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CUBIC MODULAR IDENTITIES OF RAMANUJAN, HYPERGEOMETRIC FUNCTIONS AND ANALOGUES OF THE ARITHMETIC-GEOMETRIC MEAN ITERATION

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Abstract. There are four values of s for which the hypergeometric function ${}_2F_1(\frac{1}{2}-s,\frac{1}{2}+s;1;\cdot)$ can be parametrized in terms of modular forms; namely, $s=0$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$. For the classical $s=0$ case, the parametrization is in terms of the Jacobian theta functions $\theta_3(q)$, $\theta_4(q)$ and is related to the arithmetic-geometric mean iteration of Gauss and Legendre. Analogues of the arithmetic-geometric mean are given for the remaining cases. The case $s=\frac{1}{6}$ and its relationship to the work of Ramanujan is highlighted. The work presented includes various pieces of joint work with combinations of the following: B. Berndt, S. Bhargava, J. Borwein, P. Borwein and M. Hirschhorn.

1. The AGM

The arithmetic-geometric mean iteration (or AGM) is an example of a two-term iteration. Here the means are

$$M_1(a,b) := \frac{a+b}{2}, \tag{1.1}$$

and

$$M_2(a,b) := \sqrt{ab}. \tag{1.2}$$

We iterate the means by

$$a_{n+1} := M_1(a_n,b_n), \tag{1.3}$$

$$b_{n+1} := M_2(a_n,b_n), \tag{1.4}$$

commencing with $a_0 := a$, $b_0 := b$, where a, b are positive numbers. Then, if $b \le a$, we have

$$b_n \le b_{n+1} \le a_{n+1} \le a_n, \tag{1.5}$$

$$a_{n+1}-b_{n+1} = \frac{1}{2}\left(\sqrt{a_n}-\sqrt{b_n}\right)^2 = \frac{1}{2}\left(\frac{a_n-b_n}{\sqrt{a_n}+\sqrt{b_n}}\right)^2. \tag{1.6}$$

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Many of the results of this paper will appear elsewhere in more detail.

It follows that a_n and b_n converge to a common limit which we denote by

$$M(a,b) := M_1 \otimes M_2(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{1.7}$$

We say that a two term iteration has p-th order convergence if

$$|a_{n+1} - b_{n+1}| = O((a_n - b_n)^p). \tag{1.8}$$

For the AGM we have quadratic convergence. The function M satisfies a number of nice properties:

$$M(a,b) = M(b,a), \tag{1.9}$$
$$M(\lambda a, \lambda b) = \lambda M(a,b) \qquad \text{for } \lambda > 0, \tag{1.10}$$
$$M(\frac{a+b}{2}, \sqrt{ab}) = M(a,b), \tag{1.11}$$
$$M(1,b) = \left(\frac{b+1}{2}\right) M(1, \frac{2\sqrt{b}}{1+b}). \tag{1.12}$$

Gauss was the first to see a connection between the AGM and elliptic integrals. On May 30, 1799[1] he observed that

$$\frac{1}{M(1,\sqrt{2})} \qquad \text{and} \qquad \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

agreed to at least eleven decimal places. He commented in his diary that this result "will surely open up a whole new field of analysis." Indeed, later he showed that, for $0 < x < 1$,

$$\begin{aligned} \frac{1}{M(1,x)} &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-(1-x^2)\sin^2\theta}} \\ &= {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2). \end{aligned} \tag{1.13}$$

The integral above is usually called the complete elliptic integral of the first kind. The Gaussian hypergeometric series is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!}\, x^n \qquad (|x| < 1), \tag{1.14}$$

where $(a)_0 := 1$ and for n a positive integer

$$(a)_n := a(a+1)\dots(a+n-1), \tag{1.15}$$

so that $(1)_n = n!$. The latter equality in (1.13) follows by expanding $(1-(1-x^2)\sin^2\theta)^{-1/2}$ and integrating term by term. We will provide a proof of (1.13) below. Thus the AGM provides an efficient algorithm for computing complete elliptic integrals of the first kind or values of the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \cdot)$.

[1]This tidbit was pinched from [**BB1**, p. 5]. See also [Gr, Chapter I].

