta(6\,\tau)\,\eta(\tau)}{\eta(3\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(30),\ -B=8, \quad (3.6)$$

$$G(11)\,H(1) - G(1)\,H(11) = 1,\ \Gamma_1(55),\ -B=40, \quad (3.7)$$

$$G(7)\,H(2) - G(2)\,H(7) = \frac{\eta(14\,\tau)\,\eta(\tau)}{\eta(7\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(70),\ -B=48, \quad (3.8)$$

$$G(16)\,H(1) - G(1)\,H(16) = \frac{\eta(4\,\tau)^2}{\eta(8\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(80),\ -B=64, \quad (3.9)$$

$$G(8)\,H(3) - G(3)\,H(8) = \frac{\eta(24\,\tau)\,\eta(6\,\tau)\,\eta(4\,\tau)\,\eta(\tau)}{\eta(12\,\tau)\,\eta(8\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(120),\ -B=128, \quad (3.10)$$

$$G(9)\,H(4) - G(4)\,H(9) = \frac{\eta(36\,\tau)\,\eta(6\,\tau)^2\,\eta(\tau)}{\eta(18\,\tau)\,\eta(12\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(180),\ -B=288, \quad (3.11)$$

$$G(36)\,H(1) - G(1)\,H(36) = \frac{\eta(9\,\tau)\,\eta(6\,\tau)^2\,\eta(4\,\tau)}{\eta(18\,\tau)\,\eta(12\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(180),\ -B=288, \quad (3.12) \quad \textbf{(65)}$$

We have included the relevant groups $\Gamma_1(N)$ and values of $B$ (see (2.14) and (2.16)). These identities are known and are equations (3.9), (3.5), (3.10), (3.6), (3.12), (3.14), and (3.15) in [5] respectively.

### 3.2.2 Type 2

We consider identities of the form

$$G(a)\,G(b) \pm H(a)\,H(b) = f(\tau),$$

where $f(\tau)$ is an eta-product and $a$, $b$ are positive relatively prime integers.

`findtype2(T)` — cycles through symbolic expressions

$$\_G(a)\ \_G(b)\ + c\ \_H(a)\_H(b)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $b < a$, $c \in \{-1, 1\}$, and

$$GE(a) + GE(b) - (HE(a) + HE(b)) = -\frac{1}{5}(a+b) \in \mathbb{Z}, \quad (3.13)$$

using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. Condition (3.13) eliminates the case of fractional powers of $q$, which in our case means $a \equiv -b \mod 5$. The procedure also returns a list of `[a,b,c]` which give identities .

```
> noprint:=true:
> findtype2(24);
```

$$[[1, 4, -1], [1, 4, 1], [2, 3, 1], [1, 9, 1], [1, 14, 1], [1, 24, 1]] \quad \mathbf{(66)}$$

```
> PROVEDFL2;
```

$$[[1, 4, -1, 20, -4], [1, 4, 1, 20, -4], [2, 3, 1, 30, -8], [1, 9, 1, 45, -24], [1, 14, 1, 70, -48], [1, 24, 1, 120, -128]] \quad \mathbf{(67)}$$

```
> for j from 1 to nops(PROVEDFL2) do printtype2(PROVEDFL2[j],3,13+
  j);od;
```

$$G(1)\,G(4) - H(1)\,H(4) = \frac{\eta(10\tau)^5}{\eta(20\tau)^2\,\eta(5\tau)^2\,\eta(2\tau)}, \Gamma_1(20), \text{-}B = 4, \quad (3.14)$$

$$G(1)\,G(4) + H(1)\,H(4) = \frac{\eta(2\tau)^4}{\eta(4\tau)^2\,\eta(\tau)^2}, \Gamma_1(20), \text{-}B = 4, \quad (3.15)$$

$$G(2)\,G(3) + H(2)\,H(3) = \frac{\eta(3\tau)\,\eta(2\tau)}{\eta(6\tau)\,\eta(\tau)}, \Gamma_1(30), \text{-}B = 8, \quad (3.16)$$

$$G(1)\,G(9) + H(1)\,H(9) = \frac{\eta(3\tau)^2}{\eta(9\tau)\,\eta(\tau)}, \Gamma_1(45), \text{-}B = 24, \quad (3.17)$$

