

BIRANKS FOR PARTITIONS INTO 2 COLORS

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This paper is dedicated to the memory of my friend Richard Lewis (1942 – 2007)

ABSTRACT. In 2003, Hammond and Lewis defined a statistic on partitions into 2 colors which combinatorially explains certain well known partition congruences mod 5. We give two analogs of Hammond and Lewis’s birank statistic. One analog is in terms of Dyson’s rank and the second uses the 5-core crank due to Garvan, Kim and Stanton. We discuss Andrews’s bicrank statistic and how it may be extended. We also generalize the Hammond-Lewis birank to a multirank for multipartitions and the Andrews bicrank to a multicrank for extended multipartitions. Both of these give combinatorial interpretations for multipartition congruences modulo all primes $t > 3$.

1. INTRODUCTION

Hammond and Lewis [17] found some elementary results for 2-colored partitions mod 5. Let $E(q) = \prod_{n=1}^{\infty} (1 - q^n)$, and

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{E(q)^2},$$

which is the generating function for pairs of partitions (π_1, π_2) (or 2-colored partitions). Throughout this paper we refer to such pairs of partitions as *bipartitions*. It is not hard to show that

$$(1.1) \quad p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod{5}.$$

These congruences are well-known [7], [12], [26]. Hammond and Lewis [17] found a crank for these congruences. By crank we mean a statistic that divides the relevant partitions into equinumerous classes. They define

$$(1.2) \quad \text{birank}(\pi_1, \pi_2) = \#(\pi_1) - \#(\pi_2),$$

where $\#(\pi)$ denotes the number of parts in the partition π . They show that the residue of the birank mod 5 divides the bipartitions of n into 5 equal classes provided $n \equiv 2, 3$ or $4 \pmod{5}$. The proof is elementary. It relies on Jacobi’s triple product identity and the method of [13], which uses roots of unity. We have found two other analogs.

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First Analog - The Dyson-birank

Dyson [9] defined the *rank* of a partition as the largest part minus the number of parts. We define

$$(1.3) \quad \text{Dyson-birank}(\pi_1, \pi_2) = \text{rank}(\pi_1) + 2 \text{rank}(\pi_2).$$

This statistic was independently suggested by P. R. Hammond [16, p.68], who asked whether it could explain the congruences in (1.1). In Section 3, we show that the residue of the Dyson-birank mod 5 does indeed divide the bipartitions of n into 5 equal classes provided $n \equiv 2$, or $4 \pmod{5}$. Unfortunately the Dyson-birank does not work if $n \equiv 3 \pmod{5}$. Nonetheless, for the other residue classes this is a surprising and deep result because of the nature of the rank generating function. The proof depends on known results for the rank mod 5 due to Atkin and Swinnerton-Dyer [4].

Second Analog - The 5-core-birank

In [15] new statistics were defined in terms of t -cores which gave new combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11. For example, for a partition π the *5-core-crank* is defined as

$$(1.4) \quad \text{5-core-crank}(\pi) = r_1 + 2r_2 - 2r_3 - r_4,$$

where r_k is the number of cells labelled k in the 5-residue diagram of π . In the t -residue diagram of π a cell in the i -th row and j -column is labelled by the least nonnegative residue of $j - i \pmod{t}$. Then in [15] we proved combinatorially that the residue of the 5-core-crank divides the partitions of $5n + 4$ into 5 equal classes. We define

$$(1.5) \quad \text{5-core-birank}(\pi_1, \pi_2) = \text{5-core-crank}(\pi_1) + 2(\text{5-core-crank}(\pi_2)).$$

In Section 4, we show that the 5-core-birank divides the bipartitions of n into 5 equal classes for $n \equiv 2, 3$ or $4 \pmod{5}$. This is quite a surprising result. The proof relies on the 5-dissection of the 5-core-crank generating function for 5-cores.

The crank of a partition is defined to be the largest part if it contains no ones and otherwise it is the difference between number of parts larger than the number of ones, and the number of ones. The crank gives a combinatorial interpretation of Ramanujan's partition congruences mod 5, 7 and 11 and solves a problem of Dyson [9], [10, p.52]. See [2]. This crank is different to the 5-core crank given in [15]. It is natural to ask whether there is a crank analog of the birank. This question has been answered in part by Andrews [1]. In Section 6, we consider Andrews' result and how it may be extended. In [1], Andrews also considered congruences for more general multipartitions. In Section 7, we give multipartition analogs of the Hammond-Lewis birank which explain these more general congruences. In Section 8, we extend Andrews bicrank to multicranks of what we call extended multipartitions, and give alternative explanations of our multipartition congruences. In Section 9, we close with some further problems.

Notation. For a partition π we denote the sum of parts by $|\pi|$. We will use the standard q -notation:

$$(z; q)_n = (z)_n = \begin{cases} \prod_{j=0}^{n-1} (1 - zq^j), & n > 0 \\ 1, & n = 0, \end{cases}$$

and

$$(z; q)_\infty = (z)_\infty = \lim_{n \rightarrow \infty} (z; q)_n = \prod_{n=1}^{\infty} (1 - zq^{(n-1)}),$$

where $|q| < 1$. We will also use the following notation for Jacobi-type theta-products:

$$J_{a,m}(q) := (q^a; q^m)_\infty (q^{m-a}; q^m)_\infty (q^m; q^m)_\infty.$$

2. THE HAMMOND-LEWIS BIRANK

For completeness we include some details of the Hammond-Lewis birank. For a bipartition $\pi = (\pi_1, \pi_2)$ we denote the sum of parts by

$$(2.1) \quad |\pi| = |\pi_1| + |\pi_2|.$$

We denote the Hammond-Lewis birank by

$$(2.2) \quad \text{HL-birank}(\pi) = \#(\pi_1) - \#(\pi_2),$$

where $\#(\pi)$ denotes the number of parts in the partition π . The HL-birank generating function is

$$(2.3) \quad \sum_{\pi=(\pi_1, \pi_2)} z^{\text{HL-birank}(\pi)} q^{|\pi|} = \frac{1}{(zq; q)_\infty (z^{-1}q; q)_\infty}.$$

We let $N_{\text{HL}}(m, t, n)$ denote the number of bipartitions $\pi = (\pi_1, \pi_2)$ of n with HL-birank congruent to $m \pmod{t}$. Suppose ζ is primitive 5th root of unity. By letting $z = \zeta$ in (2.3) and using Jacobi's triple product identity, Hammond and Lewis found that

$$(2.4) \quad \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k N_{\text{HL}}(k, 5, n) q^n = \frac{1}{(\zeta q; q)_\infty (\zeta^{-1} q; q)_\infty} = \frac{(\zeta^2 q; q)_\infty (\zeta^{-2} q; q)_\infty (q; q)_\infty}{(q^5; q^5)_\infty},$$

$$= (q^{25}; q^{25})_\infty \left(\frac{1}{J_{1,5}(q^5)} + (\zeta + \zeta^{-1}) \frac{q}{J_{2,5}(q^5)} \right).$$

Since the coefficient of q^n on the right side of (2.4) is zero when $n \equiv 2, 3$ or $4 \pmod{5}$, Hammond and Lewis's main result follows.

Theorem 2.1. [17] *The residue of the HL-birank mod 5 divides the bipartitions of n into 5 equal classes provided $n \equiv 2, 3$ or $4 \pmod{5}$.*

We illustrate Theorem 2.1 for the case $n = 3$.

Bipartitions of 3	HL-birank (mod 5)
(3, -)	$1 - 0 \equiv 1$
(2 + 1, -)	$2 - 0 \equiv 2$
(1 + 1 + 1, -)	$3 - 0 \equiv 3$
(2, 1)	$1 - 1 \equiv 0$
(1 + 1, 1)	$2 - 1 \equiv 1$
(1, 2)	$1 - 1 \equiv 0$
(1, 1 + 1)	$1 - 2 \equiv 4$
(-, 3)	$0 - 1 \equiv 4$
(-, 2 + 1)	$0 - 2 \equiv 3$
(-, 1 + 1 + 1)	$0 - 3 \equiv 2$

Thus

$$N_{\text{HL}}(0, 5, 3) = N_{\text{HL}}(1, 5, 3) = N_{\text{HL}}(2, 5, 3) = N_{\text{HL}}(3, 5, 3) = N_{\text{HL}}(4, 5, 3) = 2,$$

and we see that the residue of the HL-birank mod 5 divides the 10 bipartitions of 3 into 5 equal classes.

