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(p.1)

(p.15)

$$(x+y)^l + (x-y)^l = \sum_{j=0}^{l} \binom{l}{j} x^j y^{l-j} + \sum_{j=0}^{l} \binom{l}{j} x^j (-y)^{l-j}$$

$$= \sum_{j=0}^{l} \binom{l}{j} (x)^{l-j} y^j (1+(-1)^j)$$

$$= 2 \sum_{j=0}^{\lfloor l/2 \rfloor} \binom{l}{2j} x^{l-2j} y^{2j}$$

$$\left(\frac{k}{2} + \frac{1}{2}\sqrt{k^2+4x}\right)^l + \left(\frac{k}{2} - \frac{1}{2}\sqrt{k^2+4x}\right)^l$$

$$= 2 \sum_{j=0}^{\lfloor l/2 \rfloor} \binom{l}{2j} \left(\frac{k}{2}\right)^{l-2j} \left(\frac{\sqrt{k^2+4x}}{2}\right)^{2j}$$

$$= 2 \sum_{j=0}^{\lfloor l/2 \rfloor} \binom{l}{2j} k^{l-2j} (k^2+4x)^j \cdot 2^{-l}$$

$$= \sum_{j=0}^{\lfloor l/2 \rfloor} \sum_{m=0}^{j} \binom{l}{2j} \binom{j}{m} k^{l-2j} \cdot (k^2)^{j-m} (4x)^m \cdot 2^{1-l}$$

$$= \sum_{j=0}^{\lfloor l/2 \rfloor} \sum_{m=0}^{j} \binom{l}{2j} \binom{j}{m} x^m k^{l-2m} 2^{2m-l+1}$$

$$0 \le m \le j \le \lfloor l/2 \rfloor$$

$$= \sum_{m=0}^{\lfloor l/2 \rfloor} \left(\sum_{j=m}^{\lfloor l/2 \rfloor} \binom{l}{2j} \binom{j}{m} \right) x^m k^{l-2m} 2^{2m-l+1}$$

Note $\quad m \le \lfloor l/2 \rfloor, \quad 2m \le l$

If $x = km + m^2 + k^2 n(n+1) + 2mnk$,
then

$$k^2 + 4x = (k + 2m + 2kn)^2$$

$$\sqrt{k^2+4x} = k + 2m + 2kn$$

$$\frac{1}{2}\sqrt{k^2+4x} = \frac{k}{2} + m + kn$$

$$\frac{k}{2} + \frac{1}{2}\sqrt{k^2+4x} = k(n+1) + m$$

$$\frac{k}{2} - \frac{1}{2}\sqrt{k^2+4x} = -kn - m.$$

(p.15) $\mathcal{H}_k^*\left(q^{kn(n+1)/2+mn} z^m\right)$

$$= \left(k\delta_z + 2k\delta_q + \delta_z^2\right)\left(q^{kn(n+1)/2+mn} z^m\right)$$

$$= \left(km + 2k\cdot(k(n+1)n/2+mn) + m^2\right)\left(q^{(\,)} z^m\right)$$

$$= \left(km + m^2 + k^2 n(n+1) + 2mnk\right)\left(q^{(\,)} z^m\right)$$

Im Ymdef.

$$[\zeta^j]_\infty = (1-\zeta^j)\,(\zeta^j q)_\infty(\zeta^{-j}q)_\infty$$
$$= -(1-\zeta)(1+\cdots+\zeta^{j-1})\,(\zeta^j q)(\zeta^{-j}q)_\infty$$
$$= -j\,(\zeta-1)\,(q)_\infty^2 + \cdots$$

$$[\zeta^{-j}]_\infty = (1-\zeta^{-j})\,(\zeta^{-j}q)_\infty(\zeta^j q)_\infty$$
$$= \zeta^{-j}(\zeta^j-1)\cdots$$
$$= (\zeta-1)(1+\cdots+\zeta^{j-1})(\zeta^{-j})(\ )(\ )$$
$$= j\,(\zeta-1)\,(q)_\infty^2 + \cdots$$

(p.16) $$Y_m = \frac{(-1)^{m+1}(1)(2)\cdots(m+1)\,(1)(2)\cdots(m-1) \times (q)_\infty^{2(2m)} \times (\zeta-1)^{2m} + \cdots}{[z]_\infty^{2m+1}}$$

$$= (-1)^{m+1}(m+1)!\,(m-1)!\left(\frac{(q)_\infty}{[z]_\infty}\right)^{2m+1}(q)_\infty^{2m-1}\cdot(\zeta-1)^{2m} + \cdots$$

$$Y_m^{(2m)}(1,z,q) = (-1)^{2m+1}(2m)!\,(m+1)!\,(m-1)!\,[C^*(z,q)]^{2m+1}(q)_\infty^{2m-1}$$

which is eq. [Ymterm] (4.14).