$$G(1)\,G(14) + H(1)\,H(14) = \frac{\eta(7\tau)\,\eta(2\tau)}{\eta(14\tau)\,\eta(\tau)}, \Gamma_1(70), \text{-}B = 48, \quad (3.18)$$

$$G(1)\,G(24) + H(1)\,H(24) = \frac{\eta(12\tau)\,\eta(8\tau)\,\eta(3\tau)\,\eta(2\tau)}{\eta(24\tau)\,\eta(6\tau)\,\eta(4\tau)\,\eta(\tau)}, \Gamma_1(120), \text{-}B = 128, \quad (3.19) \quad \mathbf{(68)}$$

These identities are known and are equations (3.4), (3.3), (3.8), (3.7), (3.11), and (3.13) in [5] respectively.

### 3.2.3 Type 3

We consider identities of the form

$$\frac{G(a_1)G(b_1) \pm H(a_1)H(b_1)}{G(a_2)H(b_2) \pm H(a_2)G(b_2)} = f(\tau),$$

which are not a quotient of Type 1 and 2 identities, and where $f(\tau)$ is an eta-product, $a_1, b_1, a_2, b_2$ are positive relatively prime integers, and $a_1 b_1 = a_2 b_2$.

`findtype3(T)` — cycles through symbolic expressions

$$\frac{\_G(a_1)\_G(b_1) + c_1\ \_H(a_1)\_H(b_1)}{\_G(a_2)\_H(b_2) + c_2\ \_H(a_2)\_G(b_2)}$$

where $2 \le n \le T$, $a_1 b_1 = a_2 b_2 = n$, $(a_1, b_1, a_2, b_2) = 1$, $a_1 \le b_1$, $b_2 < a_2$,

$c_1, c_2 \in \{-1, 1\}$, and

`GE(a_1) + GE(b_1) - (HE(a_1) + HE(b_1)),`

`GE(a_2) + HE(b_2) - (HE(a_2) + GE (b_2))` $\in \mathbb{Z}$, (3.20)

and `[a_2, b_2, c_2]` is not an element of the list `myramtype1`

(found earlier by `findtype1`), using `CHECKRAMIDF` to check whether the expression corresponds to

a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns lists
$[a_1, b_1, c_1, a_2, b_2, c_2]$ which correspond to identities.

```
> xprint;
```

$$false \quad (69)$$

```
> qthreshold;
```

$$6000 \quad (70)$$

```
> findtype3(126);
```

[[3, 7, 1, 21, 1, −1], [2, 13, 1, 26, 1, −1], [1, 34, 1, 17, 2, −1], [1, 39, 1, 13, 3, −1], [1, 54, 1, 27, 2, −1], [7, 8, 1, 56, 1, −1], [3, 22, 1, 11, 6, −1], [2, 33, 1, 66, 1, −1], [4, 21, 1, 12, 7, −1], [1, 84, 1, 28, 3, −1], [3, 32, 1, 96, 1, −1], [7, 18, 1, 14, 9, −1], [2, 63, 1, 126, 1, −1]] **(71)**

```
> for j from 1 to nops(PROVEDFL3) do printtype3(PROVEDFL3[j],3,20+j);od;
```

$$\frac{G(3)\,G(7)+H(3)\,H(7)}{G(21)\,H(1)-H(21)\,G(1)}=1,\ \Gamma_1(105),\ -B=192, \quad (3.21)$$

$$\frac{G(2)\,G(13)+H(2)\,H(13)}{G(26)\,H(1)-H(26)\,G(1)}=1,\ \Gamma_1(130),\ -B=240, \quad (3.22)$$

$$\frac{G(1)\,G(34)+H(1)\,H(34)}{G(17)\,H(2)-H(17)\,G(2)}=\frac{\eta(17\,\tau)\,\eta(2\,\tau)}{\eta(34\,\tau)\,\eta(\tau)},\ \Gamma_1(170),\ -B=448, \quad (3.23)$$