3. THE DYSON-BIRANK

Dyson [9], [10, p.52] defined the rank of a partition as the largest part minus the number of parts. We define the Dyson-analog of the birank for bipartitions $\pi = (\pi_1, \pi_2)$ by

$$(3.1) \quad \text{Dyson-birank}(\pi) = \text{rank}(\pi_1) + 2 \text{rank}(\pi_2).$$

In this section we prove

Theorem 3.1. *The residue of the Dyson-birank mod 5 divides the bipartitions of n into 5 equal classes provided $n \equiv 2, \text{ or } 4 \pmod{5}$.*

We let $N_{\text{D}}(m, t, n)$ denote the number of bipartitions $\pi = (\pi_1, \pi_2)$ with Dyson-birank congruent to $m \pmod{t}$. We illustrate Theorem 3.1 for the case $n = 2$.

Bipartitions of 2	Dyson-birank (mod 5)
(2, -)	$1 + 0 \equiv 1$
(1 + 1, -)	$-1 + 0 \equiv 4$
(1, 1)	$0 + 0 \equiv 0$
(-, 2)	$0 + 2 \equiv 2$
(-, 1 + 1)	$0 - 2 \equiv 3$

Thus

$$N_{\text{D}}(0, 5, 2) = N_{\text{D}}(1, 5, 2) = N_{\text{D}}(2, 5, 2) = N_{\text{D}}(3, 5, 2) = N_{\text{D}}(4, 5, 2) = 1,$$

and we see that the residue of the Dyson-birank mod 5 divides the 5 bipartitions of 2 into 5 equal classes. We note that Theorem 3.1 does not hold for $n \equiv 3 \pmod{5}$. The first counterexample occurs when $n = 13$. The Dyson-birank mod 5 fails to divide the 1770 bipartitions of 13 into 5 equal classes. We have

$$N_{\text{D}}(0, 5, 13) = 358, \quad \text{but} \\ N_{\text{D}}(1, 5, 13) = N_{\text{D}}(2, 5, 13) = N_{\text{D}}(3, 5, 13) = N_{\text{D}}(4, 5, 13) = 353.$$

To prove Theorem 3.1 we need the 5-dissection of the rank generating function when $z = \zeta$. The Dyson-rank generating function is

$$(3.2) \quad f(z, q) = \sum_{\pi} z^{\text{rank}(\pi)} q^{|\pi|} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq; q)_n (z^{-1}q; q)_n}.$$

We let $N(m, t, n)$ denote the number of ordinary partitions of n with rank congruent to $m \pmod{t}$. Then

$$(3.3) \quad f(\zeta, q) = \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k N(k, 5, n) q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\zeta q; q)_n (\zeta^{-1} q; q)_n} \\ = (A(q^5) - (3 + \zeta^2 + \zeta^3) \phi(q^5)) + q B(q^5) + q^2 (\zeta + \zeta^4) C(q^5) \\ + q^3 ((1 + \zeta^2 + \zeta^3) D(q^5) + (1 + 2\zeta^2 + 2\zeta^3) \psi(q^5)),$$

where

$$(3.4) \quad A(q) = \frac{E^2(q^5) J_{2,5}(q)}{J_{1,5}^2(q)},$$

$$(3.5) \quad B(q) = \frac{E^2(q^5)}{J_{1,5}(q)},$$

$$(3.6) \quad C(q) = \frac{E^2(q^5)}{J_{2,5}(q)},$$

$$(3.7) \quad D(q) = \frac{E^2(q^5) J_{1,5}(q)}{J_{2,5}^2(q)},$$

$$(3.8) \quad \begin{aligned} \phi(q) &= -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q; q^5)_{n+1} (q^4; q^5)_n} \\ &= \frac{q}{E(q^5)} \sum_{m=-\infty}^{\infty} (-1)^m \frac{q^{\frac{15}{2}m(m+1)}}{1 - q^{5m+1}}, \end{aligned}$$

and

$$(3.9) \quad \begin{aligned} \psi(q) &= \frac{1}{q} \left\{ -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2; q^5)_{n+1} (q^3; q^5)_n} \right\} \\ &= \frac{q}{E(q^5)} \sum_{m=-\infty}^{\infty} (-1)^m \frac{q^{\frac{15}{2}m(m+1)}}{1 - q^{5m+2}}. \end{aligned}$$

Equation (3.3) has an unusual history. It is originally due to Ramanujan since it appears in the Lost Notebook. It predates, but is closely related to Dyson's conjectures on the rank [9], which were proved by Atkin and Swinnerton-Dyer [4]. As pointed out in [13] and [14], equation (3.3) is actually equivalent to one of Atkin and Swinnerton-Dyer's main results. Dyson, Atkin and Swinnerton-Dyer were unaware of Ramanujan's result.

The Dyson-birank generating function is

$$(3.10) \quad \sum_{\pi=(\pi_1, \pi_2)} z^{\text{Dyson-birank}(\pi)} q^{|\pi|} = f(z, q) f(z^2, q),$$

where $f(z, q)$ is the generating function for the Dyson rank of ordinary partitions given in (3.2). Thus we have

$$(3.11) \quad \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k N_D(k, 5, n) q^n = f(\zeta, q) f(\zeta^2, q).$$

Using only (3.3) and the fact that

$$(3.12) \quad B^2(q) = A(q) C(q), \quad C^2(q) = B(q) D(q),$$

we find that the coefficient of q^n in the q -expansion of $f(\zeta, q) f(\zeta^2, q)$ is zero if $n \equiv 2, \text{ or } 4 \pmod{5}$. Theorem 3.1 then follows from (3.11). Although Theorem 3.1 does not hold when $n \equiv 3 \pmod{5}$, there is some simplification in the product $f(\zeta, q) f(\zeta^2, q)$. We find that

$$(3.13) \quad \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k N_D(k, 5, 5n+3) q^n = 5 \phi(q) \psi(q),$$

using the fact that

$$(3.14) \quad A(q) D(q) = B(q) C(q).$$

Equation (3.13) implies that

$$(3.15) \quad N_D(0, 5, 5n + 3) \equiv N_D(k, 5, 5n + 3) \pmod{5},$$

for $1 \leq k \leq 4$, and

$$p_{-2}(5n + 3) \equiv 0 \pmod{5}.$$

This last observation is due to Byungchan Kim.

4. THE 5-CORE-BIRANK

For ordinary partitions π the 5-core-crank is defined by

$$(4.1) \quad 5\text{-core-crank}(\pi) = r_1 + 2r_2 - 2r_3 - r_4,$$

where r_j is the number of cells labelled j in the 5-residue diagram of π . See [15, Prop.1,p.7]. We define the 5-core-crank analog for bipartitions $\pi = (\pi_1, \pi_2)$ by

$$(4.2) \quad 5\text{-core-birank}(\pi) = 5\text{-core-crank}(\pi_1) + 2(5\text{-core-crank}(\pi_2)).$$

In this section we prove

Theorem 4.1. *The residue of the 5-core-birank mod 5 divides the bipartitions of n into 5 equal classes provided $n \equiv 2, 3$ or $4 \pmod{5}$.*

We let $N_{5C}(m, t, n)$ denote the number of bipartitions $\pi = (\pi_1, \pi_2)$ with 5-core-birank congruent to $m \pmod{t}$. We illustrate Theorem 4.1 for the case $n = 3$.

Bipartitions of 3	5-core-birank (mod 5)
(3, -)	$3 + 0 \equiv 3$
(2 + 1, -)	$0 + 0 \equiv 0$
(1 + 1 + 1, -)	$-3 + 0 \equiv 2$
(2, 1)	$1 + 0 \equiv 1$
(1 + 1, 1)	$-1 + 0 \equiv 4$
(1, 2)	$0 + 2 \equiv 2$
(1, 1 + 1)	$0 - 2 \equiv 3$
(-, 3)	$0 + 6 \equiv 1$
(-, 2 + 1)	$0 + 0 \equiv 0$
(-, 1 + 1 + 1)	$0 - 6 \equiv 4$

Thus

$$N_{5C}(0, 5, 3) = N_{5C}(1, 5, 3) = N_{5C}(2, 5, 3) = N_{5C}(3, 5, 3) = N_{5C}(4, 5, 3) = 2,$$

and we see that the residue of the 5-core-birank mod 5 divides the 10 bipartitions of 3 into 5 equal classes. We note that although the Dyson-birank does not in general divide the bipartitions of $5n + 3$ into 5 equal classes the 5-core-birank does.

