Check Rank-Crank PDEs and Generalized Lambert Series Paper

```
> restart;gc();
> with(qseries):
> JACP:=z->tripleprod(z,q,100)/etaq(q,1,100):
> JACPS:=L->mul(JACP(L[j]),j=1..nops(L)):
> currentdir("C:\\cygwin\\home\\fgarvan\\math\\mypapers\\krank-pde");
```

"C:\cygwin\home\fgarvan\math\mypapers\krank-pde" **(1)**

```
> ##You will need to change this to directory where you saved
> ##the program KPDEPROG.txt
> ##currentdir("H:\\math\\research\\krank-pde");
> read "KPDEPROG.txt":
```

==

Section 1

```
> ASI:=zz*JACP(zz^2)/JACP(zz)*SIGk(z,q,3,10) + JACPS([zz,zz^2])*etaq(q,1,100)^2/JACPS([z/zz,z,z*zz])-Sk(zz,z,q,3,10):
> normal(series(ASI,q,50));
```

$O(q^{50})$ **(2)**

**** This confirms [asi] (1.1).

```
> JACKSON:=zz^2*JACPS([zz^2,x*zz,x/zz])/JACPS([zz,x,x])*SIGk(z,q,5,10) + JACPS([zz,zz^2,x*zz,zz/x])*etaq(q,1,100)^2/JACPS([z/x,z/zz,z,z*zz,z*x]) + zz/x*JACPS([zz,zz^2])/JACPS([x,x^2])*Sk(x,z,q,5,10)-Sk(zz,z,q,5,10):
> normal(series(JACKSON,q,16));
```

$O(q^{16})$ **(3)**

**** This confirms [jackson] (1.2).

```
> with(rank): with(crank):
> RGEN:=add((N(0,n)+add((z^m+z^(-m))*N(m,n),m=1..n))*q^n,n=0..100):
> CGEN:=add((M(0,n)+add((z^m+z^(-m))*M(m,n),m=1..n))*q^n,n=0..100):
> RGENSTAR:=RGEN/(1-z):
> CGENSTAR:=CGEN/(1-z):
> Lrcpde:=z*etaq(q,1,50)^2*CGENSTAR^3:
> Rrcpde:=(3*QOP(RGENSTAR)+1/2*ZOP(RGENSTAR)+1/2*ZOPS(RGENSTAR,2)):
> normal(series(Lrcpde-Rrcpde,q,50));
```

$O(q^{50})$ **(4)**

**** This confirms [rcpde] (1.7) up to q^50.

```
> R3:=1/etaq(q,1,50)*add( (-1)^(n-1)*q^(n*(5*n-1)/2)*(1-q^n)*(1/(1
```

```
   -z*q^n) + 1/z*q^n/(1-q^n/z)),n=1..13):
> R3V2:=add(add(q^(n1^2+n2^2)/aqprod(q,q,n2-n1)/aqprod(z*q,q,n1)
   /aqprod(q/z,q,n1),n1=1..n2),n2=1..30):
> normal(series(R3-R3V2,q,50));
```

$$O(q^{50}) \qquad (5)$$

```
> THETAJK:=(j,k)->add((-1)^n*q^(n*((2*k-1)*n-j)/2),n=-10..10):
> LTHM11:=R3V2:
> RTHM11:=1/etaq(q,1,50)*(z^2*(1-z)*SIGk(z,q,5,10)-z*THETAJK(1,3)
   +z*(1-z)*THETAJK(3,3)):
> normal(series(LTHM11-RTHM11,q,50));
```

$$O(q^{50}) \qquad (6)$$

**** This confirms Theorem 1.1 in the case k=3.

```
> G5:=SIGk(z,q,5,10)/etaq(q,1,50)^3:
> phi3:=EISEN(3,q,50):
> MULTID:=(L,f)->QOPS(ZOPS(f,L[2]),L[1]):
> SUMD:=(BIGL,f)->add( BIGL[j][1]*MULTID([BIGL[j][2],BIGL[j][3]],
   f),j=1..nops(BIGL)):
> Rgrcpde:= 24*(1-10*phi3)*G5+SUMD([[100,1,0],[50,0,1],[100,1,1],
   [35,0,2],[20,1,2],[100,2,0],[10,0,3],[1,0,4]],G5):
> Lgrcpde:=24*etaq(q,1,50)^2*CGENSTAR^5:
> normal(series(Lgrcpde-Rgrcpde,q,50));
```

