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THE GBG-RANK AND t-CORES I. COUNTING AND 4-CORES

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Abstract

Let $r_j(\pi,s)$ denote the number of cells, colored j, in the s-residue diagram of partition π. The GBG-rank of π mod s is defined as

$$\text{GBG-rank}(\pi,s) = \sum_{j=0}^{s-1} r_j(\pi,s) e^{\frac{2\pi i}{s} j}.$$

We prove that for $(s,t)=1$

$$\nu(s,t) \le \frac{\binom{s+t}{s}}{s+t},$$

where $\nu(s,t)$ denotes the number of distinct values that the GBG-rank of a t-core mod s may assume. The above inequality becomes an equality when s is prime or when s is composite and $t \le 2p_s$, where p_s is the smallest prime divisor of s. We show that the generating functions for 4-cores with prescribed GBG-rank mod 3 value are all eta-quotients.

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1 Introduction

A partition π is a nonincreasing sequence

$$\pi = (\lambda_1, \lambda_2, \ldots, \lambda_\nu)$$

of positive integers (parts) $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots \ge \lambda_\nu > 0$. The norm of π, denoted $|\pi|$, is defined as

$$|\pi| = \sum_{i=1} \lambda_i.$$

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If $|\pi| = n$, we say that π is a partition of n. The (Young) diagram of π is a convenient way to represent π graphically: the parts of π are shown as rows of unit squares (cells). Given the diagram of π we label a cell in the i-th row and j-th column by the least nonnegative integer $\equiv j - i \mod s$. The resulting diagram is called an s-residue diagram [8]. One can also label cells in the infinite column 0 and the infinite row 0 in the same fashion. The resulting diagram is called the extended s-residue diagram of π [5]. With each π we can associate the s-dimensional vector

$$\mathbf{r}(\pi, s) = (r_0, r_1, \ldots, r_{s-1}),$$

where $r_i = r_i(\pi, s)$ $(0 \leq i \leq s-1)$ is the number of cells labelled i in the s-residue diagram of π. We shall also require

$$\mathbf{n}(\pi, s) = (n_0, n_1, \ldots, n_{s-1}),$$

where for $0 \leq i \leq s-2$

$$n_i = r_i - r_{i+1},$$

and

$$n_{s-1} = r_{s-1} - r_0.$$

Note that

$$\mathbf{n} \cdot \mathbf{l}_s = \sum_{i=0}^{s-1} n_i = 0,$$

where

$$\mathbf{l}_s = (1, 1, 1, \ldots, 1) \in \mathbb{Z}^s.$$

We recall the notions of rim hook and t-core [8]. If some cell of π shares a vertex or edge with the rim of the diagram of π, we call this cell a rim cell of π. A connected collection of rim cells of π is called a rim hook of π if $\pi \backslash$(rim hook) is a legitimate partition. A partition is a t-core if it has no rim hooks of length t. Throughout this paper we usually denote a generic t-core by $\pi_{t\text{-core}}$. We will also use this notation in a different way, but the context should be clear. Any partition π has a uniquely determined t-core which we will also denote by $\pi_{t\text{-core}}$. This partition $\pi_{t\text{-core}}$ is called the t-core of π. One can obtain $\pi_{t\text{-core}}$ from π by the successive removal of rim hooks of length t. The t-core $\pi_{t\text{-core}}$ is independent of the manner in which hooks removed. The Durfee square of π, denoted $D(\pi)$, is the largest square that fits inside the diagram of π. Reflecting the diagram of π about its main diagonal, one gets the diagram of π^* (the conjugate of π). More formally,

$$\pi^* = (\lambda_1^*, \lambda_2^*, \ldots)$$

with λ_i^* being the number of parts of $\pi \geq i$. Clearly,

$$D(\pi) = D(\pi^*).$$

In [3] we defined a new partition statistic of π

$$\text{GBG-rank}(\pi, s) := \sum_{j=0}^{s-1} r_j(\pi, s)\,\omega_s^j, \tag{1.1}$$

where

$$\omega_s = e^{\frac{2\pi i}{s}}.$$

We refer to this statistic as the GBG-rank of π mod s. The special case $s = 2$ was studied in great detail in [3] and [4]. In particular, we have shown in [3] that for any odd $t > 1$

$$\text{GBG-rank}(\pi_{t\text{-core}}, 2) = \frac{1 - \sum_{i=0}^{t-1} (-1)^{i+n_i(\pi_{t\text{-core}},t)}}{4} \tag{1.2}$$

and that

$$-\left\lfloor \frac{t-1}{4} \right\rfloor \leq \text{GBG-rank}(\pi_{t\text{-core}}, 2) \leq \left\lfloor \frac{t+1}{4} \right\rfloor, \tag{1.3}$$

where $\lfloor x \rfloor$ is the integer part of x. Our main object here is to prove the following generalisations of (1.2) and (1.3).

Theorem 1.1. *Let $t, s \in \mathbb{Z}_{>1}$ and $(t, s) = 1$. Then*

$$\textit{GBG-rank}(\pi_{t\text{-}core}, s) = \frac{\sum_{j=0}^{t-1} \omega_s^{j+1} (\omega_s^{tn_j(\pi_{t\text{-}core},t)} - 1)}{(1 - \omega_s)(1 - \omega_s^t)} \tag{1.4}$$

Theorem 1.2. *Let $\nu(s, t)$ denote the number of distinct values that GBG-rank of $\pi_{t\text{-}core}$ mod s may assume. Then*

$$\nu(s, t) \leq \frac{\binom{t+s}{t}}{t+s}, \tag{1.5}$$

provided $(s, t) = 1$.