Note $\frac{f^{(k)}(1)}{k!}$ = coeff of $(\zeta-1)^k \leftrightarrow f^{(k)}(1)$ = Coeff of $(\zeta-1)^k \cdot k!$.

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4.3 The Term $F_{0,m}(\zeta,q)$

$$F_{0,m} = \zeta^{m} \frac{[\zeta^{m+1}]_\infty}{[\zeta^{m}]_\infty}$$

$$\log F_{0,m} = m\log\zeta + \log(1-\zeta^{m+1}) - \log(1-\zeta^{m})$$

$$+ \sum_{i=1}^{\infty} \log(1-\zeta^{m+1}q^i) + \log(1-\zeta^{-m-1}q^i)$$

$$- \log(1-\zeta^{m})q^i - \log(1-\zeta^{-m}q^i)$$

$\delta_\zeta F_{0,m} = L_{0,m} F_{0,m}$ where

$$L_{0,m} = m - (m+1)\frac{\zeta^{m+1}}{1-\zeta^{m+1}} + m\frac{\zeta^{m}}{(1-\zeta^{m})}$$

$$-(m+1)\left(\sum_{i=1}^{\infty}\left(\frac{\zeta^{m+1}q^i}{1-\zeta^{m+1}q^i} - \frac{\zeta^{-m-1}q^i}{1-\zeta^{-m-1}q^i}\right)\right.$$

$$\left.+ m\left(\sum_{i=1}^{\infty}\left(\frac{\zeta^{m}q^i}{1-\zeta^{m}q^i} - \frac{\zeta^{-m}q^i}{1-\zeta^{m}q^i}\right)\right)\right)$$

$$K_{0,m} = m - \frac{(m+1)\zeta^{m+1}}{1-\zeta^{m+1}} + m\frac{\zeta^{m}}{1-\zeta^{m}}$$

$$= m + J_m(\zeta) - J_{m-1}(\zeta)$$

since

$$\delta_\zeta \log\left(\frac{1-\zeta^{m+1}}{1-\zeta^{m}}\right) = \delta_\zeta \log\left(\frac{1+\zeta+\cdots\zeta^{m}}{1+\zeta+\cdots+\zeta^{m-1}}\right)$$

$$= \delta_\zeta\left(\log(1+\cdots+\zeta^{m}) - \log(1+\cdots+\zeta^{m-1})\right)$$

$$= \frac{\zeta+2\zeta^2+\cdots+m\zeta^{m}}{1+\cdots+\zeta^{m}} - \frac{\zeta+\cdots+(m-1)\zeta^{m-1}}{1+\cdots+\zeta^{m-1}}$$

Proof of Lemma 4.2

$$\log\left(1+\zeta+\cdots+\zeta^{m}\right) = \log\left(\frac{\zeta^{m+1}-1}{\zeta-1}\right)$$

$$= \log\left(\zeta^{m+1}-1\right) - \log(\zeta-1)$$

Taking δ_{ζ} of both sides we get

$$J_m(\zeta) = \frac{(m+1)\zeta^{m+1}}{\zeta^{m+1}-1} - \frac{\zeta}{\zeta-1}$$

$$= (m+1)\frac{\left(\zeta^{m+1}-1+1\right)}{\left(\zeta^{m+1}-1\right)} - \frac{(\zeta-1+1)}{\zeta-1}$$

$$= m+1 + \frac{m+1}{\zeta^{m+1}-1} - 1 - \frac{1}{\zeta-1}$$

$$= m + (m+1)\frac{1}{\zeta^{m+1}-1} - \frac{1}{\zeta-1}.$$

$$\frac{x}{e^{x}-1} = \sum_{k=0}^{\infty} \frac{b_k\, x^{k}}{k!}$$

$$\frac{1}{e^{x}-1} = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^{k-1}$$