$$O(q^{46}) \qquad (7)$$

**** This confirms [grcpde] (1.16).

```
> SYMMULTID:=(L)->DQ^L[1]*DZ^L[2]:
> SYMSUMD:=(BIGL)->add( BIGL[j][1]*SYMMULTID([BIGL[j][2],BIGL[j]
   [3]]),j=1..nops(BIGL)):
> SYMSUMD([[100,1,0],[50,0,1],[100,1,1],[35,0,2],[20,1,2],[100,2,
   0],[10,0,3],[1,0,4]]);
```

$$100\,DQ + 50\,DZ + 100\,DQ\,DZ + 35\,DZ^2 + 20\,DQ\,DZ^2 + 100\,DQ^2 + 10\,DZ^3 + DZ^4 \qquad (8)$$

```
> factor(%);
```

$$(10\,DQ + 10 + 5\,DZ + DZ^2)\,(10\,DQ + 5\,DZ + DZ^2) \qquad (9)$$

This is the symbolic form of the second term on the right side of [grcpde].

```
> factor(%+25);
```

$$(10\,DQ + 5 + 5\,DZ + DZ^2)^2 \qquad (10)$$

This means that the right side of [grcpde] can be written as
(H^2 -1 -240*PHI[3]) G5 which gives [grcpdev2] (1.18).

```
> Llew:=JACPS([z,zz^2])*etaq(q,1,50)^2/JACPS([z*zz,zz,z/zz]):
> Rlew:=Sk(zz,z,q,1,10):
> normal(series(Llew-Rlew,q,50));
```

$$O(q^{50})$$ (11)

**** This confirms [lew] (1.24).

==

Section 4

```
> Skv2:=(zeta,z,q,k,T)->add(add((-1)^n*z^m*q^(k*n*(n+1)/2+m*n)*
  (zeta^(-k*n-m)+zeta^(k*(n+1)+m)),m=0..30),n=0..T)
> -
> add(add((-1)^n*z^(-m)*q^(k*n*(n-1)/2+m*n)*(zeta^(k*n+m)+zeta^(-
  k*n+k-m)),m=1..30),n=1..T):
```

```
> normal(series(Skv2(zeta,z,q,1,30)-Sk(zeta,z,q,1,30),q,30));
```

$$\frac{z^{31}\left(-1+z\,\zeta-\zeta^{63}+z\,\zeta^{62}\right)}{\zeta^{30}\,(-\zeta+z)\,(-1+z\,\zeta)}+O(q^{30})$$ (12)

This confirms [Skid] (4.8) at least for k=1 abs(z/zeta)<1, abs(zeta)<1.

```
> normal(series(Skv2(zeta,z,q,3,30)-Sk(zeta,z,q,3,30),q,30));
```

$$\frac{z^{31}\left(-1+z\,\zeta-\zeta^{65}+z\,\zeta^{64}\right)}{\zeta^{30}\,(-\zeta+z)\,(-1+z\,\zeta)}+O(q^{30})$$ (13)

```
> SIGkv2:=(z,q,k,T)->add(add((-1)^n*z^m*q^(k*n*(n+1)/2+m*n),m=0.
  .30),n=0..T)-add(add((-1)^n*z^(-m)*q^(k*n*(n-1)/2+m*n),m=1..30),
  n=1..T):
> normal(series(SIGk(z,q,3,30)-SIGkv2(z,q,3,30),q,30));
```

$$-\frac{z^{31}}{-1+z}+O(q^{30})$$ (14)

This confirms [Sigid] (4.9) at least for k=3 and abs(z)<1.

```
> PKLV1:=(k,l,x)->add(l*(l-m-1)!*x^m*k^(l-2*m)/(l-2*m)!/m!,m=0..
  trunc(l/2)):
> PKLV2:=(k,l,x)->expand( (k/2-1/2*sqrt(k^2+4*x))^l + (k/2+1/2*
  sqrt(k^2+4*x))^l):
> {seq(expand(PKLV1(k,l,x)-PKLV2(k,l,x)),l=1..50)};
```

$$\{0\}$$ (15)