Theorem 1.3. *Let $\nu(s, t)$ be as in Theorem 1.2 and $(s, t) = 1$. Then*

$$\nu(s, t) = \frac{\binom{t+s}{t}}{t+s}, \tag{1.6}$$

iff either s is prime or s is composite and $t < 2p_s$, where p_s is the smallest prime divisor of s.

Our proof of this theorem depends crucially on the following

Lemma 1.4. *Let $s, t \in \mathbb{Z}_{>1}$ and $(s, t) = 1$. Let $\mathbf{j} = (j_0, j_1, \ldots, j_{t-1})$, $\tilde{\mathbf{j}} = (\tilde{j}_0, \tilde{j}_1, \ldots, \tilde{j}_{t-1})$ be integer valued vectors such that*

$$0 \leq j_0 \leq j_1 \leq \cdots \leq j_{t-1} \leq s - 1, \tag{1.7}$$

$$0 \leq \tilde{j}_0 \leq \tilde{j}_1 \leq \cdots \leq \tilde{j}_{t-1} \leq s - 1, \tag{1.8}$$

and

$$\sum_{i=0}^{t-1} \omega_s^{j_i} = \sum_{i=0}^{t-1} \omega_s^{\tilde{j}_i} \tag{1.9}$$

$$\prod_{i=0}^{t-1}\omega_s^{j_i}=\prod_{i=0}^{t-1}\omega_s^{\tilde{j}_i}. \tag{1.10}$$

Then

$$\mathbf{j}=\tilde{\mathbf{j}},$$

iff either s is prime or s is composite such that $t<2p_s$, *where* p_s *is the smallest prime divisor of s.*

The rest of this paper is organised as follows. In Section 2, we collect some necessary background on t-cores and prove Theorems 1.1 and 1.2. Section 1.3 is devoted to the proof of Lemma 1.4 and Theorem 1.3. Section 4 deals with 4-cores with prescribed values of GBG-rank mod 3. There we will provide new combinatorial interpretation and proof of the Hirschhorn-Sellers identities for 4-cores [7]. We conclude with remarks connecting this development and that of [11] and [2].

2 Properties of the GBG-rank

We begin with some definitions from [5]. A region r in the extended t-residue diagram of π is the set of all cells (i,j) satisfying $t(r-1)\le j-i<tr$. A cell of π is called exposed if it is at the end of a row of π. One can construct t bi-infinite words $W_0,W_1,\dots,W_{t-1}$ of two letters N,E as follows: The rth letter of W_i is E if there is an exposed cell labelled i in the region r of π, otherwise the rth letter of W_i is N. It is easy to see that the word set $\{W_0,W_1,\dots,W_{t-1}\}$ fixes π uniquely. It was shown in [5] that π is a t-core iff each word of π is of the form:

$$\begin{array}{rcccccc} \text{Region}: & \cdots\cdots & n_{i-1} & n_i & n_{i+1} & n_{i+2} & \cdots\cdots \\ W_0: & \cdots\cdots & E & E & N & N & \cdots\cdots. \end{array} \tag{2.1}$$

For example, the word image of $\pi_{3\text{-core}}=(4,2)$ is

$$\begin{array}{rccccccc} \text{Region}: & \cdots\cdots & -1 & 0 & 1 & 2 & 3 & \cdots\cdots \\ W_0: & \cdots\cdots & E & E & E & E & N & \cdots\cdots \\ W_1: & \cdots\cdots & E & N & N & N & N & \cdots\cdots \\ W_2: & \cdots\cdots & E & N & N & N & N & \cdots\cdots, \end{array}$$

while the associated $\mathbf{r}$ and $\mathbf{n}$ vectors are $\mathbf{r}=(r_0,r_1,r_2)=(3,1,2)$, $\mathbf{n}=(n_0,n_1,n_2)=(2,-1,-1)$, respectively. In general, the map

$$\phi(\pi_{t\text{-core}})=\mathbf{n}(\pi_{t\text{-core}},t)=(n_0,n_1,\dots,n_{t-1})$$

is a bijection from the set of t-cores to the set

$$\{\mathbf{n}\in\mathbb{Z}^t:\mathbf{n}\cdot\mathbf{1}=0\}.$$

Next, we mention three more useful facts from [5].

(A) We have

$$\left|\pi_{t\text{-core}}\right| = \frac{t}{2}|\mathbf{n}|^2 + \mathbf{b}_t \cdot \mathbf{n}, \tag{2.2}$$

where $\mathbf{b}_t = (0,1,2,\ldots,t-1)$.

(B) We have

$$\sum_{i\in P_1} n_i = -\sum_{i\in P_{-1}} n_i = |D(\pi_{t\text{-core}})|, \tag{2.3}$$

where $P_\alpha = \{i \in \mathbb{Z} : 0 \le i \le t-1, \alpha n_i > 0\}$, $\alpha = -1, 1$, and $|D(\pi_{t\text{-core}})|$ denotes the number of cells on the main diagonal of the Durfee square.