$$\frac{G(1)\,G(39)+H(1)\,H(39)}{G(13)\,H(3)-H(13)\,G(3)}=\frac{\eta(13\,\tau)\,\eta(3\,\tau)}{\eta(39\,\tau)\,\eta(\tau)},\ \Gamma_1(195),\ -B=768, \quad (3.24)$$

$$\frac{G(1)\,G(54)+H(1)\,H(54)}{G(27)\,H(2)-H(27)\,G(2)}=\frac{\eta(27\,\tau)\,\eta(18\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)}{\eta(54\,\tau)\,\eta(9\,\tau)\,\eta(6\,\tau)\,\eta(\tau)},\ \Gamma_1(270),\ -B=1008, \quad (3.25)$$

$$\frac{G(7)\,G(8)+H(7)\,H(8)}{G(56)\,H(1)-H(56)\,G(1)}=\frac{\eta(28\,\tau)\,\eta(2\,\tau)}{\eta(14\,\tau)\,\eta(4\,\tau)},\ \Gamma_1(280),\ -B=1152, \quad (3.26)$$

$$\frac{G(3)\,G(22)+H(3)\,H(22)}{G(11)\,H(6)-H(11)\,G(6)}=\frac{\eta(33\,\tau)\,\eta(2\,\tau)}{\eta(66\,\tau)\,\eta(\tau)},\ \Gamma_1(330),\ -B=1600, \quad (3.27)$$

$$\frac{G(2)\,G(33)+H(2)\,H(33)}{G(66)\,H(1)-H(66)\,G(1)}=\frac{\eta(22\,\tau)\,\eta(3\,\tau)}{\eta(11\,\tau)\,\eta(6\,\tau)},\ \Gamma_1(330),\ -B=1600, \quad (3.28)$$

$$\frac{G(4)\,G(21)+H(4)\,H(21)}{G(12)\,H(7)-H(12)\,G(7)}=\frac{\eta(42\,\tau)\,\eta(28\,\tau)\,\eta(12\,\tau)\,\eta(7\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)}{\eta(84\,\tau)\,\eta(21\,\tau)\,\eta(14\,\tau)\,\eta(6\,\tau)\,\eta(4\,\tau)\,\eta(\tau)},\ \Gamma_1(420),\ -B=2688, \quad (3.29)$$

$$\frac{G(1)\,G(84)+H(1)\,H(84)}{G(28)\,H(3)-H(28)\,G(3)}=\frac{\eta(42\,\tau)\,\eta(28\,\tau)\,\eta(12\,\tau)\,\eta(7\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)}{\eta(84\,\tau)\,\eta(21\,\tau)\,\eta(14\,\tau)\,\eta(6\,\tau)\,\eta(4\,\tau)\,\eta(\tau)},\ \Gamma_1(420),\ -B=2688, \quad (3.30)$$

$$\frac{G(3)\,G(32)+H(3)\,H(32)}{G(96)\,H(1)-H(96)\,G(1)}=\frac{\eta(48\,\tau)\,\eta(12\,\tau)\,\eta(8\,\tau)\,\eta(2\,\tau)}{\eta(24\,\tau)\,\eta(16\,\tau)\,\eta(6\,\tau)\,\eta(4\,\tau)},\ \Gamma_1(480),\ -B=3072, \quad (3.31)$$

$$\frac{G(7)\,G(18)+H(7)\,H(18)}{G(14)\,H(9)-H(14)\,G(9)}=\frac{\eta(63\tau)\,\eta(42\tau)\,\eta(3\tau)\,\eta(2\tau)}{\eta(126\tau)\,\eta(21\tau)\,\eta(6\tau)\,\eta(\tau)},\Gamma_1(630),\ -B=5760,$$

*(3.32)*

$$\frac{G(2)\,G(63)+H(2)\,H(63)}{G(126)\,H(1)-H(126)\,G(1)}=\frac{\eta(42\tau)\,\eta(18\tau)\,\eta(7\tau)\,\eta(3\tau)}{\eta(21\tau)\,\eta(14\tau)\,\eta(9\tau)\,\eta(6\tau)},\Gamma_1(630),\ -B=5760, \qquad \textbf{(72)}$$

*(3.33)*

The equations marked * (see pdf) appear to be new. The other equations correspond to (3.16), (3.18), (3.35), (3.22), (3.41), (3.40) and (3.39) in [5], and (1.24) in [20] respectively.
We have corrected the statement of equation [20, (1.24)].