**** This confirms [Pellid] (4.12) for 1 <= ell <= 50.

```
> Lbinom:=(l,m)->add(binomial(l,2*j)*binomial(j,m),j=m..trunc(l/2)
  ):
> Rbinom:=(l,m)->2^(l-2*m-1)*l*(l-m-1)!/(l-2*m)!/m!;
```

$$Rbinom := (l, m) \to \frac{2^{l-2m-1}\,l\,(l-m-1)!}{(l-2\,m)!\,m!}$$ (16)

```
> checkbinom:=T->{seq(seq(Lbinom(l,m)-Rbinom(l,m),m=0..trunc(l/2)
  ),l=1..T)};
```

(17)

$$checkbinom := T \rightarrow \left\{ seq\left(seq\left(Lbinom(l, m) - Rbinom(l, m), m = 0 ..\mathrm{trunc}\left(\frac{1}{2}\, l \right) \right), l = 1 ..T \right) \right\} \tag{17}$$

```
> checkbinom(200);
```

$$\{0\} \tag{18}$$

**** This confirms [binomid] (4.13) for l<=100 and m<=trunc(l/2).

```
> x:=k*m + m^2+k^2*n*(n+1)+2*m*n*k;
```

$$x := k\, m + m^2 + k^2\, n\, (n+1) + 2\, m\, n\, k \tag{19}$$

```
> factor(k^2+4*x);
```

$$(2\, k\, n + k + 2\, m)^2 \tag{20}$$

```
> sq1:=radsimp(sqrt(%));
```

$$sq1 := 2\, k\, n + k + 2\, m \tag{21}$$

```
> k/2+sq1/2;
```

$$k + k\, n + m \tag{22}$$

```
> k/2-sq1/2;
```

$$-k\, n - m \tag{23}$$

**** This is a check on the calculations below [binomid] (4.13).

* Section 4.3

```
> F0m:=(m,zeta)->zeta^m*JACP(zeta^(m+1))/JACP(zeta^m):
> LAMBS:=(z,q,T)->add( z*q^i/(1-z*q^i) - 1/z*q^i/(1-q^i/z),i=1..T)
  :
> Jm:=(m,zeta)->add(i*zeta^i,i=1..m)/add(zeta^i,i=0..m):
> K0m:=(m,zeta)->m+Jm(m,zeta)-Jm(m-1,zeta):
> L0m:=(m,zeta)->K0m(m,zeta)-(m+1)*LAMBS(zeta^(m+1),q,50)+m*LAMBS
  (zeta^m,q,50):
> LdzetaF0:=m->NEWZETAOP(F0m(m,zeta)):
> RdzetaF0:=m->L0m(m,zeta)*F0m(m,zeta):
> seq(normal(series(LdzetaF0(m)-RdzetaF0(m),q,30)),m=1..5);
```

$$\mathrm{O}(q^{30}), \mathrm{O}(q^{30}), \mathrm{O}(q^{30}), \mathrm{O}(q^{30}), \mathrm{O}(q^{30}) \tag{24}$$

**** This confirms [dzetaF0] (4.15) for m<=5 and O(q^30).

```
> checkLem42:=(a,m)->ZETAOPS2(Jm(m,zeta),a)-bernoulli(a+1)/(a+1)*(
  (m+1)^(a+1)-1):
> {seq(seq(checkLem42(a,m),a=1..10),m=1..10)};
```

$$\{0\} \tag{25}$$

**** This confirms Lemma 4.2 [DaJm] for a,m <= 10.

```
> x:='x':
> series( x/(exp(x)-1) - add(bernoulli(k)*x^k/k!,k=0..50),x,50);
```

$$\mathrm{O}(x^{49}) \tag{26}$$

**** This confirms [Berngen] (4.21)

```
> seq(bernoulli(n),n=0..10);
```
$$1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66} \quad (27)$$

