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A NUMBER THEORETIC CRANK ASSOCIATED WITH OPEN BOSONIC STRINGS

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Abstract. A Dyson-type crank is given which explains Moreno's congruence for the number of open bosonic strings. This crank is in terms of 24-coloured partitions.

1. Introduction. Let $q = e^{2\pi i z}$, where $\mathrm{Im}(z) > 0$ so that $|q| < 1$. The well-known discriminant modular form $\Delta(z)$ has q-expansion

$$(2\pi)^{-12}\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}. \tag{1.1}$$

The reciprocal of this function is essentially the sring function associated to the affine Lie algebra $A_{24}^{(1)}$ [12, p.137], [13, §3.2]. We let $\tilde{\tau}(n)$ denote the n-th Fourier coefficient of this function so that

$$\tilde{\Delta}(z) = \sum_{n=-1}^{\infty} \tilde{\tau}(n) q^n = \frac{1}{q\prod_{n=1}^{\infty}(1-q^n)^{24}}. \tag{1.2}$$

As noted by Moreno and Rocha-Caridi [13, p.144] $\tilde{\Delta}(z)$ has a physical interpretation from the light cone formulation of string theory. In fact, the number of open string states with mass M such that $\alpha' M^2 = n$ is $\tilde{\tau}(n)$, where α' is the Regge slope [11, p.117]. The coefficients $\tilde{\tau}(n)$ can also be interpreted in terms of the weight multiplicities of the vertex algebra associated to the unique Lorentzian lattice of signature $(25, 1)$ and the No-Ghost Theorem of Brower, Goddard and Thorn [11, p.102]. Moreno and Rocha-Caridi [13] also found Hardy-Ramanujan-Rademacher expansions for string functions associated with affine Lie algebras and thus found such an expansion for $\tilde{\tau}(n)$. In [14] Moreno explored congruence and combinatorial properties of $\tilde{\tau}(n)$.

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The coefficients $\tilde{\tau}(n)$ have a natural combinatorial interpretation in terms of coloured partitions. The coefficients $p_r(n)$ are defined by

$$\sum_{n\geq 0} p_r(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^r. \tag{1.3}$$

For negative r, say $r=-d$, $p_r(n)$ counts the number of partitions of n taken from d copies of the natural numbers. We call such partitions d-coloured partitions. In view of (1.2) we see that $\tilde{\tau}(n)$ counts 24-coloured partitions:

$$\tilde{\tau}(n) = p_{-24}(n+1). \tag{1.4}$$

It is this interpretation of $\tilde{\tau}(n)$ that we use in this paper.

Moreno [14, Thm 12] proved the following congruence

$$p_{-24}(n) \equiv 0 \pmod 5 \qquad \text{for } n \equiv 1 \pmod 5 \text{ and } n \neq 1, \tag{1.5}$$

and asked for a combinatorial interpretation analogous to Dyson's [2], [3] interpretation of Ramanujan's [15] congruences for the partition function $p(n)$ $(= p_{-1}(n))$. We give such an interpretation below in Theorem 1. We also explain similar congruences modulo 2, 3 and 25. Our combinatorial interpretation is terms of the crank of 24-coloured partitions and is described in the next section. Our method is elementary and was first introduced in [5]. We note other interpretations of partition congruences have been found in [1], [6], [7], [8], [9] and [10].

2. The crank for 24-coloured partitions.

As well as (1.5) the following congruences hold

$$p_{-24}(n) \equiv 0 \pmod 2 \qquad \text{for } n \not\equiv 0 \pmod 8, \tag{2.1}$$

$$p_{-24}(n) \equiv 0 \pmod 3 \qquad \text{for } n \not\equiv 0 \pmod 3, \tag{2.2}$$

$$p_{-24}(n) \equiv 0 \pmod{25} \qquad \text{for } n \equiv 3 \text{ or } 4 \pmod 5, \tag{2.3}$$

$$p_{-24}(n) \equiv 0 \pmod 7 \qquad \text{for } n \equiv 1 \pmod 7 \text{ and } n \neq 1. \tag{2.4}$$

The congruences (2.1) and (2.2) are trivial. The proof of (2.4) is analogous to Moreno's proof of (1.5) and also follows from [4, Lemma(3.12)]. (2.3) has a very simple proof.

$$\begin{aligned} \sum_{n\geq 0} p_{-24}(n)q^n &= \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^m)^{25}} \\ &\equiv \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{5m})^5} \pmod{25} \\ &= \frac{\sum_{\infty}^{\infty}(-1)^n q^{n(3n-1)/2}}{\prod_{m=1}^{\infty}(1-q^{5m})^5}. \end{aligned} \tag{2.5}$$

The result follows since $n(3n-1)/2 \not\equiv 3,4 \pmod 5$.