The AGM and thus the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \cdot)$ may be parametrized in terms of theta functions. The Jacobian theta functions are defined for $|q| < 1$ by

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}, \tag{1.16}$$

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.17}$$

$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{1.18}$$

We note that in (1.16) the principal root is taken. We have the following

$$\theta_3^2(q^2) = \frac{\theta_3^2(q) + \theta_4^2(q)}{2}, \tag{1.19a}$$

$$\theta_4^2(q^2) = \sqrt{\theta_3^2(q)\theta_4^2(q)}, \tag{1.19b}$$

$$\theta_3^4(q) = \theta_4^4(q) + \theta_2^4(q) \qquad \text{(Jacobi's identity)}, \tag{1.19c}$$

$${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{\theta_4^4(q)}{\theta_3^4(q)}) = \theta_3^2(q), \tag{1.20}$$

(1.21) The function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2)$ satisfies

$$F(x) = \frac{2}{1+x} F(\frac{2\sqrt{x}}{1+x}).$$

Equations (1.19ab) provide the parametrizations of the AGM and (1.20) gives the parametrization of the hypergeometric function. An elementary proof of (1.19abc) using nothing more than series manipulation can be found in [BB1, pp. 34-35]. Equation (1.21) is a special case of a quadratic transformation due to Gauss [Gau]. We find that (1.21) may be written as

$${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1-x}{1+x}\right)^2) = \left(\frac{x+1}{2}\right) {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2). \tag{1.22}$$

Gauss' transformation is

$${}_2F_1(a, b; 2b; 4z(1+z)^{-2}) = (1+z)^{2a} {}_2F_1(a, a - b + \frac{1}{2}; b + \frac{1}{2}; z^2). \tag{1.23}$$

See [E, p. 111, Eq.(5)]. Equation (1.23) is also Entry 3 in Chapter 11 of Ramanujan's second notebook [Be, p. 50]. If we let $z = \frac{1-x}{1+x}$ in (1.23) we obtain

$${}_2F_1(a, a - b + \frac{1}{2}; b + \frac{1}{2}; \left(\frac{1-x}{1+x}\right)^2) = \left(\frac{x+1}{2}\right)^{2a} {}_2F_1(a, b; 2b; 1 - x^2), \tag{1.24}$$

which is a generalization of (1.22).

Before proving (1.20) we show how

$$M(1,x)=\frac{1}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x^2)}, \tag{1.25}$$

which gives (1.13), follows from (1.19ab) and (1.20). From (1.19ab) we have

$$M(\theta_3^2(q),\theta_4^2(q))=M(\theta_3^2(q^2),\theta_4^2(q^2)). \tag{1.26}$$

Iterating gives

$$M(\theta_3^2(q),\theta_4^2(q))=M(\theta_3^2(q^{2^n}),\theta_4^2(q^{2^n})). \tag{1.27}$$

and letting $n\to\infty$ gives

$$M(\theta_3^2(q),\theta_4^2(q))=1. \tag{1.28}$$

From the homogeneity property (1.10) we have

$$M(1,\tfrac{\theta_4^2(q)}{\theta_3^2(q)})=\frac{1}{\theta_3^2(q)}. \tag{1.29}$$

Now (1.25) follows by substituting $x=\frac{\theta_4^2(q)}{\theta_3^2(q)}$ in (1.20).

Borwein and Borwein [BB1] have observed the following proposition. This will complete the proof of (1.13).

Proposition 1.30. *Any two of (1.19ab), (1.20), (1.21) implies the third.*

Proof. We show how (1.19ab) and (1.21) implies (1.20), and leave the rest as an exercise for the reader. If we let $x=\theta_4^2(q)/\theta_3^2(q)$ then assuming (1.19ab) we find that

$$\frac{2\sqrt{x}}{1+x}=\frac{2\theta_4(q)/\theta_3(q)}{1+\theta_4(q)/\theta_3(q)}=\frac{\theta_3(q)\theta_4(q)}{(\theta_3^2(q)+\theta_4^2(q))/2}=\frac{\theta_4^2(q^2)}{\theta_3^2(q^2)}, \tag{1.31}$$

and

$$\frac{2}{1+x}=\frac{2}{1+\theta_4^2(q)/\theta_3^2(q)}=\frac{\theta_3^2(q)}{(\theta_3^2(q)+\theta_4^2(q))/2}=\frac{\theta_3^2(q)}{\theta_3^2(q^2)}. \tag{1.32}$$

Hence from (1.21) we have

$$\begin{aligned}{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4(q)}{\theta_3^4(q)})&=\frac{\theta_3^2(q)}{\theta_3^2(q^2)}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4(q^2)}{\theta_3^4(q^2)})\\&\;\;\vdots\\&=\frac{\theta_3^2(q)}{\theta_3^2(q^{2^m})}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4(q^{2^m})}{\theta_3^4(q^{2^m})}),\end{aligned} \tag{1.33}$$

on iterating. Finally (1.20) follows after letting $m\to\infty$. $\square$

For the hypergeometric functions

$$F_s(x) := {}_2F_1(\frac{1}{2} - s, \frac{1}{2} + s; 1; x) \tag{1.34}$$

there are three other values of s for which similar results hold. These values are $s = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$. The hypergeometric functions given explicitly are

$$F_0(x) = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = \sum_{n=0}^{\infty} \frac{(2n)!^2}{n!^4} \left(\frac{x}{2^4}\right)^n, \tag{1.35}$$

$$F_{1/6}(x) = {}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \left(\frac{x}{3^3}\right)^n, \tag{1.36}$$

$$F_{1/4}(x) = {}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x) = \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!\, n!^2} \left(\frac{x}{4^3}\right)^n, \tag{1.37}$$

$$F_{1/3}(x) = {}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!\,(2n)!\,n!} \left(\frac{x}{3^3 2^4}\right)^n. \tag{1.38}$$

There are analogues of the AGM in each case.