**** Sequence of Bernoulli numbers

```
> eisG:=m->-bernoulli(m)/m/2 + EISEN(m-1,q,100):
> RDaL0:=proc(a,m)
>    if modp(a,2)=1 then
>       2*(m^(a+1)-(m+1)^(a+1))*eisG(a+1):
>    elif a=0 then
>        m+1/2:
>    else
>        0:
>    fi:
> end:
> series(RDaL0(1,4),q,5);
```
$$\frac{3}{4} - 18\, q - 54\, q^2 - 72\, q^3 - 126\, q^4 + \mathrm{O}(q^5) \quad (28)$$

```
> LDaL0:=(a,m)->ZETAOPS2(L0m(m,zeta),a):
> series(LDaL0(1,4),q,5);
```
$$\frac{3}{4} - 18\, q - 54\, q^2 - 72\, q^3 - 126\, q^4 + \mathrm{O}(q^5) \quad (29)$$

```
> {seq(seq(series(LDaL0(a,m)-RDaL0(a,m),q,30),a=0..10),m=1..10)};
```
$$\{0, \mathrm{O}(q^{30})\} \quad (30)$$

**** This confirms Corollary 4.3 [cor:DaL0m] for a,m <= 10.

```
> seq(bernoulli(2*k-1),k=1..10);
```
$$-\frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0 \quad (31)$$

```
> checkDaF0m:=(a,m,T)->ZETAOPS2(normal(series(F0m(m,zeta),q,T)),a)
  ;
```
$$checkDaF0m := (a, m, T) \rightarrow ZETAOPS2(normal(series(F0m(m, \zeta), q, T)), a) \quad (32)$$

```
> series(checkDaF0m(2,2,10),q,10);;
```
$$10 - 15\, q - 45\, q^2 - 60\, q^3 - 105\, q^4 - 90\, q^5 - 180\, q^6 - 120\, q^7 - 225\, q^8 - 195\, q^9 + \mathrm{O}(q^{10}) \quad (33)$$

```
> series(qDaF0m(2,2),q,10);
```
$$10 - 15\, q - 45\, q^2 - 60\, q^3 - 105\, q^4 - 90\, q^5 - 180\, q^6 - 120\, q^7 - 225\, q^8 - 195\, q^9 + \mathrm{O}(q^{10}) \quad (34)$$

```
> {seq(seq(series(checkDaF0m(a,m,30)-qDaF0m(a,m),q,30),a=0..6),m=
  1..6)};
```

$$\{O(q^{30})\} \quad (35)$$

**** This checks [rec1] (4.23) for a,m <= 6.

```
> for a from 0 to 4 do
> print(a,expand(symbDaF0m(a,2)),"=",expand(symbDaF0mv2(a,2)));
> od;
```

$$0, \frac{3}{2}, "=", \frac{3}{2}$$

$$1, \frac{15}{4}, "=", \frac{15}{4}$$

$$2, \frac{75}{8} - 15\, G_2, "=", 10 - 15\, \Phi_1$$

$$3, \frac{375}{16} - \frac{225}{2}\, G_2, "=", \frac{225}{8} - \frac{225}{2}\, \Phi_1$$

$$4, \frac{1875}{32} - \frac{1125}{2}\, G_2 + 450\, G_2^2 - 195\, G_4, "=", 82 - 600\, \Phi_1 + 450\, \Phi_1^2 - 195\, \Phi_3 \quad (36)$$

**** This confirms the Table of values preceding Section 4.4.

* Section 4.4

```
> Fjm:=(j,m,zeta)->mul(JACP(zeta^(k-m))/JACP(zeta^(m+k+1)),k=1..j)
  :
> checkDaFjm:=(a,j,m,T)->ZETAOPS2(normal(series(Fjm(j,m,zeta),q,T)
  ),a);
```

$$checkDaFjm := (a, j, m, T) \rightarrow ZETAOPS2(normal(series(Fjm(j, m, \zeta), q, T)), a) \quad (37)$$

```
> series(checkDaFjm(2,1,2,10),q,10);;
```

$$-\frac{5}{4} - \frac{15}{2}\, q - \frac{45}{2}\, q^2 - 30\, q^3 - \frac{105}{2}\, q^4 - 45\, q^5 - 90\, q^6 - 60\, q^7 - \frac{225}{2}\, q^8 - \frac{195}{2}\, q^9 + O(q^{10}) \quad (38)$$

```
> series(qDaFjm(2,1,2),q,10);
```

$$-\frac{5}{4} - \frac{15}{2}\, q - \frac{45}{2}\, q^2 - 30\, q^3 - \frac{105}{2}\, q^4 - 45\, q^5 - 90\, q^6 - 60\, q^7 - \frac{225}{2}\, q^8 - \frac{195}{2}\, q^9 + O(q^{10}) \quad (39)$$

```
> {seq(seq(seq(series(checkDaFjm(a,j,m,30)-qDaFjm(a,j,m),q,30),a=
  0..6),j=1..3),m=1..6)};
```