To prove Theorem 4.1 we need the 5-dissection of the 5-core-crank generating function when $z = \zeta$. The 5-core-crank generating function is

$$(4.3) \quad \Phi(z, q) = \sum_{\pi} z^{5\text{-core-crank}(\pi)} q^{|\pi|} = \frac{1}{E^5(q^5)} T(z, q),$$

where

$$(4.4) \quad T(z, q) := \sum_{\pi \text{ a 5-core}} z^{5\text{-core-crank}(\pi)} q^{|\pi|} = \sum_{\substack{\vec{n} \in \mathbb{Z}^5 \\ \vec{n} \cdot \vec{1} = 0}} z^{n_1 + 3n_2 + n_3} q^{\frac{5}{2} \|\vec{n}\|^2 + \vec{b} \cdot \vec{n}},$$

where $\vec{1} = (1, 1, 1, 1, 1)$ and $\vec{b} = (0, 1, 2, 3, 4)$. Equation (4.3) can be proved combinatorially and in a straightforward manner using Bijections 1 and 2 from [15, pp.2-3] and [15, (4.2), p.6].

We need the 5-dissection of $T(\zeta, q)$:

$$(4.5) \quad T(\zeta, q) = W(q^5)(1 + qR(q^5) + q^2(\zeta^2 + \zeta^3)R(q^5)^2 - q^3(\zeta^2 + \zeta^3)R(q^5)^3),$$

where

$$(4.6) \quad W(q) := J_{2,5}(q)^3(J_{10,25}(q) - q(1 + \zeta^2 + \zeta^3)J_{5,25}(q)),$$

and

$$(4.7) \quad R(q) := \frac{J_{1,5}(q)}{J_{2,5}(q)}.$$

We will prove (4.5) in the next section. Theorem 4.1 follows easily from (4.5).

The 5-core-birank generating function is

$$(4.8) \quad \sum_{\pi=(\pi_1, \pi_2)} z^{5\text{-core-birank}(\pi)} q^{|\pi|} = \frac{1}{E^{10}(q^5)} T(z, q) T(z^2, q),$$

where $T(z, q)$ is the generating function for the 5-core-crank of partitions that are 5-cores given in (4.4). Thus we have

$$(4.9) \quad \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k N_{5C}(k, 5, n) q^n = \frac{1}{E^{10}(q^5)} T(\zeta, q) T(\zeta^2, q).$$

From (4.5) we find that

$$(4.10) \quad T(\zeta, q) T(\zeta^2, q) = Z(q^5)(1 + 2q^5 R^5(q^5) + qR(q^5)(2 - q^5 R^5(q^5))),$$

where

$$(4.11) \quad Z(q) = J_{2,5}(q)^6(J_{10,25}(q)^2 - qJ_{10,25}(q)J_{5,25}(q) - q^2J_{5,25}(q)^2).$$

Since the coefficient of q^n in the q -expansion of $T(\zeta, q) T(\zeta^2, q)$ is zero when $n \equiv 2, 3$ or $4 \pmod{5}$, Theorem 4.1 then follows from (4.9).

5. A THETA-FUNCTION IDENTITY

In this section we will prove the following theta-function identity.

$$(5.1) \quad U(z, q) = F_0(q) S_0(z, q) + F_1(q) S_1(z, q) + F_2(q) S_2(z, q) + F_3(q) S_3(z, q) + F_4(q) S_4(z, q),$$

where

$$(5.2) \quad F_0(q) = W(q^{10})(1 + q^2 R(q^{10}) + (\zeta^2 + \zeta^3) q^4 R(q^{10})^2 - (\zeta^2 + \zeta^3) q^6 R(q^{10})^3),$$

$$(5.3) \quad F_1(q) = W(q^{10})(\zeta^4 + \zeta q^2 R(q^{10}) + (1 + \zeta) q^4 R(q^{10})^2 - (\zeta^2 + \zeta^3) q^6 R(q^{10})^3),$$

$$(5.4) \quad F_2(q) = W(q^{10})(1 + \zeta^4 q^2 R(q^{10}) + (1 + \zeta) q^4 R(q^{10})^2 - (1 + \zeta^4) q^6 R(q^{10})^3),$$

(5.5)

$$F_3(q) = W(q^{10}) (1 + \zeta q^2 R(q^{10}) + (1 + \zeta^4) q^4 R(q^{10})^2 - (1 + \zeta) q^6 R(q^{10})^3),$$

(5.6)

$$F_4(q) = W(q^{10}) (\zeta + \zeta^4 q^2 R(q^{10}) + (1 + \zeta^4) q^4 R(q^{10})^2 - (\zeta^2 + \zeta^3) q^6 R(q^{10})^3),$$

(5.7)

$$S_0(z, q) = \sum_{n=-\infty}^{\infty} z^{5n} q^{25n^2+20n},$$

(5.8)

$$S_1(z, q) = \sum_{n=-\infty}^{\infty} z^{5n+1} q^{25n^2+30n+5},$$

(5.9)

$$S_2(z, q) = \sum_{n=-\infty}^{\infty} z^{5n+2} q^{25n^2+40n+12},$$

(5.10)

$$S_3(z, q) = \sum_{n=-\infty}^{\infty} z^{5n+3} q^{25n^2+50n+21},$$

(5.11)

$$S_4(z, q) = \sum_{n=-\infty}^{\infty} z^{5n+4} q^{25n^2+60n+32},$$

and

(5.12)

$$U(z, q) = \sum_{\vec{n} \in \mathbb{Z}^5} z^{\vec{n} \cdot \vec{1}} \zeta^{n_1+3n_2+n_3} q^{5\|\vec{n}\|^2+2\vec{b} \cdot \vec{n}},$$

with $W(q)$ and $R(q)$ defined in (4.6) and (4.7) respectively, and the vectors $\vec{n} = (n_0, n_1, n_2, n_3, n_4)$, $\vec{1} = (1, 1, 1, 1, 1)$ and $\vec{b} = (0, 1, 2, 3, 4)$ as before. We note that (4.5) follows from (5.1) by taking the coefficient of z^0 and replacing q by $q^{1/2}$. Equation (4.5) was the crucial identity needed in the proof of Theorem 4.1.

We prove the identity (5.1) using standard techniques. We show that both sides satisfy the same functional equation and both sides agree for enough values of the parameter z . Most of these evaluations can be proved by elementary means using Jacobi's triple product. For one evaluation we will need the theory of modular functions.

We define the following Jacobi theta function

(5.13)

$$\Theta(z, q) = \sum_{n=-\infty}^{\infty} z^n q^{n^2},$$

for $z \neq 0$ and $|q| < 1$. We will need Jacobi's triple product identity

(5.14)

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_{\infty} (-z^{-1}q; q^2)_{\infty} (q^2; q^2)_{\infty},$$

and the well-known functional equation

(5.15)

$$\Theta(zq^2, q) = z^{-1}q^{-1}\Theta(z, q),$$

for $z \neq 0$ and $0 < |q| < 1$. From the definition (5.12) we have

(5.16)

$$U(z, q) = \Theta(z\zeta^4, q^5) \Theta(zq^2\zeta, q^5) \Theta(zq^4, q^5) \Theta(z\zeta q^6, q^5) \Theta(z\zeta^4 q^8, q^5).$$

From (5.15) and (5.16) we have

(5.17)

$$U(zq^{10}, q) = z^{-5}q^{-45}U(z, q),$$

and

$$(5.18) \quad U(z, q) = 0 \quad \text{for } z = -q^5 \zeta, \quad -q^3 \zeta^4, -q, -\zeta^4 q^9, -\zeta q^7.$$