$$= \sum_{k=-1}^{\infty} \frac{b_{k+1}}{(k+1)!} x^{k}$$

$$= \frac{1}{x} - \frac{1}{2} + \sum_{k=0}^{\infty} \frac{b_{k+1}}{(k+1)!} x^{k}$$

$$J_m(e^{x}) = m + (m+1)\frac{1}{e^{(m+1)x}-b_0} - \frac{1}{e^{x}-1}$$

$$= m + \frac{(m+1)}{(m+1)x} - \frac{1}{x} + \sum_{k=0} \frac{b_{k+1}}{(k+1)!}\left((m+1)^{k+1}-1\right)x^{k}$$

$$= m + \sum_{k=0}^{\infty} \frac{b_{k+1}}{(k+1)!}\left((m+1)^{k+1}-1\right)x^{k}.$$

$$D_a(J_m(\zeta)) = \left(\frac{d}{dx}\right)^a J_m(e^x)\Big|_{x=0}$$

$$= \text{Coeff of } x^a \times a!$$

$$= \frac{b_{a+1}}{(a+1)!}\left((m+1)^{a+1}-1\right) a!$$

$$= \frac{b_{a+1}}{a+1}\left((m+1)^{a+1}-1\right). \quad \square$$

4.4 The Terms $F_{j,m}(\zeta,q)$ $(1 \leq j \leq m-1)$

$$F_{j,m} = \frac{[\zeta^{-(m-1)}, \zeta^{-(m-2)}, \ldots, \zeta^{-(m-j)}]_\infty}{[\zeta^{m+2}, \ldots, \zeta^{m+j+1}]_\infty}$$

$$= \prod_{k=1}^{j} \frac{[\zeta^{-(m-k)}]_\infty}{[\zeta^{m+k+1}]_\infty}$$

$$\log F_{j,m} = \sum_{k=1}^{j} \log[\zeta^{-(m-k)}]_\infty - \log[\zeta^{m+k+1}]_\infty$$

$$= \sum_{k=1}^{j} \log(1-\zeta^{-(m-k)}) - \log(1-\zeta^{m+k+1})$$

$$+ \sum_{k=1}^{j}\sum_{i=1}^{\infty} \log(1-\zeta^{-(m-k)}q^i) + \log(1-\zeta^{(m-k)}q^i)$$

$$- \log(1-\zeta^{m+k+1}q^i) - \log(1-\zeta^{-m-k-1}q^i)$$

$$\log\left(\frac{1-\zeta^{-(m-k)}}{1-\zeta^{m+k+1}}\right)$$

$$= \log\left(\frac{\zeta^{m-k}-1}{\zeta^{m-k}(1-\zeta^{m+k+1})}\right)$$

$$= \log(\zeta^{m-k}-1) - \log\zeta^{m-k} - \log(1-\zeta^{m+k+1})$$

$$\delta_\zeta = (m-k)\frac{\zeta^{m-k}}{\zeta^{m-k}-1} + (k-m) + \frac{(m+k+1)\zeta^{m+k+1}}{1-\zeta^{m+k+1}}$$

$$= (m-k)\frac{(\zeta^{m-k}-1+1)}{(\zeta^{m-k}-1)} + (k-m) + (m+k+1)\frac{(\zeta^{m+k+1}-1+1)}{(1-\zeta^{m+k+1})}$$

$$= (m-k) - (m-k)\times\frac{1}{1-\zeta^{m-k}} + (k-m)$$

$$- (m+k+1) + (m+k+1)\frac{1}{(1-\zeta^{m+k+1})}$$

$$= (m+k+1)\frac{1}{1-\zeta^{m+k+1}} - (m-k)\frac{1}{1-\zeta^{m-k}} - (m+k+1)$$

Hence,

$$\delta_\zeta F_{j,m}(\zeta,q) = L_{j,m}(\zeta,q)\, F_{j,m}(\zeta,q)$$

where

$$L_{j,m}(\zeta,q) = K_{j,m}(\zeta)$$
$$\cdot \sum_{k=1}^{j} \left(-(m-k) \sum_{i=1}^{\infty} \left(\frac{\zeta^{m-k} q^i}{1-\zeta^{m-k} q^i} - \frac{\zeta^{-(m-k)} q^i}{1-\zeta^{-(m-k)} q^i} \right) \right.$$
$$\left. + (m+k+1) \sum_{i=1}^{\infty} \left(\frac{\zeta^{m+k+1} q^i}{1-\zeta^{m+k+1} q^i} - \frac{\zeta^{-(m+k+1)} q^i}{1-\zeta^{-(m+k+1)} q^i} \right) \right)$$