(C) Under conjugation, $\phi(\pi_{t\text{-core}})$ transforms as

$$(n_0, n_1, n_2, \ldots, n_{t-1}) \to (-n_{t-1}, -n_{t-2}, \ldots, -n_0). \tag{2.4}$$

We begin our proof of the Theorem 1.1 by observing that under conjugation GBG-rank transforms as

$$\text{GBG-rank}(\pi, s) = \sum_{i=0}^{s-1} r_i \omega_s^i \Longrightarrow \text{GBG-rank}(\pi^*, s) = \sum_{i=0}^{s-1} r_i \omega_s^{-i}. \tag{2.5}$$

Next, we use that

$$\text{GBG-rank}(\pi, s) = \text{GBG-rank}(\pi_1, s) + \text{GBG-rank}(\pi_2, s) - D. \tag{2.6}$$

Here, π_1 is obtained from the diagram of $\pi_{t\text{-core}}$ by removing all cells strictly below the main diagonal of $\pi_{t\text{-core}}$. Similarly, π_2 is obtained from $\pi_{t\text{-core}}$ by removing the cells strictly to the right of the main diagonal.

Recalling (2.1) and (2.3) we find that

$$\text{GBG-rank}(\pi_1, s) = \sum_{i\in P_1} \sum_{k=1}^{n_i} \sum_{j=0}^{i+t(k-1)} \omega_s^j = \frac{D}{1-\omega_s} - \sum_{i\in P_1} \frac{\omega_s^{i+1}(1-\omega_s^{tn_i})}{(1-\omega_s)(1-\omega_s^t)}. \tag{2.7}$$

Analogously,

$$\text{GBG-rank}(\pi_2^*, s) = \frac{D}{1-\omega_s} - \sum_{i\in P_{-1}} \frac{\omega_s^{t-i}(1-\omega_s^{-tn_i})}{(1-\omega_s)(1-\omega_s^t)}, \tag{2.8}$$

where we made use of (2.4).

Clearly, (2.5) and (2.8) imply that

$$\text{GBG-rank}(\pi_2, s) = -\frac{D\omega_s}{1-\omega_s} - \sum_{i\in P_{-1}} \frac{\omega_s^{1+i}(1-\omega_s^{tn_i})}{(1-\omega_s)(1-\omega_s^t)}. \tag{2.9}$$

Next, we combine (2.6), (2.7) and (2.9) to find that

$$\text{GBG-rank}(\pi_{t\text{-core}}, s) = -\sum_{i\in P_{-1}\bigcup P_1} \frac{\omega_s^{1+i}(1-\omega_s^{tn_i})}{(1-\omega_s)(1-\omega_s^t)} = \sum_{i=0}^{t-1} \frac{\omega_s^{1+i}(\omega_s^{tn_i}-1)}{(1-\omega_s)(1-\omega_s^t)},$$

as desired.

Our proof of Theorem 1.2 involves three observations, which we now proceed to discuss.

Observation 1. *Let $a_r(s,t)$ denote the number of vectors $\mathbf{j}=(j_0,j_1,\ldots,j_{t-1})$ such that*

$$0\le j_0\le j_1\le j_2\le\cdots\le j_{t-1}<s,$$
$$\sum_{k=0}^{t-1} j_k\equiv r \textit{ mod } s.$$

Then

$$\nu(s,t)\le a_{\frac{t(t+1)}{2}}(s,t), \tag{2.10}$$

provided $(s,t)=1$.

Proof. Suppose $(s,t)=1$. It is clear that the number of values of the GBG-rank of t-cores mod s is the number of distinct values of

$$\sum_{i=0}^{t-1}\omega_s^{1+i+tn_i},$$

where $\mathbf{n}\in\mathbb{Z}^t$ and $\mathbf{n}\cdot\mathbf{1}_t=0$. Given any such $\mathbf{n}$-vector we reduce the exponents $1+i+tn_i$ mod s and reorder to obtain a $\mathbf{j}$-vector such that

$$\sum_{k=0}^{t-1} j_k\equiv\sum_{i=0}^{t-1}1+i+tn_i\equiv\frac{t(t+1)}{2}\pmod{s}.$$

It follows that

$$\nu(s,t)\le a_{\frac{t(t+1)}{2}}(s,t).$$

□

Observation 2. *We have*

$$\sum_{r=0}^{s-1}a_r(s,t)=\binom{t+s-1}{t}. \tag{2.11}$$

This result is well known and we omit the proof. Finally, we need

Observation 3. *If* $(s,t)=1$ *then*

$$a_0(s,t)=a_1(s,t)=\cdots=a_{s-1}(s,t). \tag{2.12}$$

Proof. There exists an integer T such that $T\cdot t\equiv 1$ mod s, because s and t are coprime. This implies that

$$\sum_{i=0}^{t-1}(j_i+T)\equiv 1+\sum_{i=0}^{t-1}j_i \text{ mod } s.$$

Consequently, $a_r(s,t)=a_{r+1}(s,t)$, as desired. Combining (2.10), (2.11) and (2.12) we see that

$$\nu(s,t)\le a_{\frac{t(t+1)}{2}}=\frac{\binom{s-1+t}{t}}{s}=\frac{\binom{s+t}{t}}{s+t},$$

and we have Theorem 1.2. □

3 Roots of Unity and the Number of Values of the GBG-rank

It is clear from our proof of Theorem 1.2 that

$$\nu(s,t) = \frac{\binom{s+t}{t}}{s+t}.$$

iff each $\mathbf{j} = (j_0, j_1, \ldots, j_{t-1})$ such that

$$0 \le j_0 \le j_1 \le \cdots \le j_{t-1} < s,$$

$$\prod_{i=0}^{t-1} \omega_s^{j_i} = \omega_s^{\frac{t(t+1)}{2}}$$

is associated with a distinct complex number $\sum_{i=0}^{t-1} \omega_s^{j_i}$. Lemma 1.4 tells us when this is exactly the case. This means that Theorem 1.3 is an immediate corollary of this Lemma. To prove it we need to consider six cases.

Case 1. s is prime, $(s,t) = 1$.