Let $V(z, q)$ denote the function of the right side of (5.1). Each $S_j(z, q)$ can be written in terms of the theta function $\Theta(z, q)$ and we find that $S_j(zq^{10}, q) = z^{-5}q^{-45}S_j(z, q)$ for each j so that

$$(5.19) \quad V(zq^{10}, q) = z^{-5}q^{-45}V(z, q).$$

Hence the left and right sides of (5.1) satisfy the same functional equation (i.e. (5.17), (5.19)). In view of [4, Lemma 2] or [19, Lemma 1], it suffices to show that (5.1) holds for 6 distinct values of z with $|q|^{10} < |z| \leq 1$. We claim that

$$(5.20) \quad V(z, q) = 0 \quad \text{for } z = -q^5 \zeta, \quad -q^3 \zeta^4, -q, -\zeta^4 q^9, -\zeta q^7.$$

Using (5.14) we can easily evaluate each $S_j(z, q)$ for these values of z .

$$(5.21) \quad S_0(-\zeta q^5, q) = -q^{-20} J_{2,5}(q^{10}),$$

$$(5.22) \quad S_1(-\zeta q^5, q) = \zeta q^{-20} J_{2,5}(q^{10}),$$

$$(5.23) \quad S_2(-\zeta q^5, q) = -\zeta^2 q^{-18} J_{1,5}(q^{10}),$$

$$(5.24) \quad S_3(-\zeta q^5, q) = 0,$$

$$(5.25) \quad S_4(-\zeta q^5, q) = \zeta^4 q^{-18} J_{1,5}(q^{10}),$$

$$(5.26) \quad S_0(-q^3 \zeta^4, q) = -q^{-10} J_{1,5}(q^{10}),$$

$$(5.27) \quad S_1(-q^3 \zeta^4, q) = \zeta^4 q^{-12} J_{2,5}(q^{10}),$$

$$(5.28) \quad S_2(-q^3 \zeta^4, q) = -\zeta^3 q^{-12} J_{2,5}(q^{10}),$$

$$(5.29) \quad S_3(-q^3 \zeta^4, q) = \zeta^2 q^{-10} J_{1,5}(q^{10}),$$

$$(5.30) \quad S_4(-q^3 \zeta^4, q) = 0,$$

$$(5.31) \quad S_0(-q, q) = 0,$$

$$(5.32) \quad S_1(-q, q) = q^{-4} J_{1,5}(q^{10}),$$

$$(5.33) \quad S_2(-q, q) = -q^{-6} J_{2,5}(q^{10}),$$

$$(5.34) \quad S_3(-q, q) = q^{-6} J_{2,5}(q^{10}),$$

$$(5.35) \quad S_4(-q, q) = -q^{-4} J_{1,5}(q^{10}),$$

$$(5.36) \quad S_0(-\zeta^4 q^9, q) = -q^{-40} J_{1,5}(q^{10}),$$

$$(5.37) \quad S_1(-\zeta^4 q^9, q) = 0,$$

$$(5.38) \quad S_2(-\zeta^4 q^9, q) = \zeta^3 q^{-40} J_{1,5}(q^{10}),$$

$$(5.39) \quad S_3(-\zeta^4 q^9, q) = -\zeta^2 q^{-42} J_{2,5}(q^{10}),$$

$$(5.40) \quad S_4(-\zeta^4 q^9, q) = \zeta q^{-42} J_{2,5}(q^{10}),$$

$$(5.41) \quad S_0(-\zeta q^7, q) = -q^{-30} J_{2,5}(q^{10}),$$

$$(5.42) \quad S_1(-\zeta q^7, q) = \zeta q^{-28} J_{1,5}(q^{10}),$$

$$(5.43) \quad S_2(-\zeta q^7, q) = 0,$$

$$(5.44) \quad S_3(-\zeta q^7, q) = -\zeta^3 q^{-28} J_{1,5}(q^{10}),$$

$$(5.45) \quad S_4(-\zeta q^7, q) = \zeta^4 q^{-30} J_{2,5}(q^{10}).$$

The verification of (5.20) is just a routine calculation.

Thus both sides of (5.1) agree for 5 distinct values of z in the region $|q|^{10} < |z| \leq 1$. We show that both sides agree for $z = -1$, and then our identity (5.1) will follow. To achieve this we use the theory of modular functions. Since this is a standard technique we just sketch some of the details.

First, we calculate the 5-dissection of each theta function on the right side of (5.16) when $z = -1$. By (5.14) we find that

$$(5.46) \quad \Theta(-\zeta^4, q^5) = J_{1,2}(q^{125}) + (1 + \zeta^2 + \zeta^3)q^5 J_{3,10}(q^{25}) + (\zeta^2 + \zeta^3)q^{20} J_{1,10}(q^{25}),$$

$$(5.47) \quad \Theta(-\zeta q^2, q^5) = J_{23,50}(q^5) + \zeta^3 q^{16} J_{7,50}(q^5) - \zeta q^7 J_{13,50}(q^5) - \zeta^4 q^3 J_{17,50}(q^5) \\ + q^{24} \zeta^2 J_{3,50}(q^5),$$

$$(5.48) \quad \Theta(-q^4, q^5) = J_{21,50}(q^5) - q J_{19,50}(q^5) + q^{12} J_{9,50}(q^5) + q^{28} J_{1,50}(q^5) \\ - q^9 J_{11,50}(q^5),$$

$$(5.49) \quad q \Theta(-\zeta q^6, q^5) = -\zeta^4 J_{21,50}(q^5) + q J_{19,50}(q^5) - \zeta q^{12} J_{9,50}(q^5) - \zeta^2 q^{28} J_{1,50}(q^5) \\ + \zeta^3 q^9 J_{11,50}(q^5),$$

$$(5.50) \quad q^3 \Theta(-\zeta^4 q^8, q^5) = -\zeta J_{23,50}(q^5) - \zeta^4 q^{16} J_{7,50}(q^5) + \zeta^2 q^7 J_{13,50}(q^5) + q^3 J_{17,50}(q^5) \\ - \zeta^3 q^{24} J_{3,50}(q^5).$$

Next, we evaluate each $S_j(-1, q)$ using (5.14)

$$(5.51) \quad S_0(-1, q) = S_1(-1, q) = J_{1,10}(q^5),$$

$$(5.52) \quad S_2(-1, q) = S_4(-1, q) = -q^{-3} J_{3,10}(q^5),$$

$$(5.53) \quad S_3(-1, q) = q^{-4} J_{1,2}(q^{25}).$$

For $0 \leq r \leq 4$, we define the operator $\mathcal{U}_{r,5}$ by

$$(5.54) \quad \mathcal{U}_{r,5} \left(\sum_n a(n) q^n \right) = \sum_n a(5n+r) q^n.$$

To show that (5.1) holds for $z = -1$ we need to prove 5 identities

$$(5.55) \quad \mathcal{U}_{r,5} (q^4 U(-1, q)) = \mathcal{U}_{r,5} (q^4 V(-1, q)),$$

for $0 \leq r \leq 4$. It turns out that each of these identities is equivalent to a modular function identity for the group $\Gamma_1(50)$. We provide some detail for the case $r = 0$. Using (5.51)–(5.53) we find that

$$(5.56) \quad \mathcal{U}_{0,5} (q^4 V(-1, q)) = (J_{2,5}(q^{10}) - q^2(1 + \zeta^2 + \zeta^3)J_{1,5}(q^{10})) \\ \times (J_{1,2}(q^5)J_{4,10}^3(q) + q(-1 + \zeta^2 + \zeta^3)J_{2,10}^2(q)J_{3,10}(q)J_{4,10}(q) \\ - 2q^2(\zeta^2 + \zeta^3)J_{2,10}^3 J_{1,10}(q)).$$

We can utilize (5.46)–(5.50) to write the left side of the $r = 0$ case of (5.55) as a sum of 135 explicit theta products

$$(5.57) \quad \begin{aligned} \mathcal{U}_{0,5}(q^4 U(-1, q)) &= q^{10} J_{3,50}^2(q) J_{19,50}^2(q) J_{25,50}(q) + \cdots \\ &\quad + 2q^{10} (\zeta^2 + \zeta^3) J_{1,50}(q) J_{9,50}(q) J_{13,50}(q) J_{17,50}(q) J_{25,50}(q). \end{aligned}$$