Theorem 1.39. *(*[BB2],[BB3],[BB4],[BBG2]*) For $s = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$ there exist (algebraic) means M_1 and M_2 such that*

$$M_1 \otimes M_2(1, x) = \frac{1}{[F_s(1 - x^\mu)]^\nu} \tag{1.40}$$

with quadratic convergence, and cubic convergence (except for $s = \frac{1}{3}$). The means M_1, M_2 are given explicitly in §4. Here $\nu = 1$ or 2 and

$$\mu = \frac{2}{\nu(1 - 2s)}. \tag{1.41}$$

A table of μ, ν and the contributors to the theorem is given below.

s	Quadratic convergence	Cubic convergence
0	Gauss $(\mu = 2, \nu = 1)$	Borwein-Borwein-Garvan $(\mu = 2, \nu = 1)$
$\frac{1}{6}$	Borwein-Borwein-Garvan $(\mu = 3, \nu = 1)$	Borwein-Borwein $(\mu = 3, \nu = 1)$
$\frac{1}{4}$	Borwein-Borwein $(\mu = \nu = 2)$	Borwein-Borwein-Garvan $(\mu = 4, \nu = 1)$
$\frac{1}{3}$	Borwein-Borwein-Garvan $(\mu = 3, \nu = 2)$	

It should be noted that in Theorem 1.39 above we are not saying that there do not exist means with cubic convergence for the case $s = \frac{1}{3}$. On the contrary, we have found such means. However, they are too horrible to include.

Since [BBG2] we have found that $M_1 \otimes M_2(x,1)$ can also be written in terms of the hypergeometric function. We define

$$G_s(x) := {}_2F_1(\frac{1}{2}-s, \frac{1}{2}-s; 1; x). \tag{1.42}$$

We need Pffaf's [P], [E, p. 109 Eq.(6)] transformation

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a, c-b; c; z/(z-1)). \tag{1.43}$$

From (1.43) we have

$$\left[{}_2F_1(a, 1-a; 1; 1-\frac{1}{x^\mu})\right]^\nu = x^{a\mu\nu}\, \left[{}_2F_1(a,a;1;1-x^\mu)\right]^\nu . \tag{1.44}$$

Corollary 1.45. *For $s = 0$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$ let $M_1 = M_{1,s}$, $M_2 = M_{2,s}$ be as in Theorem 1.39. Then*

$$M_1 \otimes M_2(x,1) = \frac{1}{\left[G_s(1-x^\mu)\right]^\nu}. \tag{1.46}$$

Proof. By homogeneity we have

$$\begin{aligned} M_1 \otimes M_2(x,1) &= x M_1 \otimes M_2(1, \frac{1}{x}) \\ &= \frac{x}{\left[F_s(1-x^{-\mu})\right]^\nu} \qquad \text{(by Theorem 1.39)} \\ &= \frac{1}{\left[G_s(1-x^\mu)\right]^\nu} \end{aligned}$$

by (1.44) with $a = \frac{1}{2} - s$ and noting that $a\mu\nu = 1$ by (1.41). □

2. A cubic analogue of the AGM

The AGM converges quadratically and the limit $M(1,x)$ can be written in terms of the hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2)$.

$$M(1,x) = 1 \big/ F_0(1-x^2)\ , \tag{2.1}$$

where $F_s(\cdot)$ is defined in (1.35). Borwein and Borwein [BB3],[BB4] discovered a cubic analogue of the AGM. Their iteration converged cubically and the limit was identified as $1 \big/ F_{1/6}(1-x^3)$. Their iteration is defined as follows. The means are

$$M_1(a,b) := \frac{a+2b}{3}, \tag{2.2}$$

$$M_2(a,b) := \sqrt[3]{\frac{b(a^2+ab+b^2)}{3}}. \tag{2.3}$$

As with the AGM we iterate the means by

$$a_{n+1} := M_1(a_n, b_n), \tag{2.4}$$

$$b_{n+1} := M_2(a_n, b_n), \tag{2.5}$$

commencing with $a_0 := a$, $b_0 := b$, where a, b are positive numbers. Then, if $b \le a$, we have