Note that

$$\Phi_s(x) := 1 + x + x^2 + \cdots + x^{s-1}$$

is a minimal polynomial of ω_s over $\mathbb{Q}$. Let us now define

$$p_1(x) := \sum_{i=0}^{t-1} (x^{j_i} - x^{\tilde{j}_i}), \tag{3.1}$$

where $\mathbf{j}$ and $\tilde{\mathbf{j}}$ satisfy the constraints (1.7) - (1.10). It is clear that

$$p_1(\omega_s) = 0,$$

$$p_1(1) = 0,$$

and that $\deg(p_1(x)) < s$. But $(x-1)\Phi_s(x)$ divides $p_1(x)$. This implies that $p_1(x)$ is identically zero and $\mathbf{j} = \tilde{\mathbf{j}}$, as desired.

Case 2. s is composite, $(s,t) = 1$ and $t < 2p_s$, where p_s is the smallest prime divisor of s.

Once again (1.9) implies that

$$p_1(\omega_s) = 0.$$

Moreover, the sth cyclotomic polynomial, defined as

$$\Phi_s(x) := \prod_{\substack{0<j<s,\\ (j,s)=1}} (x - \omega_s^j), \tag{3.2}$$

is a minimal polynomial of ω_s over $\mathbb{Q}$. This means that

$$p_1(\omega_s^m) = 0, \tag{3.3}$$

for any $1 \le m < s$ such that $(s,m) = 1$. In particular, we have that

$$p_1(\omega_s^k) = 0, \qquad 1 \le k \le p_s - 1. \tag{3.4}$$

At this point, it is expedient to rewrite (3.4) as

$$h_k(\omega_s^{j_0}, \ldots, \omega_s^{j_{t-1}}) = h_k(\omega_s^{\tilde{j}_0}, \ldots, \omega_s^{\tilde{j}_{t-1}}), \qquad 1 \le k \le p_s - 1, \tag{3.5}$$

where

$$h_k(x_1, x_2, \ldots, x_t) = x_1^k + x_2^k + \cdots + x_t^k.$$

Next, we use Newton's theorem on symmetric polynomials to convert (3.5) into $p_s - 1$ identities

$$\sigma_k(\omega_s^{j_0}, \ldots, \omega_s^{j_{t-1}}) = \sigma_k(\omega_s^{\tilde{j}_0}, \ldots, \omega_s^{\tilde{j}_{t-1}}), \qquad 1 \le k \le p_s - 1, \tag{3.6}$$

where the kth elementary symmetric polynomials σ_k's in $x_1, x_2, \ldots, x_t$ are defined in a standard way as

$$\sigma_k(x_1, x_2, \ldots, x_t) = \sum_{1 \le i_1 \le i_2 < \ldots < i_k \le t} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad 1 \le k \le t. \tag{3.7}$$

Note that we can rewrite (1.10) now as

$$\sigma_t(\omega_s^{j_0}, \ldots, \omega_s^{j_{t-1}}) = \sigma_t(\omega_s^{\tilde{j}_0}, \ldots, \omega_s^{\tilde{j}_{t-1}}). \tag{3.8}$$

But

$$\sigma_t \sigma_k^* = \sigma_{t-k}, \tag{3.9}$$

where

$$\sigma_k^*(x_1, x_2, \ldots, x_t) = \sigma_k(x_1^{-1}, x_2^{-1}, \ldots, x_t^{-1}).$$

This fortunate fact enables us to convert (3.6) into $p_s - 1$ identities

$$\sigma_k(\omega_s^{j_0}, \ldots, \omega_s^{j_{t-1}}) = \sigma_k(\omega_s^{\tilde{j}_0}, \ldots, \omega_s^{\tilde{j}_{t-1}}), \qquad t - p_s + 1 \le k \le t - 1. \tag{3.10}$$

But $t < 2p_s$, and so, $t - p_s + 1 \le p_s$. This means that we have the following t identities

$$\sigma_k(\omega_s^{j_0}, \ldots, \omega_s^{j_{t-1}}) = \sigma_k(\omega_s^{\tilde{j}_0}, \ldots, \omega_s^{\tilde{j}_{t-1}}), \qquad 1 \le k \le t. \tag{3.11}$$

Consequently,

$$\prod_{i=0}^{t-1}(x - \omega_s^{j_i}) = \prod_{i=0}^{t-1}(x - \omega_s^{\tilde{j}_i}).$$

Recalling that $\mathbf{j}, \tilde{\mathbf{j}}$ satisfy (1.7) and (1.8), we conclude that $\mathbf{j} = \tilde{\mathbf{j}}$.

Let us summarize. If s is a prime or if s is a composite number such that $t < 2p_s$, then $\mathbf{j} = \tilde{\mathbf{j}}$, provided that $(s,t) = 1$ and $\mathbf{j}, \tilde{\mathbf{j}}$ satisfy (1.7)–(1.10).