### 3.2.4 Type 4

We consider identities of the form

$$G^*(a)\,H^*(b)\,\pm G^*(b)\,H^*(a)=f(\tau),$$

where $f(\tau)$ is an eta-product and $a$, $b$ are positive relatively prime integers, and at least one of *a,b* is even.

`findtype4(T)` — cycles through symbolic expressions

$$\_GM(a)\ \_HM(b)\ +c\ \_GM(b)\_HM(a)$$

where $2\le n\le T,\ ab=n,\ (a,b)=1, b<a, c\in\{-1,1\}$, and

$$GE(a)+HE(b)-(GE(b)+HE(a))\ \in\mathbb{Z},\quad (3.34)$$

and at least one of *a,b* is even, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities .

```
> findtype4(24);
```

$$[[6,1,-1]] \qquad \textbf{(73)}$$

```
> read xprogs:
> printtype4(PROVEDFL4[1],3,35);
```

$$G^*(6)\,H^*(1)-G^*(1)\,H^*(6)=\frac{\eta(24\tau)\,\eta(6\tau)^3\,\eta(4\tau)^3\,\eta(\tau)}{\eta(12\tau)^3\,\eta(8\tau)\,\eta(3\tau)\,\eta(2\tau)^3},\Gamma_1(120),\ -B=128, \qquad \textbf{(74)}$$

*(3.35)*

### 3.2.5 Type 5

We consider identities of the form

$$G^*(a)\,G^*(b)\,\pm H^*(a)\,H^*(b)=f(\tau),$$

where $f(\tau)$ is an eta-product and $a$, $b$ are positive relatively prime integers, and at least one of *a,b* is even.

`findtype5(T)` — cycles through symbolic expressions

$$\_GM(a)\ \_GM(b)\ +c\ \_HM(a)\_HM(b)$$

where $2\le n\le T,\ ab=n,\ (a,b)=1, b<a, c\in\{-1,1\}$, and

$$GE(a)+GE(b)-(HE(a)+HE(b))\ \in\mathbb{Z},\quad (3.36)$$

and at least one of *a,b* is even, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities .

```
> findtype5(24);
```

$$[[1, 4, 1], [2, 3, 1]] \tag{75}$$

```
> printtype5(PROVEDFL5[1],3,37);
```

$$G^*(1)\ G^*(4) + H^*(1)\ H^*(4) = \frac{\eta(4\,\tau)^2}{\eta(8\,\tau)\ \eta(2\,\tau)}, \Gamma_1(80), -B = 64, \quad (3.37) \tag{76}$$

```
> printtype5(PROVEDFL5[2],3,38);
```

$$G^*(2)\ G^*(3) + H^*(2)\ H^*(3) = \frac{\eta(12\,\tau)^3\ \eta(8\,\tau)\ \eta(3\,\tau)\ \eta(2\,\tau)^3}{\eta(24\,\tau)\ \eta(6\,\tau)^3\ \eta(4\,\tau)^3\ \eta(\tau)}, \Gamma_1(120), -B = 128, \quad (3.38) \tag{77}$$

These correspond to equations (3.26) and (3.27) in [5].

### 3.2.6 Type 6

We consider identities of the form

$$G(a)\ H^*(b)\ \pm G^*(a)\ H(b) = f(\tau),$$

where $f(\tau)$ is an eta-product and $a$, $b$ are positive relatively prime integers.

`findtype6(T)` — cycles through symbolic expressions

$$\_G(a)\ \_HM(b)\ + c\ \_GM(a)\_H(b)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $b \le a$, $c \in \{-1, 1\}$,

using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities .

```
> read xprogs:
Warning, `L` is implicitly declared local to procedure `printtypelist`
> findtype6(24);
WARNING: There were 41 ebasethreshold problems.
         See the global array EBL.
```

$$[[1, 1, -1], [1, 1, 1]] \tag{78}$$

NOTE: This may mean there identities giving theta-functions that are not eta-products.

```
> PROVEDFL6;
```