We have to prove that the right side of (5.56) equals the right side of (5.57). After dividing both sides by $J_{2,5}(q^{10}) J_{1,2}(q^5) J_{4,10}^3(q)$ we find that this is equivalent to showing that a certain linear combination of 140 generalized eta-quotients simplifies to the constant 1

$$(5.58) \quad \begin{aligned} (1 + \zeta^2 + \zeta^3) \eta_{50,10} \eta_{50,20}^{-1} + \cdots \\ + \eta_{50,4}^{-3} \eta_{50,5}^{-2} \eta_{50,6}^{-3} \eta_{50,10}^{-4} \eta_{50,14}^{-3} \eta_{50,15}^{-2} \eta_{50,16}^{-3} \eta_{50,20}^{-5} \eta_{50,24}^{-3} \eta_{50,23}^2 \eta_{50,21}^2 = 1. \end{aligned}$$

Here

$$(5.59) \quad \begin{aligned} \eta_{m,n} &= \eta_{m,n}(\tau) = \exp(\pi i P_2(m/n) n \tau) \prod_{k \equiv \pm m \pmod{n}} (1 - \exp(2\pi i k \tau)) \\ &= q^{n P_2(m/n)/2} J_{m,n}(q), \end{aligned}$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$, and $q = \exp(2\pi i \tau)$. Using [8, Theorem 2.9, p.7], [25, Theorem 3, p.126] we check that each generalised eta-quotient in (5.58) is indeed a modular function on $\Gamma_1(50)$. As usual we need the valence formula

$$(5.60) \quad \sum_{z \in \mathcal{F}} \text{ORD}(f; z, \Gamma) = 0,$$

provided f is a nontrivial modular function on Γ , and \mathcal{F} is a fundamental set for Γ . Using MAGMA, the following is a complete set of inequivalent cusps for $\Gamma_1(50)$

$$(5.61) \quad \begin{aligned} \mathcal{C} = \{ \infty, \frac{0}{1}, \frac{1}{10}, \frac{1}{9}, \frac{2}{17}, \frac{3}{25}, \frac{1}{8}, \frac{2}{15}, \frac{3}{22}, \frac{4}{29}, \frac{5}{36}, \frac{6}{43}, \frac{7}{50}, \frac{3}{20}, \frac{2}{13}, \frac{3}{19}, \frac{4}{25}, \frac{1}{6}, \frac{6}{35}, \frac{9}{52}, \frac{13}{75}, \\ \frac{4}{23}, \frac{7}{39}, \frac{9}{50}, \frac{7}{38}, \frac{3}{16}, \frac{5}{26}, \frac{1}{5}, \frac{27}{125}, \frac{7}{32}, \frac{11}{50}, \frac{12}{53}, \frac{17}{75}, \frac{6}{25}, \frac{1}{4}, \frac{13}{50}, \frac{53}{200}, \frac{67}{250}, \frac{27}{100}, \frac{11}{40}, \frac{18}{65}, \\ \frac{7}{25}, \frac{3}{10}, \frac{7}{20}, \frac{9}{25}, \frac{11}{30}, \frac{19}{50}, \frac{49}{125}, \frac{2}{5}, \frac{21}{50}, \frac{11}{25}, \frac{11}{20}, \frac{26}{45}, \frac{3}{5}, \frac{59}{85}, \frac{7}{10} \}. \end{aligned}$$

with corresponding widths

$$(5.62) \quad \begin{aligned} \{1, 50, 5, 50, 50, 2, 25, 10, 25, 50, 25, 50, 1, 5, 50, 50, 2, 25, 10, 25, 2, 50, 50, 1, \\ 25, 25, 25, 10, 2, 25, 1, 50, 2, 2, 25, 1, 1, 1, 1, 5, 10, 2, 5, 5, 2, 5, 1, 2, 10, 1, \\ 2, 5, 10, 10, 10, 5\}. \end{aligned}$$

Using known results for the invariant order of generalized eta-quotients at cusps [8, (2.3), p.7], [25, pp.127-128] we have calculated the order at each cusp of every function in (5.58). As a check we verified that the total order of each function is zero. With \mathcal{J} being the set of generalized eta-quotients occurring in (5.58) we calculated

$$(5.63) \quad \sum_{c \in \mathcal{C} \setminus \{\infty\}} \min(\text{ORD}(f; c; \Gamma_1(50)), 0) = -145.$$

Hence, by the valence formula (5.60) it suffices to verify (5.58) (or equivalently (5.55) with $r = 0$) up to q^{145} , since generalized eta-quotients have no poles or zeros in the upper-half plane. We have actually verified the result up to q^{200} . All calculations,

except for (5.61), and (5.62), were done using MAPLE. The calculations needed to verify (5.55) for $r = 1, 2, 3, 4$ are similar and have been carried out. This completes our proof of (5.1).

6. THE ANDREWS BICRANK AND EXTENSIONS

For a partition π , let $\ell(\pi)$ denote the largest part of π , $\varpi(\pi)$ denote the number of ones in π , and $\mu(\pi)$ denote the number of parts of π larger than $\varpi(\pi)$. The crank of π is given by

$$(6.1) \quad \text{crank}(\pi) = \begin{cases} \ell(\pi), & \text{if } \varpi(\pi) = 0, \\ \mu(\pi) - \varpi(\pi), & \text{if } \varpi(\pi) > 0. \end{cases}$$

The crank gives a combinatorial interpretation of Ramanujan's partition congruences mod 5, 7 and 11 and solves a problem of Dyson [9], [10, p.52]. See [2].

In [1], Andrews gave a combinatorial interpretation of the congruence

$$(6.2) \quad p_{-2}(5n+3) \equiv 0 \pmod{5},$$

in terms of the crank. This result is a crank analog of the Dyson-birank but is more complicated since it involves positive and negative weights. This complication is there because of the nature of the generating function for the crank. Let $M(m, n)$ denote the number of partitions of n with crank m . Then

$$(6.3) \quad \sum_{n \geq 0} \sum_m M(m, n) z^m q^n = (1-z)q + \frac{(q; q)_\infty}{(zq; q)_\infty (z^{-1}q; q)_\infty}.$$

Define $M'(m, n)$ by

$$(6.4) \quad \sum_{n \geq 0} \sum_m M'(m, n) z^m q^n = \frac{(q; q)_\infty}{(zq; q)_\infty (z^{-1}q; q)_\infty} = 1 + (z-1+z^{-1})q + (z^2+z^{-2})q^2 + \dots$$

We need to interpret $M'(m, n)$ combinatorially. To do this we need to extend the definition of partition. To the set of partitions we need to add two additional partitions of 1 which we denote by 1_a and 1_b . We call this new set \mathcal{E} , the set of extended partitions.

$$(6.5) \quad \mathcal{E} = \{(-), 1_a, 1_b, 1, 2, 1+1, 3, 2+1, 1+1+1, \dots\}.$$

We have $|1_a| = |1_b| = 1$. Here as usual $(-)$ is the empty partition of 0. For these extended partitions define a weight function $w(\pi)$ defined by

$$(6.6) \quad w(\pi) = \begin{cases} -1, & \text{if } \pi = 1_b. \\ |\pi|, & \text{otherwise.} \end{cases}$$

Thus for the three extended partitions of 1 we have $w(1) = w(1_a) = 1$, and $w(1_b) = -1$, and the total weight is still $p(1) = 1$. Therefore

$$(6.7) \quad \sum_{\substack{\pi \in \mathcal{E} \\ |\pi|=n}} w(\pi) = p(n).$$

We also extend the definition of crank by $\text{crank}(1_a) = 1$, and $\text{crank}(1_b) = 0$. Recall that for ordinary partition of 1 we have $\text{crank}(1) = -1$. We now have our desired

combinatorial interpretation of $M'(m, n)$.

(6.8)

$$F(z, q) = \sum_{\pi \in \mathcal{E}} w(\pi) z^{\text{crank}(\pi)} q^{|\pi|} = \sum_{n \geq 0} \sum_m M'(m, n) z^m q^n = \frac{(q; q)_\infty}{(zq; q)_\infty (z^{-1}q; q)_\infty}.$$

In other words,

$$(6.9) \quad M'(m, n) = \sum_{\substack{\pi \in \mathcal{E} \\ |\pi|=n, \text{crank}(\pi)=m}} w(\pi).$$

We note that the function $F(z, q)$ (at least as an infinite product) occurred in Ramanujan's Lost Notebook.