It remains to show that $\mathbf{j} = \tilde{\mathbf{j}}$ does not have to be true if s is a composite number and $t \ge 2p_s$. To this end consider

$$\mathbf{j} := (0, 0, \ldots, 0, 1, 1, 3, 3) \in \mathbb{Z}^t,$$
$$\tilde{\mathbf{j}} := (0, 0, \ldots, 0, 0, 0, 2, 2) \in \mathbb{Z}^t,$$

if $s = 4$, $t \geq 4$,

$$\mathbf{j} := (0,0,\ldots,0,1,1,4,4) \in \mathbb{Z}^t,$$
$$\tilde{\mathbf{j}} := (0,0,\ldots,0,0,2,3,5) \in \mathbb{Z}^t,$$

if $s = 6$, $t \geq 4$ and

$$\mathbf{j} := (0,0,\ldots,0,3,3,6,6) \in \mathbb{Z}^t,$$
$$\tilde{\mathbf{j}} := (0,0,\ldots,0,1,2,4,5,7,8) \in \mathbb{Z}^t,$$

if $s = 9$, $t \geq 6$, respectively. It is not hard to verify that $\mathbf{j}, \tilde{\mathbf{j}}$ satisfy (1.7)–(1.10) and that $\mathbf{j} \neq \tilde{\mathbf{j}}$ in these cases. It remains to consider the last case where s is a composite number $\neq 4,6,9$, $t \geq 2p_s$. In this case $s > 3p_s$. And, as a result,

$$3 + \frac{s}{p_s}(p_s - 1) < s.$$

Let us now consider

$$\mathbf{j} := \left(0,0,\ldots,0,2,2,2+\frac{s}{p_s},2+\frac{s}{p_s},\ldots,2+\frac{s}{p_s}(p_s-1),2+\frac{s}{p_s}(p_s-1)\right) \in \mathbb{Z}^t,$$
$$\tilde{\mathbf{j}} := \left(0,0,\ldots,0,1,3,1+\frac{s}{p_s},3+\frac{s}{p_s},\ldots,1+\frac{s}{p_s}(p_s-1),3+\frac{s}{p_s}(p_s-1)\right) \in \mathbb{Z}^t.$$

Again, it is straightforward to check that $\mathbf{j}, \tilde{\mathbf{j}}$ satisfy (1.7)–(1.10) and that $\mathbf{j} \neq \tilde{\mathbf{j}}$. This completes our proof of Lemma 1.4.

It is clear that Theorem 1.3 follows from the above.

To illustrate the usefulness of Theorem 1.3, consider the following example: $s = 3$, $t = 4$. In this case we should have exactly $\binom{4+3}{3}/(4+3) = 5$ distinct values of GBG-rank $(\pi_{4\text{-core}}, 3)$. To determine these distinct values we substitute the following **n**-vectors $(0,-1,1,0)$, $(0,0,0,0)$, $(-1,0,0,1)$, $(0,0,-1,1)$, $(-1,1,0,0)$ into (1.4) to obtain $-1,0,1$, $-\omega_3, -\omega_3^2$, respectively. To verify this we note that there are exactly 27 vectors such that

$$\mathbf{n} \in \mathbb{Z}_3^4 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{1}_4 \equiv 0 \bmod 3.$$

In Table 1 we list all these vectors together with the associated GBG-rank mod 3 values, determined by (1.4). These vectors will come in handy later.

4 The GBG-rank of 4-cores mod 3

Let $G_t(q)$ denote the generating function for t-cores.

$$G_t(q) := \sum_{\pi_{t\text{-core}}} q^{|\pi_{t\text{-core}}|}. \tag{4.1}$$

Let P be the set of all partitions and $P_{t\text{-core}}$ be the set of all t-cores. There is a well-known bijection

$$\tilde{\phi} : P \to P_{t\text{-core}} \times P \times P \times P \ldots \times P$$

n vectors	GBG-rank values
$\mathbf{n}_1 = (0,-1,1,0)$	-1
$\mathbf{n}_2 = (0,0,0,0)$	0
$\mathbf{n}_3 = (1,1,-2,0)$	0
$\mathbf{n}_4 = (-1,-1,1,1)$	0
$\mathbf{n}_5 = (0,-1,-1,2)$	0
$\mathbf{n}_6 = (1,-1,0,0)$	0
$\mathbf{n}_7 = (0,1,-2,1)$	0
$\mathbf{n}_8 = (2,-1,-1,0)$	0
$\mathbf{n}_9 = (0,0,1,-1)$	0
$\mathbf{n}_{10} = (0,1,-1,0)$	0
$\mathbf{n}_{11} = (-1,0,1,0)$	0
$\mathbf{n}_{12} = (1,-1,1,-1)$	0
$\mathbf{n}_{13} = (0,-1,0,1)$	0
$\mathbf{n}_{14} = (1,1,0,-2)$	1
$\mathbf{n}_{15} = (-1,1,-1,1)$	1
$\mathbf{n}_{16} = (2,0,-1,-1)$	1
$\mathbf{n}_{17} = (1,0,0,-1)$	1
$\mathbf{n}_{18} = (1,1,-1,-1)$	1
$\mathbf{n}_{19} = (-1,0,0,1)$	1
$\mathbf{n}_{20} = (1,0,-1,0)$	$-\omega_3$
$\mathbf{n}_{21} = (1,0,-2,1)$	$-\omega_3$
$\mathbf{n}_{22} = (1,-1,-1,1)$	$-\omega_3$
$\mathbf{n}_{23} = (0,0,-1,1)$	$-\omega_3$
$\mathbf{n}_{24} = (-1,1,0,0)$	$-\omega_3^2$
$\mathbf{n}_{25} = (-1,1,1,-1)$	$-\omega_3^2$
$\mathbf{n}_{26} = (-1,2,0,-1)$	$-\omega_3^2$
$\mathbf{n}_{27} = (0,1,0,-1)$	$-\omega_3^2$