& $$K_{j,m}(\zeta) = \sum_{k=1}^{j} \left(-(m+k+1) + (m+k+1) \frac{1}{1-\zeta^{m+k+1}} - (m-k) \frac{1}{1-\zeta^{m-k}} \right)$$

$$\frac{1}{e^x-1} = \frac{1}{x} + \sum_{n=0}^{\infty} \frac{b_{n+1}}{(n+1)!} x^n$$

$$K_{j,m}(e^x) = \sum_{k=1}^{j} -(m+k+1) + (m-k) \frac{1}{e^{x(m-k)}-1} - (m+k+1) \frac{1}{e^{x(m+k+1)}-1}$$

$$= \sum_{k=1}^{j} -(m+k+1) + \frac{(m-k)}{x(m-k)} - \frac{(m+k+1)}{x(m+k+1)}$$
$$+ \sum_{n=0}^{\infty} \frac{b_{n+1}}{(n+1)!} \left((m-k)^{n+1} - (m+k+1)^{n+1} \right) x^n$$

$$= \sum_{k=1}^{j} -(m+k+1) + \left(-\tfrac{1}{2}\right)\Big((m-k) - (m+k+1)\Big)$$

$$+ \sum_{n=1}^{\infty} \left(\sum_{k=1}^{j} (m-k)^{n+1} - (m+k+1)^{n+1}\right) x^n \cdot \frac{b_{n+1}}{(n+1)!}$$

$$= \sum_{k=1}^{j} -(m+k+1) - \tfrac{1}{2}(-2k-1)$$

$$+ \sum_{n=1}^{\infty} \left(\sum_{k=1}^{j} (m-k)^{n+1} - (m+k+1)^{n+1}\right) \frac{x^n \cdot b_{n+1}}{(n+1)!}$$

$$= \sum_{k=1}^{j} -m - \tfrac{1}{2} + \ldots$$

$$= -j\left(m+\tfrac{1}{2}\right) + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{j} (m-k)^{n+1} - (m+k+1)^{n+1}\right) \frac{x^n \cdot b_{n+1}}{(n+1)!}$$

$$D_a K_{j,m}(\zeta) = \left(\frac{d}{dx}\right)^a K_{j,m}(e^x)\Big|_{x=0}$$

$$= \text{coeff of } x^a \cdot a!$$

$$= \begin{cases} -j\left(m+\tfrac{1}{2}\right) & \text{if } a=0 \\ 0 & \text{if } a \text{ even } a>0 \\ \dfrac{b_{a+1}}{a+1} \displaystyle\sum_{k=1}^{j} \left((m-k)^{a+1} - (m+k+1)^{a+1}\right) & \text{if } a \text{ is odd} \end{cases}$$

Hence,

$P_a(\Lambda_{j,m}(\zeta, q))$

$$= \begin{cases} -j(m+\frac{1}{2}) & \text{if } a=0 \\ 0 & \text{if } a \text{ even } \& \, a>0 \\ 2\sum_{i=1}^{j}\left((m+i+1)^{a+1} - (m-i)^{a+1}\right) G_{a+1} & \text{if } a \text{ odd} \end{cases}$$

since $G_{a+1} = -\dfrac{b_{a+1}}{2(a+1)} + \phi_a$

<u>NOTE:</u> $D_a\left(\alpha\left(\sum_{i,n\geq 1} (\zeta^{\alpha} q^{i})^{n} - \sum_{i,n\geq 1} (\zeta^{-\alpha} q^{i})^{n}\right)\right)$

$$= \alpha \sum_{i,n\geq 1} \left((\alpha n)^a - (-\alpha n)^a\right) q^{in}$$

$$= \alpha^{a+1} \sum_{i,n\geq 1} \left(n^a - (-1)^a n^a\right) q^{in}$$

$$= \begin{cases} 0 & \text{if } a \text{ is even} \\ 2\cdot \alpha^{a+1} \sum_{m=1}^{\infty} \sigma_a(m) q^m & \text{if } a \text{ odd} \end{cases}$$

$$= \begin{cases} 0 & \text{if } a \text{ even} \\ 2\cdot \alpha^{a+1} \phi_a & \text{if } a \text{ odd} \end{cases}$$