We define the set of extended bipartitions by $\mathcal{E} \times \mathcal{E}$, i.e. an extended bipartition is simply a pair of extended partitions. For an extended bipartition $\pi = (\pi_1, \pi_2)$ we define a sum of parts function and a weight function in the natural way

$$(6.10) \quad |\pi| = |\pi_1| + |\pi_2|, \quad \text{and} \quad w(\pi) = w(\pi_1) w(\pi_2).$$

We denote Andrews's bicrank function by bicrank_1 . We give a variant which we call bicrank_2 . For an extended bipartition $\pi = (\pi_1, \pi_2)$ we define

$$(6.11) \quad \text{bicrank}_1(\pi) = \text{crank}(\pi_1) + \text{crank}(\pi_2),$$

and

$$(6.12) \quad \text{bicrank}_2(\pi) = \text{crank}(\pi_1) + 2 \text{crank}(\pi_2).$$

Amazingly together these two bicrank functions give a new interpretation for all three congruences in (1.1). For $j = 1, 2$ we define $M_j(m, t, n)$ by

$$(6.13) \quad M_j(m, t, n) = \sum_{\substack{\pi \in \mathcal{E} \times \mathcal{E} \\ |\pi|=n, \text{bicrank}_j(\pi) \equiv m \pmod{t}}} w(\pi).$$

In other words, $M_j(m, t, n)$ is the number of extended bipartitions of n with bicrank_j congruent to m mod t counted by the weight w .

In this section we prove

Theorem 6.1. *We have the following two results:*

- (i) *The residue of the $\text{bicrank}_1(\pi)$ mod 5 divides the extended bipartitions of n into 5 classes of equal weight provided $n \equiv 3 \pmod{5}$.*
- (ii) *The residue of the $\text{bicrank}_2(\pi)$ mod 5 divides the extended bipartitions of n into 5 classes of equal weight provided $n \equiv 2$ or $4 \pmod{5}$.*

We illustrate the first case of Theorem 6.1 (i). There are 18 extended bipartitions of 3 giving a total weight of $p_{-2}(3) = 10$.

Extended bipartitions of 3	bicrank ₁ (mod 5)	weight = w
(2 + 1, -)	$0 + 0 \equiv 0$	1
(-, 2 + 1)	$0 + 0 \equiv 0$	1
(2, 1)	$2 - 1 \equiv 1$	1
(1, 2)	$-1 + 2 \equiv 1$	1
(1 + 1 + 1, -)	$-3 + 0 \equiv 2$	1
(2, 1 _b)	$2 + 0 \equiv 2$	-1
(1 + 1, 1)	$-2 - 1 \equiv 2$	1
(1 _b , 2)	$0 + 2 \equiv 2$	-1
(1, 1 + 1)	$-1 - 2 \equiv 2$	1
(-, 1 + 1 + 1)	$0 - 3 \equiv 2$	1
(3, -)	$3 + 0 \equiv 3$	1
(2, 1 _a)	$2 + 1 \equiv 3$	1
(1 + 1, 1 _b)	$-2 + 0 \equiv 3$	-1
(1 _a , 2)	$1 + 2 \equiv 3$	1
(1 _b , 1 + 1)	$0 - 2 \equiv 3$	-1
(-, 3)	$0 + 3 \equiv 3$	1
(1 + 1, 1 _a)	$-2 + 1 \equiv 4$	1
(1 _a , 1 + 1)	$1 - 2 \equiv 4$	1

Thus

$$M_1(0, 5, 3) = M_1(1, 5, 3) = M_1(2, 5, 3) = M_1(3, 5, 3) = M_1(4, 5, 3) = 2,$$

and we see that the residue of the bicrank₁ mod 5 divides the 18 bipartitions of 3 into 5 classes of equal total weight 2.

We illustrate the first case of Theorem 6.1 (ii). There are 13 extended bipartitions of 2 giving a total weight of $p_{-2}(2) = 5$.

Extended bipartitions of 2	bicrank ₂ (mod 5)	weight = w
(2, -)	$2 + 0 \equiv 2$	1
(1 + 1, -)	$-2 + 0 \equiv 3$	1
(1, 1)	$-1 - 2 \equiv 2$	1
(1, 1 _a)	$-1 + 2 \equiv 1$	1
(1, 1 _b)	$-1 + 0 \equiv 4$	-1
(1 _a , 1)	$1 - 2 \equiv 4$	1
(1 _a , 1 _a)	$1 + 2 \equiv 3$	1
(1 _a , 1 _b)	$1 + 0 \equiv 1$	-1
(1 _b , 1)	$0 - 2 \equiv 3$	-1
(1 _b , 1 _a)	$0 + 2 \equiv 2$	-1
(1 _b , 1 _b)	$0 + 0 \equiv 0$	1
(-, 2)	$0 + 4 \equiv 4$	1
(-, 1 + 1)	$0 - 4 \equiv 1$	1

Thus

$$M_2(0, 5, 2) = M_2(1, 5, 2) = M_2(2, 5, 2) = M_2(3, 5, 2) = M_2(4, 5, 2) = 1,$$

and we see that the residue of the bicrank₂ mod 5 divides the 13 bipartitions of 2 into 5 classes of equal total weight 1.

Theorem 6.1 (i) is due to Andrews [1]. Theorem 6.1 (ii) is a natural extension, and its proof is analogous. For completeness we include a sketch of the proof. We define

$$(6.14) \quad M'(r, t, n) = \sum_{m \equiv r \pmod{t}} M'(m, n),$$

which is the number of ordinary partitions of n with crank congruent to $r \pmod{t}$ when $n \neq 1$. When $n = 1$ it is counting extended partitions. Then

$$(6.15) \quad \begin{aligned} F(\zeta, q) &= \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k M'(k, 5, n) q^n = \frac{(q; q)_{\infty}}{(\zeta q; q)_{\infty} (\zeta^{-1} q; q)_{\infty}} \\ &= A(q^5) - q(\zeta + \zeta^4)^2 B(q^5) + q^2(\zeta^2 + \zeta^3) C(q^5) - q^3(\zeta + \zeta^4) D(q^5), \end{aligned}$$

where $F(z, q)$ is given in (6.8), $A(q)$, $B(q)$, $C(q)$, and $D(q)$ are given in (3.4)–(3.7). Equation (6.15) appears in Ramanujan's Lost Notebook [24, p.20] and is proved in [13, (1.30)].

The two bicrank generating functions are given by

$$(6.16) \quad \sum_{\pi=(\pi_1, \pi_2)} z^{\text{bicrank}_1(\pi)} q^{|\pi|} = F(z, q)^2,$$

$$(6.17) \quad \sum_{\pi=(\pi_1, \pi_2)} z^{\text{bicrank}_2(\pi)} q^{|\pi|} = F(z, q) F(z^2, q),$$

where $F(z, q)$ is the generating function for the crank of extended partitions (6.8). Thus we have

$$(6.18) \quad \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k M_1(k, 5, n) q^n = F(\zeta, q)^2,$$

$$(6.19) \quad \sum_{n=0}^{\infty} \sum_{k=0}^4 \zeta^k M_1(k, 5, n) q^n = F(\zeta, q) F(\zeta^2, q).$$

Using only (6.15) and equations (3.12) and (3.14) we easily find that the coefficient of q^n in the q -expansion of $F(\zeta, q)^2$ is zero if $n \equiv 3 \pmod{5}$, and that the coefficient of q^n in the q -expansion of $F(\zeta, q) F(\zeta^2, q)$ is zero if $n \equiv 2$, or $4 \pmod{5}$. Both parts of Theorem 6.1 then follow from equations (6.18) and (6.19).

7. A MULTIRANK ANALOG OF THE HAMMOND-LEWIS BIRANK

Let \mathcal{P} denote the set of partitions. A multipartition with r components or an r -colored partition of n is simply an r -tuple

$$(7.1) \quad \vec{\pi} = (\pi_1, \pi_2, \dots, \pi_r) \in \mathcal{P} \times \mathcal{P} \times \dots \times \mathcal{P} = \mathcal{P}^r,$$

where

$$(7.2) \quad \sum_{k=1}^r |\pi_k| = n.$$

It is clear that the number of r -colored partitions of n is $p_{-r}(n)$ where

$$(7.3) \quad \sum_{n \geq 0} p_{-r}(n) q^n = \frac{1}{E(q)^r}.$$

Congruences for $p_{-r}(n)$ were first considered by Ramanujan. In [23], Ramanathan includes a copy of letter from Ramanujan to Hardy in which Ramanujan alludes to congruences for $p_{-r}(n)$. Ramanathan [23, pp.153-4] also mentions some of these congruences that appear in the Lost Notebook. Below in Theorem 7.1 we state two elementary and well-known congruences. See for instance [12]. We note that deeper congruences have been found by Atkin [3], Boylan [6], Forbes [11], Newman [21] and Ramanathan [22].