Table 1:

which goes back to D.E. Littlewood [10]

$$\tilde{\phi}(\pi) = (\pi_t\text{-core}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$$

such that

$$|\pi| = |\pi_t\text{-core}| + t\sum_{i=0}^{t-1} |\hat{\pi}_i|.$$

The multipartition $(\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$ is called the t-quotient of π. An immediate corollary of the Littlewood bijection is

$$G_t(q) = \frac{E^t(q^t)}{E(q)}, \tag{4.2}$$

where

$$E(q) := \prod_{j\geq 1}(1-q^j). \tag{4.3}$$

The function $E(q)$ is related to Dedekind's eta-function $\eta(\tau)$ by

$$\eta(\tau) = q^{\frac{1}{24}} E(q),$$

where $q = \exp(2\pi i\tau)$ and $\Im(\tau) > 0$. Quotients of the functions $E(q^a)$ are called *eta*-quotients. On the other hand, formula (2.2) implies [5] that

$$G_t(q) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^t,\\ \mathbf{n}\cdot\mathbf{1}_t=0}} q^{\frac{t}{2}|\mathbf{n}|^2+\mathbf{n}\cdot\mathbf{b}_t}, \tag{4.4}$$

so that

$$\sum_{\substack{\mathbf{n}\in\mathbb{Z}^t,\\ \mathbf{n}\cdot\mathbf{1}_t=0}} q^{\frac{t}{2}|\mathbf{n}|^2+\mathbf{n}\cdot\mathbf{b}_t} = \frac{E^t(q^t)}{E(q)}. \tag{4.5}$$

The above identity was first obtained by Klyachko [9], who observed that it is a special case of Macdonald's identity for the root system A_{t-1}. An elementary proof of (4.5) can be found in [3]. Next we define

$$g_c(q) = \sum_{\substack{\pi_{\text{4-core}},\\ \text{GBG-rank}(\pi_{\text{4-core}},3)=c}} q^{|\pi_{\text{4-core}}|} \tag{4.6}$$

In other words, $g_c(q)$ is the generating function for 4-cores with a given value c of the GBG-rank mod 3. From the discussion at the end of the last section it is clear that

$$\frac{E^4(q^4)}{E(q)} = g_{-1}(q) + g_0(q) + g_1(q) + g_{-\omega_3}(q) + g_{-\omega_3^2}(q). \tag{4.7}$$

It turns out that each $g_c(q)$ is an *eta*-quotient.

$$g_{-1}(q) = q^5\frac{E^4(q^{36})}{E(q^9)}, \tag{4.8}$$

$$g_0(q) = \frac{E^6(q^6)E^2(q^{18})}{E^3(q^3)E(q^{12})E(q^{36})}, \tag{4.9}$$

$$g_1(q) = q\frac{E^2(q^9)E^4(q^{12})}{E(q^3)E(q^6)E(q^{18})}, \tag{4.10}$$

$$g_{-\omega_3}(q) = q^2\frac{E^2(q^9)E(q^{12})E(q^{36})}{E(q^3)}, \tag{4.11}$$

$$g_{-\omega_3^2}(q) = q^2\frac{E^2(q^9)E(q^{12})E(q^{36})}{E(q^3)}. \tag{4.12}$$

Hence

$$\frac{E^4(q^4)}{E(q)} = \frac{E^6(q^6)E^2(q^{18})}{E^3(q^3)E(q^{12})E(q^{36})} + q\frac{E^2(q^9)E^4(q^{12})}{E(q^3)E(q^6)E(q^{18})} + 2q^2\frac{E^2(q^9)E(q^{12})E(q^{36})}{E(q^3)} + q^5\frac{E^4(q^{36})}{E(q^9)}. \tag{4.13}$$

We note that the identity

$$g_{-\omega_3}(q) = g_{-\omega_3^2}(q)$$

follows from (2.5) and the fact that π is a t-core if and only if the conjugate π^* is a t-core. The identities equivalent to (4.13) were first proven by Hirschhorn and Sellers [7]. However, the combinatorial identities (4.7)–(4.12) given here are brand new. The proof of (4.8) is rather simple. Indeed, data in Table 1, implies that

$$\begin{aligned} g_{-1}(q) &= \sum_{\substack{\mathbf{n}\cdot\mathbf{1}_4=0,\\ \mathbf{n}\equiv\mathbf{n}_1 \bmod 3}} q^{2|\mathbf{n}|^2+\mathbf{b}_4\cdot\mathbf{n}} \\ &= q^5 \sum_{\tilde{\mathbf{n}}\cdot\mathbf{1}_4=\tilde{n}_0+\tilde{n}_1+\tilde{n}_2+\tilde{n}_3=0} q^{9(2|\tilde{\mathbf{n}}|^2-\tilde{n}_1+2\tilde{n}_2+\tilde{n}_3)} \\ &= q^5 \sum_{\tilde{\mathbf{n}}\cdot\mathbf{1}_4=0} q^{9(2|\tilde{\mathbf{n}}|^2+\tilde{n}_0+2\tilde{n}_3+3\tilde{n}_2)} \\ &= q^5 \frac{E^4(q^{36})}{E(q^9)} \end{aligned} \tag{4.14}$$

where in the last step we relabelled the summation variables and used (4.5) with $t=4$ and $q \to q^9$.