$$[[1, 1, -1, 20, -4], [1, 1, 1, 20, -4]] \tag{79}$$

```
> read xprogs:
> printtypelist(printtype6,PROVEDFL6,3,39);
```

$$G(1)\ H^*(1) - G^*(1)\ H(1) = \frac{2\ \eta(20\,\tau)^2}{\eta(10\,\tau)\ \eta(2\,\tau)}, \Gamma_1(20), -B = 4, \quad (3.39)$$

$$G(1)\ H^*(1) + G^*(1)\ H(1) = \frac{2\ \eta(4\,\tau)^2}{\eta(2\,\tau)^2}, \Gamma_1(20), -B = 4, \quad (3.40) \tag{80}$$

These are equivalent to equations (3.25) and (3.24) in [5].

### 3.2.7 Type 7

We consider identities of the form

$$G^*(a)\,G(b)\,\pm H^*(a)\,H(b)=f(\tau),$$

where $f(\tau)$ is an eta-product and $a$, $b$ are positive relatively prime integers.

findtype7(T) — cycles through symbolic expressions

$$\_GM(a)\ \_G(b)\ +c\ \_HM(a)\_H(b)$$

where $2\le n\le T$, $ab=n$, $(a,b)=1$, $b\le a$, $c\in\{-1,1\}$, and both $a$, $b$ are odd, using CHECKRAMIDF to check whether the expression corresponds to a likely eta-product, and if so uses provemodfuncidBATCH to prove it. The procedure also returns a list of [a,b,c] which give identities .

```
> findtype7(24);
```

$$[[1,9,-1]] \tag{81}$$

```
> printtypelist(printtype7,PROVEDFL7,3,41);
```

$$G^*(1)\,G(9)-H^*(1)\,H(9)=\frac{\eta(18\tau)^2\,\eta(12\tau)\,\eta(\tau)}{\eta(36\tau)\,\eta(9\tau)\,\eta(6\tau)\,\eta(2\tau)},\ \Gamma_1(180),\ -B=288,\quad (3.41) \tag{82}$$

This corresponds to (3.29) in [5].

### 3.2.8 Type 8

We consider identities of the form

$$G(1)^a\,H(a)\,\pm H(1)^a\,G(a)=f(\tau),$$

where $f(\tau)$ is an eta-product, and $a>1$ is an integer.

findtype8(T) — cycles through symbolic expressions

$$\_G(1)^a\ \_H(a)\ +c\ \_H(1)^a\_G(a)$$

where $2\le a\le T$, and $c\in\{-1,1\}$, using CHECKRAMIDF to check whether the expression corresponds to a likely eta-product, and if so uses provemodfuncidBATCH to prove it. The procedure also returns a list of [a,c] which give identities.

```
> findtype8(24);
WARNING: There were 2 ebasethreshold problems.
         See the global array EBL.
```

$$[[3,-1]] \tag{83}$$

```
> printtypelist(printtype8,PROVEDFL8,3,42);
```

$$G(1)^3\,H(3)-H(1)^3\,G(3)=\frac{3\,\eta(15\tau)^3}{\eta(5\tau)\,\eta(3\tau)\,\eta(\tau)},\ \Gamma_1(15),\ -B=4,\quad (3.42) \tag{84}$$

This is equivalent to equation (1.27) in Robin's thesis [20].

### 3.2.9 Type 9

We consider identities of the form

$$G(1)^a\,H(1)^b - H(1)^a\,G(1)^b + x=f(\tau),$$

where $f(\tau)$ is an eta-product, and $a$, $b$ are positive integers, and $x=0$ or $x=-1$.

findtype8() — determines whether

$$\_G(1)^a\ \_H(1)^b\ -\ \_H(1)^a\_G(1)^b\ +\ x$$

is a likely eta-product for $x=0$ or $x=-1$ with $a$, $b$ the smallest such positive integers, using CHECKRAMIDF, and if so uses provemodfuncidBATCH to prove it. The procedure also returns a list of [a,b,x] which give identities.

```
> findtype9();
```

$$[[11, 1, 1]] \quad (85)$$

```
> printtypelist(printtype9,PROVEDFL9,3,43);
```

$$G(1)^{11} H(1) - H(1)^{11} G(1) - 1 = \frac{11\,\eta(5\tau)^6}{\eta(\tau)^6}, \Gamma_1(5), -B=2, \quad (3.43) \quad (86)$$

This is equation (3.1) in [5].