Theorem 7.1. *Let $t > 3$ be prime.*

(i) *If $24n + 1$ is a quadratic nonresidue mod t , then*

$$(7.4) \quad p_{1-t}(n) \equiv 0 \pmod{t}.$$

(ii) *If $8n + 1$ is not a quadratic residue mod t , then*

$$(7.5) \quad p_{3-t}(n) \equiv 0 \pmod{t}.$$

These results follow easily from identities of Euler and Jacobi. Theorem 7.1 (i) follows from

$$(7.6) \quad \sum_{n \geq 0} p_{1-t}(n) q^{24n+1} = \frac{qE(q^{24})}{E(q^{24})^t} \equiv \frac{1}{E(q^{24t})} \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \pmod{t}.$$

Here we have used Euler's Pentagonal Number Theorem [18, Thm 353]

$$(7.7) \quad E(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

Theorem 7.1 (ii) follows from

$$(7.8) \quad \sum_{n \geq 0} p_{3-t}(n) q^{8n+1} = \frac{qE^3(q^8)}{E(q^8)^t} \equiv \frac{1}{E(q^{8t})} \sum_{n \geq 0} (-1)^n (2n+1) q^{(2n+1)^2} \pmod{t},$$

where we have used Jacobi's Identity [18, Thm 237]

$$(7.9) \quad E(q)^3 = \sum_{n \geq 0} (-1)^n (2n+1) q^{n(n+1)/2}.$$

Theorem 7.2 (ii) is Theorem 1 in [1].

In this section we construct analogs of the Hammond-Lewis birank to combinatorially explain the two congruences in Theorem 7.1. Andrews's bicrank [1] (see also equation (6.11)) gave a combinatorial interpretation of Theorem 7.1 (ii) for the case $t = 5$, and $n \equiv 3 \pmod{5}$. The Hammond-Lewis birank gave a combinatorial interpretation of Theorem 7.1 (ii) for the case $t = 5$, and all relevant n .

For even r , we define the generalized Hammond-Lewis multirank by

$$(7.10) \quad \text{gHL-multirank}(\vec{\pi}) = \sum_{k=1}^{r/2} k (\#\pi_k - \#\pi_{r+1-k}),$$

for $\vec{\pi} = (\pi_1, \pi_2, \dots, \pi_r)$ a multipartition with r components. The $r = 2$ case corresponds to the Hammond-Lewis birank. The $r = 4$ case was considered by Hammond [16, p.67], who proved the $t = 5$ case of Theorem 7.2(i) below.

In this section we prove

Theorem 7.2. *Let $t > 3$ be prime.*

- (i) *The residue of the generalized-Hammond-Lewis-multirank mod t divides the multipartitions of n with $r = t - 1$ components into t equal classes provided $24n + 1$ is a quadratic nonresidue mod t .*
- (ii) *The residue of the generalized-Hammond-Lewis-multirank mod t divides the multipartitions of n with $r = t - 3$ components into t equal classes provided $8n + 1$ is not a quadratic residue mod t .*

We illustrate Theorem 7.2 (ii) for $t = 7$ and $n = 2$.

Multipartitions of 2 with 4 components	generalized-HL-multirank (mod 7)
$(-, -, -, 1 + 1)$	$-2 \equiv 5$
$(-, -, -, 2)$	$-1 \equiv 6$
$(-, -, 1, 1)$	$-3 \equiv 4$
$(-, -, 1 + 1, -)$	$-4 \equiv 3$
$(-, -, 2, -)$	$-2 \equiv 5$
$(-, 1, -, 1)$	$1 \equiv 1$
$(-, 1, 1, -)$	$0 \equiv 0$
$(-, 1 + 1, -, -)$	$4 \equiv 4$
$(-, 2, -, -)$	$2 \equiv 2$
$(1, -, -, 1)$	$0 \equiv 0$
$(1, -, 1, -)$	$-1 \equiv 6$
$(1, 1, -, -)$	$3 \equiv 3$
$(1 + 1, -, -, -)$	$2 \equiv 2$
$(2, -, -, -)$	$1 \equiv 1$

We see that the residue of generalized-Hammond-Lewis-multirank mod 7 divides the 14 4-colored partitions of 2 into 7 equal classes.

Both parts of Theorem 7.2 are easy to prove. For (i), we need only Euler's pentagonal number theorem (7.7). We let ζ_t be a primitive t -th root of unity. We have

$$(7.11) \quad \sum_{\vec{\pi} \in \mathcal{P}^{t-1}} \zeta_t^{\text{gHL-multirank}(\vec{\pi})} q^{|\vec{\pi}|} = \prod_{k=1}^{(t-1)/2} \frac{1}{(\zeta_t^k q; q)_\infty (\zeta_t^{-k} q; q)_\infty} = \frac{(q; q)_\infty}{(q^t; q^t)_\infty}.$$

From (7.7) we have

$$(7.12) \quad \sum_{\vec{\pi} \in \mathcal{P}^{t-1}} \zeta_t^{\text{gHL-multirank}(\vec{\pi})} q^{24|\vec{\pi}|+1} = \frac{1}{(q^{24t}; q^{24t})_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2}.$$

We see that in the q -expansion on the right side of (7.12) the coefficient of q^n is zero when n is a quadratic nonresidue mod t . Theorem 7.2 (i) follows.

For part (ii) of Theorem 7.2 we only need Jacobi's triple product identity (5.14). We have

$$\begin{aligned}
(7.13) \quad \sum_{\vec{\pi} \in \mathcal{P}^{t-3}} \zeta_t^{\text{gHL-multirank}(\vec{\pi})} q^{|\vec{\pi}|} &= \prod_{k=1}^{(t-3)/2} \frac{1}{(\zeta_t^k q; q)_\infty (\zeta_t^{-k} q; q)_\infty} \\
&= \frac{(\zeta_t^{(t-1)/2} q; q)_\infty (\zeta_t^{-(t-1)/2} q; q)_\infty (q; q)_\infty}{(q^t; q^t)_\infty} \\
&= \frac{\sum_{m=-\infty}^{\infty} (-1)^{m+1} (\zeta_t^{(m+1)(t-1)/2} - \zeta_t^{-m(t-1)/2}) q^{m(m+1)/2}}{(1 - \zeta_t^{(t-1)/2})(q^t; q^t)_\infty},
\end{aligned}$$

and

$$\begin{aligned}
(7.14) \quad \sum_{\vec{\pi} \in \mathcal{P}^{t-3}} \zeta_t^{\text{gHL-multirank}(\vec{\pi})} q^{8|\vec{\pi}|+1} \\
= \frac{\sum_{m=-\infty}^{\infty} (-1)^{m+1} \zeta_t^{-m(t-1)/2} (\zeta_t^{(2m+1)(t-1)/2} - 1) q^{(2m+1)^2}}{(1 - \zeta_t^{(t-1)/2})(q^{8t}; q^{8t})_\infty}.
\end{aligned}$$

We see that in the q -expansion on the right side of (7.14) the coefficient of q^n is zero when n is not a quadratic residue mod t , i.e. when n is either a quadratic nonresidue or $n \equiv 0 \pmod{t}$. Theorem 7.2 (ii) follows.