In what follows we shall require the Jacobi triple product identity ([6],II.28)

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n = E(q^2)[zq;q^2]_\infty, \tag{4.15}$$

where

$$[z;q]_\infty := \prod_{j=0}^{\infty} (1-zq^j)\left(1-\frac{q^{1+j}}{z}\right)$$

and the formula ([6],ex.5.21)

$$\left[ux,\frac{u}{x},vy,\frac{v}{y};q\right]_\infty = \left[uy,\frac{u}{y},vx,\frac{v}{x};q\right]_\infty + \frac{v}{x}\left[xy,\frac{x}{y},uv,\frac{u}{v};q\right]_\infty, \tag{4.16}$$

where

$$[z_1,z_2,\ldots,z_n;q]_\infty := \prod_{j=1}^{n} [z_i;q]_\infty.$$

Setting $u=q^5$, $v=q^3$, $x=q^2$, $y=q$ and replacing q by q^{12} in (4.16) we find that

$$[q^2,q^3;q^{12}]_\infty([q^5;q^{12}]_\infty - q[q;q^{12}]_\infty) = [q,q^5,q^6;q^{12}]_\infty. \tag{4.17}$$

Analogously, (4.16) with $u=q^5$, $v=q^2$, $x=q$, $y=1$ and $q \to q^{12}$ becomes

$$[q^5;q^{12}]_\infty + q[q;q^{12}]_\infty = \frac{[q^2,q^2,q^4,q^6;q^{12}]_\infty}{[q,q^3,q^5;q^{12}]_\infty}. \tag{4.18}$$

Finally, setting $u=q^6$, $v=q^4$, $x=q^3$, $y=1$ and $q \to q^{12}$ in (4.16) yields

$$[q^3,q^4;q^{12}]_\infty^2 = [q,q^5,q^6,q^6;q^{12}]_\infty + q[q^2,q^3;q^{12}]_\infty^2. \tag{4.19}$$

Next, we use again Table 1 to rewrite (4.9) as

$$\sum_{j=2}^{13} \sum_{\substack{\mathbf{n}\cdot\mathbf{1}_4=0,\\ \mathbf{n}\equiv\mathbf{n}_j \bmod 3}} q^{2|\mathbf{n}|^2+\mathbf{b}_4\cdot\mathbf{n}} = R_1(q), \tag{4.20}$$

where

$$R_1(q) = \frac{E^6(q^6)E^2(q^{18})}{E^3(q^3)E(q^{12})E(q^{36})}. \tag{4.21}$$

Remarkably, (4.20) is the constant term in z of the following identity

$$\sum_{j=2}^{13} s_j(z,q) = R_1(q) \sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}} z^n, \tag{4.22}$$

where

$$s_j(z,q) := \sum_{\mathbf{n}\equiv\mathbf{n}_j \bmod 3} q^{2|\mathbf{n}|^2+\mathbf{b}_4\cdot\mathbf{n}} z^{\frac{\mathbf{n}\cdot\mathbf{1}_4}{3}}, \qquad j = 1,2,\ldots,27. \tag{4.23}$$

Using simple changes of variables, it is straightforward to check that

$$zq^9 s_i(zq^9,q) = s_j(z,q), \tag{4.24}$$

holds true for the following (i,j) pairs: $(2,3)$, $(3,4)$, $(4,5)$, $(5,2)$, $(6,7)$, $(7,8)$, $(8,9)$, $(9,6)$, $(10,11)$, $(11,12)$, $(12,13)$, $(13,10)$, and that

$$zq^9 \sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}} (zq^9)^n = \sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}} z^n. \tag{4.25}$$

Consequently, both sides of (4.22) satisfy the same first order functional equation

$$zq^9 f(zq^9,q) = f(z,q). \tag{4.26}$$

Thus to prove (4.22) it is sufficient to verify it at one nontrivial point, say $z = z_0 := -q^{-6}$. It is not hard to check that

$$s_4(z_0,q) = s_8(z_0,q) = s_{11}(z_0,q) = 0, \tag{4.27}$$

and that

$$s_3(z_0,q) + s_9(z_0,q) = s_5(z_0,q) + s_{12}(z_0,q) = 0. \tag{4.28}$$

We see that (4.22) with $z = z_0$ becomes

$$s_2(z_0,q) + s_6(z_0,q) + s_7(z_0,q) + s_{10}(z_0,q) + s_{13}(z_0,q) = R_1(q) \sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}} z_0^n. \tag{4.29}$$

Upon making repeated use of (4.15) and replacing q^3 by q we find that (4.9) is equivalent to

$$\begin{aligned}
[q^4,q^5,q^5,q^6;q^{12}]_\infty &+ q[q^2,q^3,q^4;q^{12}]_\infty([q^5;q^{12}]_\infty - q[q;q^{12}]_\infty) \\
&+ q[q,q^4,q^5,q^6;q^{12}]_\infty + q^2[q,q,q^4,q^6;q^{12}]_\infty \\
&= \frac{E^2(q^6)E^6(q^2)}{E^5(q^{12})E(q^4)E^2(q)}.
\end{aligned} \tag{4.30}$$

eq We can simplify (4.30) with the aid of (4.17) as

$$[q^4,q^6;q^{12}]_\infty([q^5;q^{12}]_\infty+q[q;q^{12}]_\infty)^2=\frac{E^2(q^6)E^6(q^2)}{E^5(q^{12})E(q^4)E^2(q)}. \tag{4.31}$$

Next, we use (4.18) to reduce (4.31) to the following easily verifiable identity

$$\frac{[q^2;q^{12}]_\infty^4[q^4,q^6;q^{12}]_\infty^3}{[q,q^3,q^5;q^{12}]_\infty^2}=\frac{E^2(q^6)E^6(q^2)}{E^5(q^{12})E(q^4)E^2(q)}. \tag{4.32}$$

This completes our proof of (4.22), (4.20). We have (4.9), as desired.