### 3.2.10 Type 10

We consider identities of the form

$$\frac{G(a_1)H(b_1) \pm H(a_1)G(b_1)}{G(a_2)H^*(b_2) \pm H(a_2)G^*(b_2)} = f(\tau),$$

in which the numerator is not a Type 1 identity, and where $f(\tau)$ is an eta-product, $a_1$, $b_1$, $a_2$, $b_2$ are positive relatively prime integers, and $a_1 b_1 = a_2 b_2$.

findtype10(T) — cycles through symbolic expressions

$$\frac{\_G(a_1)\_H(b_1) + c_1\,\_H(a_1)\_G(b_1)}{\_G(a_2)\_HM(b_2) + c_2\_H(a_2)\_GM(b_2)}$$

where $2 \le n \le T$, $a_1 b_1 = a_2 b_2 = n$, $(a_1, b_1, a_2, b_2) = 1$, $a_1 > b_1$, $b_2 < a_2$,

$c_1, c_2 \in \{-1, 1\}$, and

$\mathrm{GE}(a_1) + \mathrm{HE}(b_1) - (\mathrm{HE}(a_1) + \mathrm{GE}(b_1))$,

$\mathrm{GE}(a_2) + \mathrm{HE}(b_2) - (\mathrm{HE}(a_2) + \mathrm{GE}(b_2)) \in \mathbb{Z}$, (3.44)

and $[a_1, b_1, c_1]$ is not an element of the list myramtype1

(found earlier by findtype1), using CHECKRAMIDF to check whether the expression corresponds to a likely eta-product, and if so uses provemodfuncidBATCH to prove it. The procedure also returns lists

$[a_1, b_1, c_1, a_2, b_2, c_2]$ which correspond to identities.

```
> qthreshold:=3000:
> findtype10(120);
```

$$[[19, 4, -1, 76, 1, 1], [28, 3, -1, 12, 7, 1], [12, 7, -1, 28, 3, 1]] \quad (87)$$

```
> printtypelist(printtype10,PROVEDFL10,3,45);
```

$$\frac{G(19)\,H(4) - H(19)\,G(4)}{G(76)\,H^*(1) + H(76)\,G^*(1)} = \frac{\eta(76\tau)\,\eta(2\tau)}{\eta(38\tau)\,\eta(4\tau)}, \Gamma_1(380), -B=2160, \quad (3.45)$$

$$\frac{G(28)\,H(3) - H(28)\,G(3)}{G(12)\,H^*(7) + H(12)\,G^*(7)} = \frac{\eta(21\tau)\,\eta(14\tau)^2\,\eta(6\tau)\,\eta(4\tau)\,\eta(\tau)}{\eta(42\tau)\,\eta(28\tau)\,\eta(7\tau)\,\eta(3\tau)\,\eta(2\tau)^2}, \Gamma_1(420), -B=2400, \quad (3.46)$$

$$(88)$$

$$\frac{G(12)\,H(7)-H(12)\,G(7)}{G(28)\,H^*(3)+H(28)\,G^*(3)}=\frac{\eta(84\,\tau)\,\eta(21\,\tau)\,\eta(14\,\tau)\,\eta(6\,\tau)^2\,\eta(\tau)}{\eta(42\,\tau)^2\,\eta(12\,\tau)\,\eta(7\,\tau)\,\eta(3\,\tau)\,\eta(2\,\tau)},\ \Gamma_1(420),\ -B \tag{88}$$

$=2400$, *(3.47)*

Equation is (3.38) in [5]. The other type 10 identities appear to be new.

# 4 More Generalized Ramanujan-Robins identities

We consider generalized Ramanujan-Robins identities associated with non-principal real Dirichlet characters $\chi$ mod $N$ for $N \leq 60$, that satisfy $\chi(-1)=-1$. We found David Ireland's *Dirichlet Character Table Generator* [14] useful. See the website

http: //www.di-mgt.com.au/dirichlet-character-generator.html