$$\{O(q^{120}), O(q^{30})\} \quad (40)$$

**** This confirms [rec2] (4.27) for these values of a,j,m.

```
> for a from 0 to 4 do
> print(a,expand(symbDaFjm(a,1,2)),"=",expand(symbDaFjmv2(a,1,2)))
  ;
> od;
```

$$0, -\frac{1}{4}, "=", -\frac{1}{4}$$

$$1, \frac{5}{8}, "=", \frac{5}{8}$$

$$2, -\frac{25}{16} - \frac{15}{2} G_2, "=", -\frac{5}{4} - \frac{15}{2} \Phi_1$$

$$3, \frac{125}{32} + \frac{225}{4} G_2, "=", \frac{25}{16} + \frac{225}{4} \Phi_1$$

$$4, -\frac{625}{64} - \frac{1125}{4} G_2 - 675 G_2^2 - \frac{255}{2} G_4, "=", \frac{1}{4} - 225 \Phi_1 - 675 \Phi_1^2 - \frac{255}{2} \Phi_3 \quad \textbf{(41)}$$

**** This confirms the table before Section 4.5.

Section 4.5: No checking

Section 4.6

```
> for k from 3 by 2 to 9 do
> lprint("k=",k);
> checkPDE(k,10,50);
> od;
"k=", 3
```

$$4\, CS^3\, QINF^3 = 4 + 2\, \_H_3 + 12\, \Phi_1$$

$$3$$

$$1$$

```
"C2 computed"
```

$$O(q^{50})$$

```
"k=", 5
```

$$-144\, CS^5\, QINF^5 = -6\, \_H_5^2 + \left(-360\, \Phi_1 - 60\right) \_H_5 - 144 - 1800\, \Phi_1^2 - 60\, \Phi_3 - 2100\, \Phi_1$$

$$5$$

$$2$$

$$1$$

```
"C2 computed"
```

$$O(q^{50})$$

```
"k=", 7
```

$$34560\, CS^7\, QINF^7 = 48\, \_H_7^3 + \left(10080\, \Phi_1 + 1344\right) \_H_7^2 + \left(211680\, \Phi_1 + 423360\, \Phi_1^2 + 12096 + 10080\, \Phi_3\right) \_H_7 + 34560 + 1091328\, \Phi_1 + 141120\, \Phi_1 \Phi_3 + 117600\, \Phi_3 + 4939200\, \Phi_1^2 + 1975680\, \Phi_1^3 + 672\, \Phi_5$$

$$7$$

$$3$$

$$1$$

```
"C2 computed"
```

$$O(q^{50})$$

```
"k=", 9
```

$$-29030400\ CS^9\ QINF^9 = -720\ \_H_9^4 + \left(-362880\ \Phi_1 - 43200\right) \_H_9^3 + \left(-17962560\ \Phi_1 - 907200\ \Phi_3 - 941760 - 48988800\ \Phi_1^2\right) \_H_9^2 + \left(-97977600\ \Phi_1\ \Phi_3 - 1763596800\ \Phi_1^2 - 8766720 - 290666880\ \Phi_1 - 32659200\ \Phi_3 - 1763596800\ \Phi_1^3 - 362880\ \Phi_5\right) \_H_9 - 29030400 - 1530887040\ \Phi_1 - 1910563200\ \Phi_1\ \Phi_3 - 881798400\ \Phi_1^2\ \Phi_3 - 6531840\ \Phi_1\ \Phi_5 - 8164800\ \Phi_3^2 - 34390137600\ \Phi_1^3 - 7936185600\ \Phi_1^4 - 7076160\ \Phi_5 - 290939040\ \Phi_3 - 15710708160\ \Phi_1^2 - 12960\ \Phi_7$$