The proof of (4.10) is analogous. Again, we view this identity as the constant term in z of the following

$$\sum_{j=14}^{19} s_j(z,q)=R_2(q)\sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}}z^n, \tag{4.33}$$

where

$$R_2(q)=q\frac{E^2(q^9)E^4(q^{12})}{E(q^3)E(q^6)E(q^{18})}. \tag{4.34}$$

Again, (4.24) holds true for the following (i,j) pairs: $(14,15)$, $(15,16)$, $(16,17)$, $(17,14)$, $(18,19)$, $(19,18)$.

And so, both sides of (4.33) satisfy (4.26). Again it remains to show that (4.33) holds at one nontrivial point, say $z_1=-q^{-3}$. Observing that

$$s_{14}(z_1,q)=s_{15}(z_1,q)=s_{19}(z_1,q)=0, \tag{4.35}$$

we find that (4.33) with $z=z_1$ becomes

$$s_{16}(z_1,q)+s_{17}(z_1,q)+s_{18}(z_1,q)=R_2(q)\sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}}z_1^n, \tag{4.36}$$

Again, making repeated use of (4.15) and replacing q^3 by q, we can rewrite (4.36) as

$$\begin{aligned}&[q^3,q^4,q^6;q^{12}]_\infty([q^5;q^{12}]_\infty-q[q;q^{12}]_\infty)+q[q^2,q^3,q^3,q^4;q^{12}]_\infty\\&\qquad=\frac{E^4(q^4)E^2(q^3)}{E^4(q^{12})E(q^6)E(q^2)}.\end{aligned} \tag{4.37}$$

If we multiply both sides of (4.37) by $\dfrac{[q^2;q^{12}]_\infty}{[q^4;q^{12}]_\infty}$ and take advantage of (4.17) we find that

$$[q,q^5,q^6,q^6;q^{12}]_\infty+q[q^2,q^3;q^{12}]_\infty^2=\frac{E^4(q^4)E^2(q^3)}{E^4(q^{12})E(q^6)E(q^2)}\frac{[q^2;q^{12}]_\infty}{[q^4;q^{12}]_\infty}, \tag{4.38}$$

which is easy to recognize as (4.19). This completes our proof of (4.33) and (4.10).

To prove (4.11), (4.12) we will follow a well trodden path and observe that these identities are just constant terms in z of

$$\sum_{j=20+4\alpha}^{23+4\alpha} s_j(z,q)=q^2\frac{E^2(q^9)E(q^{12})E(q^{36})}{E(q^3)}\cdot\sum_{n=-\infty}^{\infty} q^{9\frac{n(n+1)}{2}}z^n, \tag{4.39}$$

with $\alpha = 0$ and 1, respectively.

To prove that both sides of (4.39) satisfy (4.26) we verify that (4.24) holds for the following (i, j) pairs: $(20+4\alpha, 21+4\alpha)$, $(21+4\alpha, 22+4\alpha)$, $(22+4\alpha, 23+4\alpha)$, $(23+4\alpha, 20+4\alpha)$ with $\alpha = 0, 1$. It remains to verify (4.39) at

$$\tilde{z}_\alpha = -q^{6(1-2\alpha)}, \qquad \alpha = 0, 1.$$

Taking into account that

$$s_{j+4\alpha}(\tilde{z}_\alpha, q) = 0$$

for $j = 20, 21, 22$ and $\alpha = 0, 1$, we find that

$$s_{23+4\alpha}(\tilde{z}_\alpha, q) = (-1)^{\alpha+1} q^{4+6\alpha} E^2(q^9) E(q^{12}) E(q^{36}),$$

which is easy to prove with the aid of (4.15).

This completes our proof of (4.11) and (4.12).

5 Concluding Remarks

Making use of the Littlewood decomposition of $\pi_{t\text{-core}}$ into its s-core and s-quotient,

$$\tilde{\phi}(\pi_{t\text{-core}}) = (\pi_{s\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{s-1}),$$

together with

$$1 + \omega_s + \omega_s^2 + \cdots + \omega_s^{s-1} = 0,$$

it is not hard to see that

$$\text{GBG-rank}(\pi_{t\text{-core}}, s) = \text{GBG-rank}(\pi_{s\text{-core}}, s).$$

In a recent paper [11], Olsson proved a somewhat unexpected result:

Theorem 5.1. *Let s, t be relatively prime positive integers. Then the s-core of a t-core is, again, a t-core.*

A partition is called an (s,t)-core if it is simultaneously an s-core and a t-core.

In [2], Anderson established

Theorem 5.2. *Let s, t be relatively prime positive integers. Then the number of (s,t)-cores is $\binom{s+t}{s}/(s+t)$.*

Remarkably, the three observations above imply our Theorem 1.2. Moreover, our Theorem 1.3 implies

Corollary 5.3. *Let s, t be relatively prime positive integers. Then no two distinct (s,t)-cores share the same value of GBG-rank mod s, when s is prime, or when s is composite and $t < 2p_s$, where p_s is the smallest prime divisor of s.*

On the other hand, when the conditions on s and t in the corollary above are not met, two distinct (s,t)-cores may, in fact, share the same value of GBG-rank mod s. For example, consider two relatively prime integers s and t such that $2 \mid s$, $s > 2$, $t > 1 + \frac{s}{2}$, $t \neq s+1$. In this case partition $[1^{\frac{s}{2}-1}, 2, 1+\frac{s}{2}]$ and empty partition $[\ ]$ are two distinct (s,t)-cores such that

$$\text{GBG-rank}\left(\left[1^{\frac{s}{2}-1}, 2, 1+\frac{s}{2}\right], s\right) = \text{GBG-rank}([\ ], s) = 0.$$

Here we are using the partition notation of Andrews [1, p.1].

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