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$$\delta_q((q)_\infty) = -\Phi_1 (q)_\infty$$

$$\delta_q(\Phi_1) = \frac{1}{6}\Phi_1 - 2\Phi_1^2 + \frac{5}{6}\Phi_3$$

(see [At-G]).

$$\mathcal{H}_5^* = 5\delta_z + 10\delta_q + \delta_z^2$$

$$\mathcal{H}_5^*(f(z,q)\,g(q)) = 10\delta_q(f(z,q)\,g(q)) + (5\delta_z + \delta_z^2)(f(z,q)\,g(q))$$

$$= (10\delta_q(g(q)))\,f(z,q) + g(q)\,\mathcal{H}_5^*(f(z,q)).$$

Therefore,

$$\mathcal{H}_5^*(\Sigma^{(5)}) = \mathcal{H}_5^*(G^{(5)}\,(q)_\infty^3)$$

$$= \delta_q((q)_\infty^3)\,G^{(5)} + (q)_\infty^3\,\mathcal{H}_5^*(G^{(5)})$$

$$= -30\,\Phi_1\,G^{(5)}(q)_\infty^3 + (q)_\infty^3\,\mathcal{H}_5^*(G^{(5)})$$

$$(\mathcal{H}_5^*)^2(\Sigma^{(5)}) = \mathcal{H}_5^*(-30\,\Phi_1\,G^{(5)}(q)_\infty^3) + \mathcal{H}_5^*((q)_\infty^3\,\mathcal{H}_5^*(G^{(5)}))$$

$$\mathcal{H}_5^*(-30\Phi_1(q)_\infty^3\,G^{(5)}) = -300\,\delta_q(\Phi_1\,(q)_\infty^3)\,G^{(5)} - 30\Phi_1\,(q)_\infty^3\,\mathcal{H}_5^*(G^{(5)})$$

$$\delta_q(\Phi_1\,(q)_\infty^3) = \delta_q(\Phi_1)\,(q)_\infty^3 + \Phi_1\,\delta_q((q)_\infty^3)$$

$$= (q)_\infty^3\left(\frac{1}{6}\Phi_1 - 2\Phi_1^2 + \frac{5}{6}\Phi_3 - 3\Phi_1^2\right)$$

$$= (q)_\infty^3\left(\frac{1}{6}\Phi_1 - 5\Phi_1^2 + \frac{5}{6}\Phi_3\right)$$

$$\mathcal{H}_5^*\left(-30\Phi_1 (q)_\infty^3 G^{(5)}\right)$$
$$= (q)_\infty^3\left(-50\Phi_1 + 1500\Phi_1^2 - 250\Phi_3 - 30\Phi_1\right) G^{(5)}$$

$$\mathcal{H}_5^*\left((q)_\infty^3 \mathcal{H}_5^*(G^{(5)})\right)$$
$$= \left(10\delta_q\left((q)_\infty^3\right)\right)\mathcal{H}_5^*(G^{(5)}) + (q)_\infty^3 (\mathcal{H}_5^*)^2(G^{(5)})$$
$$= -30\Phi_1 (q)_\infty^3 \mathcal{H}_5^*(G^{(5)}) + (q)_\infty^3 (\mathcal{H}_5^*)^2(G^{(5)})$$

$$(\mathcal{H}_5^*)^2(\Sigma^{(5)}) = (q)_\infty^3\left(-50\Phi_1 + 1500\Phi_1^2 - 250\Phi_3\right) G^{(5)}$$
$$- (q)_\infty^3\, 60\Phi_1 \mathcal{H}_5^*(G^{(5)})$$
$$+ (q)_\infty^3 (\mathcal{H}_5^*)^2(G^{(5)})$$

$$\left((\mathcal{H}_5^*)^2 + (60\Phi_1 + 10)\mathcal{H}_5^* + 300\Phi_1^2 + 10\Phi_3 + 350\Phi_1 + 24\right)\Sigma^{(5)}$$
$$= (q)_\infty^3\left[24 + 10\mathcal{H}_5^* + (\mathcal{H}_5^*)^2 - 240\Phi_3\right](G^{(5)})$$

This gives $(H_* = \mathcal{H}_5^* + 5)$

$$\left(H_*^2 - E_4\right) G^{(5)} = 24 (C^*)^5 (q)_\infty^2$$

which is [grcpdev2] (1.18).