9

4

2

1

"C2 computed"

$$O(q^{50}) \quad \textbf{(42)}$$

**** This confirms Theorem 4.4 [maintheorem] for k = 3, 5, 7,9 (up to q^50).

Section 4.7

```
> SYMBOLPDEGEN2(3);
```

$$4\ CS^3\ QINF^3 = 9 + 2\ \_H_3 - \_D_2(\_F_{0,1}) \quad \textbf{(43)}$$

```
> SYMBOLPDEGEN3(3);
```

$$4\ CS^3\ QINF^3 = 9 + 2\ \_H_3 - \left\{5 - 12\ \Phi_1\right\} \quad \textbf{(44)}$$

```
> SYMBOLPDEGEN(3);
```

$$4\ CS^3\ QINF^3 = 4 + 2\ \_H_3 + 12\ \Phi_1 \quad \textbf{(45)}$$

```
> flistgen(3);
```

$$2\ CS^3\ QINF^3 = 2 + \_H_3 + 6\ \Phi_1$$

$$\_f_1, "=", 2 + 6\ \Phi_1$$

$$\_f_0, "=", 1 \quad \textbf{(46)}$$

**** This confirms Example m=1.

```
> SYMBOLPDEGEN2(5);
```

$$-144\ CS^5\ QINF^5 = 625 + 100\ \_H_5 + 2\ \_H_5^2 + 16\ \_D_0(\_F_{1,2})\left(625 + 100\ \_H_5 + 2\ \_H_5^2\right) + 32\ \_D_1(\_F_{1,2})\left(125 + 15\ \_H_5\right) + 24\ \_D_2(\_F_{1,2})\left(25 + 2\ \_H_5\right) + 40\ \_D_3(\_F_{1,2}) + 2\ \_D_4(\_F_{1,2}) - \_D_4(\_F_{0,2}) \quad \textbf{(47)}$$

```
> SYMBOLPDEGEN3(5);
```

$$-144\ CS^5\ QINF^5 = 625 + 100\ \_H_5 + 2\ \_H_5^2 + \{-4\}\left(625 + 100\ \_H_5 + 2\ \_H_5^2\right) + \{20\}\left(125 + 15\ \_H_5\right) + \left\{-30 - 180\ \Phi_1\right\}\left(25 + 2\ \_H_5\right) + 5\left\{\frac{25}{2} + 450\ \Phi_1\right\} + 2\left\{\frac{1}{4} - 225\ \Phi_1\right. \quad \textbf{(48)}$$

$$-675\,\Phi_1^2 - \frac{255}{2}\,\Phi_3\Big\} - \Big\{82 - 600\,\Phi_1 + 450\,\Phi_1^2 - 195\,\Phi_3\Big\}$$

```
> SYMBOLPDEGEN(5);
```

$$-144\,CS^5\,QINF^5 = -6\,\_H_5^2 + \left(-360\,\Phi_1 - 60\right)\,\_H_5 - 144 - 1800\,\Phi_1^2 - 60\,\Phi_3 - 2100\,\Phi_1 \quad \textbf{(49)}$$

```
> flistgen(5);
```

$$24\,CS^5\,QINF^5 = \_H_5^2 + \left(60\,\Phi_1 + 10\right)\,\_H_5 + 24 + 300\,\Phi_1^2 + 10\,\Phi_3 + 350\,\Phi_1$$

$$\_f_2, \text{"="}, 24 + 300\,\Phi_1^2 + 10\,\Phi_3 + 350\,\Phi_1$$

$$\_f_1, \text{"="}, 60\,\Phi_1 + 10$$

$$\_f_0, \text{"="}, 1 \quad \textbf{(50)}$$

```
> EP1:=etaq(q,1,100):
> series(QOP(EP1)+phi1*EP1,q,50);
```

$$\mathrm{O}\left(q^{50}\right) \quad \textbf{(51)}$$

```
> series(QOP(phi1)-1/6*phi1+2*phi1^2-5/6*phi3,q,50);
```

$$\mathrm{O}\left(q^{50}\right) \quad \textbf{(52)}$$

```
> findhomcombo(QOP(phi1*EP1^3),[phi1*EP1^3,phi1^2*EP1^3,phi3*
  EP1^3],q,1,0,no);
```

# of terms , 24

-----*possible linear combinations of degree*------, 1

$$\left\{\frac{1}{6}\,X_1 - 5\,X_2 + \frac{5}{6}\,X_3\right\} \quad \textbf{(53)}$$