We now define a *crank* that explains (1.5), (2.1), (2.2), (2.3) but not (2.4). In this section we concentrate on the more interesting congruences (1.5), (2.3) and leave the remaining congruences (2.1) and (2.2) to §3. We number the 24 colours $1, 2, \ldots, 24$. For a 24-coloured partition $\tilde{\pi}$ let

$$c_i(\tilde{\pi}) := \text{the number of parts of } \tilde{\pi} \text{ coloured } i.$$

We define the crank of $\tilde{\pi}$ as

$$\operatorname{crank}(\tilde{\pi}) := \sum_{i=1}^{24} i c_i(\tilde{\pi}). \tag{2.6}$$

Let $N_{24}(m, n)$ denote the number of 24-coloured partitions $\tilde{\pi}$ of n with crank m and let $N_{24}(k, t, n)$ denote the number of 24-coloured partitions $\tilde{\pi}$ of n with crank congruent to k modulo t. Unfortunately it is *not* true that

$$N_{24}(-m, n) = N_{24}(m, n). \tag{2.7}$$

We could have defined our crank differently so that (2.7) holds. However, this other crank explains (1.5) and (2.3) but fails to explain (2.1) and (2.2). Luckily an analog of (2.7) holds for $N_{24}(k, t, n)$ for the values of t we are concerned with. By considering each of the following two recolouring involutions:

$$\begin{array}{lll} \text{Involution}_1: & i \mapsto 25 - i & \text{for } 1 \le i \le 24 \\ \text{Involution}_2: & i \mapsto 24 - i & \text{for } 1 \le i \le 23 \text{ and } 24 \mapsto 24, \end{array}$$

we have

$$N_{24}(-k, t, n) = N_{24}(k, t, n) \qquad \text{when } t \text{ is a divisor of 24 or 25.} \tag{2.8}$$

Then we have

Theorem 1. *For $n \equiv 1, 3$ or $4 \pmod 5$ and $n \neq 1$ we have*

$$N_{24}(k, 25, n) = \tfrac{1}{5} N_{24}(k, 5, n). \tag{2.9}$$

This provides a natural way of dividing the 24-coloured partitions of n into 5 equal classes for $n \equiv 1, 3$ or $4 \pmod 5$ and $n \neq 1$.

Corollary 1. *For $0 \le j \le 4$, let $M_{24}(j, n)$ denote the number of 24-coloured partitions of n with crank congruent to $5j$, $5j + 1$, $5j + 2$, $5j + 3$, or $5j + 4 \pmod{25}$. Then*

$$M_{24}(j, n) = \tfrac{1}{5} p_{-24}(n) \tag{2.10}$$

when $n \equiv 1, 3$ or $4 \pmod 5$ and $n \neq 1$.

For $n \equiv 3$ or $4 \pmod 5$ more is true.

Theorem 2. *For $n \equiv 3$ or $4 \mod 5$ we have*

$$N_{24}(k,5,n) = \tfrac{1}{5}p_{-24}(n) \qquad 0 \le k \le 4. \tag{2.11}$$

Combining the results of Theorems 1 and 2 we find that the residue of the crank mod 25 divides the partitions of n (for $n \equiv 3$ or $4 \mod 5$) into 25 equal classes.

Theorem 3. *For $n \equiv 3$ or $4 \mod 5$ we have*

$$N_{24}(k,25,n) = \tfrac{1}{25}p_{-24}(n) \qquad 0 \le k \le 24. \tag{2.12}$$

Proof of Theorem 1. Let $\zeta = e^{2\pi i/25}$ so that $\zeta^{25} = 1$ and $1 + \zeta^5 + \zeta^{10} + \zeta^{15} + \zeta^{20} = 0$. The generating function for $N_{24}(m,n)$ is

$$\sum_{n\ge 0}\sum_{m=0}^{24n} N_{24}(m,n) z^m q^n = \prod_{i=1}^{24}(z^i q;q)_\infty^{-1}, \tag{2.13}$$

where $(a;q)_\infty = \prod_{m=1}^{\infty}(1 - aq^{m-1}),\quad |q| < 1$. Substituting $z = \zeta$ in (2.13) and proceeding as in [6, §2] we find that

$$\sum_{n\ge 0}\left(\sum_{k=0}^{24} N_{24}(k,25,n)\zeta^k\right) q^n = \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{25n})}. \tag{2.14}$$