8. MULTICRANKS

In this section we give some extensions of the bicrank to multipartitions and provide alternative interpretations for some of the congruences given in Theorem 7.1. We define two multicranks. These multicranks are defined in terms of cartesian products of extended partitions and ordinary partitions. In Section 6, we defined the set of extended partitions \mathcal{E} and its associated crank and weight function. Recall from Section 7 that \mathcal{P} denotes the set of ordinary partitions, and $\mathcal{P} \subset \mathcal{E}$. Let r be a positive even integer. For an extended multipartition

$$(8.1) \quad \vec{\pi} = (\pi_1, \pi_2, \dots, \pi_r) \in \mathcal{E} \times \dots \times \mathcal{E} \times \mathcal{P} \times \dots \times \mathcal{P} = \mathcal{E}^{r/2} \times \mathcal{P}^{r/2},$$

we define multicrank-I by

$$(8.2) \quad \text{multicrank-I}(\vec{\pi}) = \sum_{k=1}^{r/2} k \cdot \text{crank}(\pi_k).$$

For an extended multipartition

$$(8.3) \quad \vec{\pi} = (\pi_1, \pi_2, \dots, \pi_r) \in \mathcal{E} \times \mathcal{E} \times \mathcal{P} \times \dots \times \mathcal{P} = \mathcal{E} \times \mathcal{E} \times \mathcal{P}^{r-2},$$

we define multicrank-II by

$$(8.4) \quad \text{multicrank-II}(\vec{\pi}) = \sum_{k=1}^2 k \cdot \text{crank}(\pi_k) + \sum_{k=3}^r k (\#\pi_k - \#\pi_{r-k+3}).$$

We note that the bicrank₂ corresponds to the multicrank-II when $r = 2$.

For both types of extended multipartitions we define a sum of parts function and a weight function in the natural way

$$(8.5) \quad |\vec{\pi}| = \sum_{k=1}^r |\pi_k|, \quad \text{and} \quad w(\vec{\pi}) = \prod_{k=1}^r w(\pi_k).$$

We have

$$(8.6) \quad \sum_{\substack{\vec{\pi} \in \mathcal{E}^{r/2} \times \mathcal{P}^{r/2} \\ |\vec{\pi}|=n}} w(\vec{\pi}) = \sum_{\substack{\vec{\pi} \in \mathcal{E}^2 \times \mathcal{P}^{r-2} \\ |\vec{\pi}|=n}} w(\vec{\pi}) = p_{-r}(n).$$

Theorem 8.1. *Let $t > 3$ be prime.*

- (i) *The residue of the multicrank-I mod t divides the extended multipartitions of n from $\mathcal{E}^{(t-1)/2} \times \mathcal{P}^{(t-1)/2}$ into t equal classes of equal weight provided $24n + 1$ is a quadratic nonresidue mod t .*
- (ii) *The residue of the multicrank-I mod t divides the extended multipartitions of n from $\mathcal{E}^{(t-3)/2} \times \mathcal{P}^{(t-3)/2}$ into t equal classes of equal weight provided $8n + 1$ is not a quadratic residue mod t .*
- (iii) *The residue of the multicrank-II mod t divides the extended multipartitions of n from $\mathcal{E}^2 \times \mathcal{P}^{t-5}$ into t equal classes of equal weight provided $8n + 1$ is a quadratic nonresidue mod t .*

In view of (8.6), Theorem 8.1 (i), (ii) provides alternative combinatorial interpretations of our congruences for multipartitions given in Theorem 7.1 (i), (ii). The result in part (iii) is weaker than (ii). We include it since it is a generalization of the bicrank₂ result.

The proof of Theorem 8.1 is very similar to Theorem 7.2. We have

$$(8.7) \quad \sum_{\vec{\pi} \in \mathcal{E}^{(t-1)/2} \times \mathcal{P}^{(t-1)/2}} \zeta_t^{\text{multicrank-I}(\vec{\pi})} w(\vec{\pi}) q^{|\vec{\pi}|} = \left(\prod_{k=1}^{(t-1)/2} F(\zeta_t^k, q) \right) \frac{1}{E^{(t-1)/2}(q)}$$

$$= \prod_{k=1}^{(t-1)/2} \frac{1}{(\zeta_t^k q; q)_\infty (\zeta_t^{-k} q; q)_\infty}$$

$$= \frac{(q; q)_\infty}{(q^t; q^t)_\infty},$$

where $F(z, q)$ is the crank generating function given in (6.8). Theorem 8.1 (i) then follows from (7.12).

Similarly,

$$(8.8) \quad \sum_{\vec{\pi} \in \mathcal{E}^{(t-3)/2} \times \mathcal{P}^{(t-3)/2}} \zeta_t^{\text{multicrank-I}(\vec{\pi})} w(\vec{\pi}) q^{|\vec{\pi}|} = \left(\prod_{k=1}^{(t-3)/2} F(\zeta_t^k, q) \right) \frac{1}{E^{(t-3)/2}(q)}$$

$$= \prod_{k=1}^{(t-3)/2} \frac{1}{(\zeta_t^k q; q)_\infty (\zeta_t^{-k} q; q)_\infty}.$$

Theorem 8.1 (ii) then follows from (7.13), (7.14).

We have

$$(8.9) \quad \sum_{\vec{\pi} \in \mathcal{E}^2 \times \mathcal{P}^{t-5}} \zeta_t^{\text{multicrank-II}(\vec{\pi})} w(\vec{\pi}) q^{|\vec{\pi}|} = F(\zeta_t, q) F(\zeta_t^2, q) \prod_{k=3}^{(t-1)/2} \frac{1}{(\zeta_t^k q; q)_\infty (\zeta_t^{-k} q; q)_\infty} \\ = \frac{(q)_\infty^3}{(q^t; q^t)_\infty}.$$

Then

$$(8.10) \quad \sum_{\vec{\pi} \in \mathcal{E}^2 \times \mathcal{P}^{t-5}} \zeta_t^{\text{multicrank-II}(\vec{\pi})} w(\vec{\pi}) q^{8|\vec{\pi}|+1} = \frac{1}{E(q^{8t})} \sum_{n \geq 0} (-1)^n (2n+1) q^{(2n+1)^2},$$

and Theorem 8.1 (iii) follows.

9. CONCLUDING REMARKS

The two main results of this paper are the combinatorial interpretations of the 2-colored partition congruences (1.1) in terms of the Dyson-birank and the 5-core-birank. The author has been unable to extend these two results to higher dimensional multipartitions. The extensions of the Hammond-Lewis birank and Andrews bicrank are much easier because the generating functions involved are simple infinite products.

It seems unlikely that a combinatorial proof of (4.5) is possible. This identity gives the 5-dissection of the 5-core-crank generating function when $z = \zeta_5$. The proof given in the paper relies on a heavy use of the theory of modular functions. A more elementary proof is desirable. In [15], a combinatorial proof is given that the residue of 5-core-crank mod 5 divides the 5-cores of $5n+4$ into 5 equal classes. It would be interesting to see if the methods of [15] could be extended to give a combinatorial proof of Theorem 4.1, which is our result for the 5-core-birank.

It is clear that the generalized-Hammond-Lewis multiranks and our multicranks are related. For instance, from equations (7.11), (8.7), (7.13), and (8.8) we have

$$(9.1) \quad \sum_{\vec{\pi} \in \mathcal{P}^{t-1}} \zeta_t^{\text{gHL-multirank}(\vec{\pi})} q^{|\vec{\pi}|} = \sum_{\vec{\pi} \in \mathcal{E}^{(t-1)/2} \times \mathcal{P}^{(t-1)/2}} \zeta_t^{\text{multicrank-I}(\vec{\pi})} w(\vec{\pi}) q^{|\vec{\pi}|},$$

$$(9.2) \quad \sum_{\vec{\pi} \in \mathcal{P}^{t-3}} \zeta_t^{\text{gHL-multirank}(\vec{\pi})} q^{|\vec{\pi}|} = \sum_{\vec{\pi} \in \mathcal{E}^{(t-3)/2} \times \mathcal{P}^{(t-3)/2}} \zeta_t^{\text{multicrank-I}(\vec{\pi})} w(\vec{\pi}) q^{|\vec{\pi}|}.$$

It would be interesting to find a combinatorial proof of these identities. However what would be more interesting is to find bijective proofs of Theorems 6.1 and 7.2. This is a reasonable problem since the generating functions involved are simple infinite products.

Throughout this paper we have assumed t is prime. Houser [20] has considered the analogue of Theorem 7.1 for prime powers. Although he does not give rank-type results he does find combinatorial proofs. Dennis Eichhorn has observed that equations (7.12), (7.13), and (7.14) also hold when t is composite. In this case, there will be relations between the number of multipartitions in each gHL-multirank class mod t .

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