**** This confirms the results for delta[q] in line before [H5S] (4.44).

```
> H2:=-50*Phi[1]+1500*Phi[1]^2 -250*Phi[3] - 60*Phi[1]*_H[5] + _H
  [5]^2;
```

$$H2 := -50\,\Phi_1 + 1500\,\Phi_1^2 - 250\,\Phi_3 - 60\,\Phi_1\,\_H_5 + \_H_5^2 \quad \textbf{(54)}$$

```
> H1:=-30*Phi[1]+_H[5];
```

$$H1 := -30\,\Phi_1 + \_H_5 \quad \textbf{(55)}$$

```
> _f1:=60*Phi[1]+10: _f2:=300*Phi[1]^2 + 10*Phi[3]+350*Phi[1]+24;
```

$$\_f2 := 300\,\Phi_1^2 + 10\,\Phi_3 + 350\,\Phi_1 + 24 \quad \textbf{(56)}$$

```
> expand(H2+_f1*H1+_f2);
```

$$-240\,\Phi_3 + \_H_5^2 + 10\,\_H_5 + 24 \quad \textbf{(57)}$$

**** This confirms Example m=2.

```
> flistgen(7);
```

$$720\,CS^7\,QINF^7 = \_H_7^3 + \left(210\,\Phi_1 + 28\right)\,\_H_7^2 + \left(4410\,\Phi_1 + 8820\,\Phi_1^2 + 252 + 210\,\Phi_3\right)\,\_H_7$$

$$+ 720 + 22736\,\Phi_1 + 2940\,\Phi_1\,\Phi_3 + 2450\,\Phi_3 + 102900\,\Phi_1^2 + 41160\,\Phi_1^3 + 14\,\Phi_5$$

$\_f_3$, "=", $720 + 22736\,\Phi_1 + 2940\,\Phi_1\,\Phi_3 + 2450\,\Phi_3 + 102900\,\Phi_1^2 + 41160\,\Phi_1^3 + 14\,\Phi_5$

$\_f_2$, "=", $4410\,\Phi_1 + 8820\,\Phi_1^2 + 252 + 210\,\Phi_3$

$\_f_1$, "=", $210\,\Phi_1 + 28$

$\_f_0$, "=", 1 **(58)**

**** This confirms Example m=3.

```
> flistgen(9);
```

$$40320\,CS^9\,QINF^9 = \_H_9^4 + \left(504\,\Phi_1 + 60\right)\_H_9^3 + \left(1260\,\Phi_3 + 24948\,\Phi_1 + 1308 + 68040\,\Phi_1^2\right)\_H_9^2 + \left(136080\,\Phi_1\,\Phi_3 + 2449440\,\Phi_1^2 + 504\,\Phi_5 + 45360\,\Phi_3 + 2449440\,\Phi_1^3 + 12176 + 403704\,\Phi_1\right)\_H_9 + 40320 + 2126232\,\Phi_1 + 2653560\,\Phi_1\,\Phi_3 + 1224720\,\Phi_1^2\,\Phi_3 + 9072\,\Phi_1\,\Phi_5 + 11340\,\Phi_3^2 + 18\,\Phi_7 + 404082\,\Phi_3 + 21820428\,\Phi_1^2 + 47764080\,\Phi_1^3 + 11022480\,\Phi_1^4 + 9828\,\Phi_5$$

$\_f_4$, "=", $40320 + 2126232\,\Phi_1 + 2653560\,\Phi_1\,\Phi_3 + 1224720\,\Phi_1^2\,\Phi_3 + 9072\,\Phi_1\,\Phi_5 + 11340\,\Phi_3^2 + 18\,\Phi_7 + 404082\,\Phi_3 + 21820428\,\Phi_1^2 + 47764080\,\Phi_1^3 + 11022480\,\Phi_1^4 + 9828\,\Phi_5$

$\_f_3$, "=", $136080\,\Phi_1\,\Phi_3 + 2449440\,\Phi_1^2 + 504\,\Phi_5 + 45360\,\Phi_3 + 2449440\,\Phi_1^3 + 12176 + 403704\,\Phi_1$

$\_f_2$, "=", $1260\,\Phi_3 + 24948\,\Phi_1 + 1308 + 68040\,\Phi_1^2$

$\_f_1$, "=", $504\,\Phi_1 + 60$

$\_f_0$, "=", 1 **(59)**

**** This confirms Example m=4.

===