But the coefficient of q^n in the left side of (2.14) is

$$\begin{aligned}
&(n_0 - n_5) + (n_1 - n_4)\zeta + (n_2 - n_3)\zeta^2 + (n_3 - n_2)\zeta^3 + (n_4 - n_1)\zeta^4 \\
&\quad + (n_6 - n_4)\zeta^6 + (n_7 - n_3)\zeta^7 + (n_8 - n_2)\zeta^8 + (n_9 - n_1)\zeta^9 \\
&+ (n_{10} - n_5)\zeta^{10} + (n_{11} - n_4)\zeta^{11} + (n_{12} - n_3)\zeta^{12} + (n_{12} - n_2)\zeta^{13} + (n_{11} - n_1)\zeta^{14} \\
&+ (n_{10} - n_5)\zeta^{15} + (n_9 - n_4)\zeta^{16} + (n_8 - n_3)\zeta^{17} + (n_7 - n_2)\zeta^{18} + (n_6 - n_1)\zeta^{19},
\end{aligned}$$

where $n_i = n_i(n) = N_{24}(i,25,n)$. The result then follows from

$$\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{25n})} = \prod_{n=1}^{\infty}\frac{(1-q^{25n-15})(1-q^{25n-10})}{(1-q^{25n-20})(1-q^{25n-5})} - q - q^2\prod_{n=1}^{\infty}\frac{(1-q^{25n-20})(1-q^{25n-5})}{(1-q^{25n-15})(1-q^{25n-10})}, \tag{2.15}$$

which is [6, Lemma (3.18)], was known to Ramanujan and has been generalised by Atkin and Swinnerton-Dyer [2, Lemma 6]. □

Proof of Theorem 2. Let $\eta = e^{2\pi i/5}$. We substitute $z = \eta$ into (2.13) to find

$$\sum_{n\ge 0}\left(\sum_{k=0}^{4} N_{24}(k,5,n)\eta^k\right) q^n = \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-q^{5n})^5}. \tag{2.16}$$

Since the series expansion of $\prod_{n=1}^{\infty}(1-q^n)$ has no terms with exponent congruent to either 3 or 4 modulo 5, as in the proof of (2.3), the result follows. □

Now Theorem 3 follows immediately from Theorems 1 and 2.

3. Remarks. We remark that our crank also explains the congruences (2.1) and (2.2). We omit the proof.

Theorem 4. *We have*

$$\sum_{k=0}^{3} N_{24}(k,8,n) = \tfrac{1}{2}p_{-24}(n) \qquad \textit{for } n \not\equiv 0 \mod 8, \tag{4.1}$$

and

$$N_{24}(k,3,n) = \tfrac{1}{3}p_{-24}(n) \qquad \textit{for } n \not\equiv 0 \mod 3 \textit{ and } 0 \le k \le 2. \tag{4.2}$$

Unfortunately our crank fails to explain the mod 7 congruence (2.4).

For restricted n stronger congruences than (2.1), (2.2) hold:

$$p_{-24}(n) \equiv 0 \mod 2^7 \qquad \text{for } n \equiv 1 \mod 2 \text{ and } n \ne 1, \tag{4.3}$$

$$p_{-24}(n) \equiv 0 \mod 3^3 \qquad \text{for } n \not\equiv 0 \mod 3 \text{ and } n \ne 1. \tag{4.4}$$

(4.3) follows from the following q-series identity

$$\sum_{n\ge 0} p_{-24}(2n+1)q^n = 24\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{24}}{(1-q^n)^{48}} + 2^{11}q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{48}}{(1-q^n)^{72}}, \tag{4.5}$$

which follows from a certain well known quadratic modular equation ([16], page 470). (4.4) follows first by observing that the generating function for $p_{-24}(n)$ is congruent to $(q;q)_\infty^3(q^9;q^9)_\infty^{-9} \mod 27$ and then by using Jacobi's identity for $(q;q)_\infty^3$ ([2], (3.6)). Our crank fails to explain either (4.3) or (4.4). For (4.4) the best we can do is

$$N_{24}(k,8,n) = \tfrac{1}{8}p_{-24}(n) \qquad \text{for } n \equiv 1 \mod 2 \text{ and } 0 \le k \le 7. \tag{4.6}$$

We can explain a weaker form of (4.4) but with a different crank. If we define crank$'$ by

$$\text{crank}'(\tilde{\pi}) := \sum_{i=1}^{24}(i+1)c_i(\tilde{\pi}) \tag{4.7}$$

and define N'_{24} in the obvious way then

(4.8)

$$N_{24}(k,27,n) = \tfrac{1}{27}p_{-24}(n) \qquad \text{for } n \not\equiv 0 \mod 3 \text{ and } n \not\equiv 1 \mod 9,\ 0 \le k \le 26.$$

We omit the proof.

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