$$a_n - a_{n+1} = \frac{2}{3}(a_n - b_n), \tag{2.6}$$

$$b_{n+1}^3 - b_n^3 = \frac{b_n(a_n + 2b_n)(a_n - b_n)}{3}, \tag{2.7}$$

$$a_{n+1}^3 - b_{n+1}^3 = \frac{(a_n - b_n)^3}{3}, \tag{2.8}$$

and

$$b_n \le b_{n+1} \le a_{n+1} \le a_n. \tag{2.9}$$

It follows that a_n and b_n converge cubically to a common limit which we denote by

$$M_1 \otimes M_2(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{2.10}$$

Then for $0 < x < 1$

$$\frac{1}{M_1 \otimes M_2(1,x)} = F_{1/6}(1-x^3) = {}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3). \tag{2.11}$$

See [BB3, Theorem 1 p. 28]. In [BB4] Borwein and Borwein were able to find analogues of (1.19)–(1.21). We define

(2.12)
$$a(q) := \sum_{n,m=-\infty}^{\infty} q^{n^2+nm+m^2},$$

(2.13)
$$b(q) := \sum_{n,m=-\infty}^{\infty} \omega^{n-m} q^{n^2+nm+m^2} \qquad \text{(where } \omega = exp(2\pi i/3)\text{)},$$

(2.14)
$$c(q) := \sum_{n,m=-\infty}^{\infty} q^{(n+\frac{1}{3})^2+(n+\frac{1}{3})(m+\frac{1}{3})+(m+\frac{1}{3})^2}.$$

We have the following analogues of (1.19)–(1.21):

$$a(q^3) = \frac{a(q) + 2b(q)}{3}, \tag{2.15a}$$

$$b(q^3) = \sqrt[3]{\frac{b(q)(a^2(q) + a(q)b(q) + b^2(q))}{3}}, \tag{2.15b}$$

$$a^3(q) = b^3(q) + c^3(q), \tag{2.15c}$$

$${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3(q)}{a^3(q)}) = a(q), \tag{2.16}$$

(2.17) The function ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3)$ satisfies

$$F(x) = \frac{3}{1+2x} F\left(\frac{3}{1+2x}\sqrt[3]{\frac{x(1+x+x^2)}{3}}\right).$$

Equations (2.15ab) provide the parametrizations of the cubic analogue of the AGM and (2.16) gives the parametrization of the corresponding hypergeometric function. In [BBG1] we found elementary proofs of (2.15abc) which we sketch below. As in the classical case (see Proposition 1.30) any two of (2.15ab), (2.16), (2.17) implies the third. So (2.16) will follow from (2.15ab) and (2.17). We find that (2.17) may be written as

$$ {}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x^3) = \frac{3}{1+2x}\,{}_2F_1(\frac{1}{3},\frac{2}{3};1;\left(\frac{1-x}{1+2x}\right)^3). \tag{2.18}$$

It was a surprise that this cubic transformation was missed by the classical analysts of last century. It should be noted however that an identity equivalent to (2.18) appears on page 258 of Ramanujan's second notebook [R]. Recall that the classical analogue (1.22) is a special case of Gauss' transformation (1.23) which is an identity involving two parameters. With the aid of the computer algebra package MAPLE we have found a one-parameter generalization of (2.18):

$$ {}_2F_1(a,a+\frac{1}{3};\frac{a}{2}+\frac{5}{6};\left(\frac{1-x}{1+2x}\right)^3) = \left(\frac{2x+1}{3}\right)^{3a}\,{}_2F_1(a,a+\frac{1}{3};\frac{3a}{2}+\frac{1}{2};1-x^3). \tag{2.19}$$

With a little help from MAPLE the proof of (2.19) is straightforward. We have found that both sides of (2.19) satisfy the following second order linear differential equation:

$$ \begin{aligned} &2x(1-x^3)(1+2x)^2y'' - (1+2x)[(4x^3-1)(3a+2x+1)+18ax]\,y' \\ &\quad - 6a(3a+1)(1-x)^2y = 0. \end{aligned} \tag{2.20}$$

Since $x=1$ is a regular singular point the result (2.19) follows easily.

We have found an analogue of (2.19) for the $s=\frac{1}{4}$ case.

$$ {}_2F_1(a,a+\frac{1}{2};\frac{2a}{3}+\frac{5}{6};\left(\frac{1-x}{1+3x}\right)^2) = \left(\frac{3x+1}{4}\right)^{2a}\,{}_2F_1(a,a+\frac{1}{2};\frac{4a}{3}+\frac{2}{3};1-x^2).$$

This identity can be proved easily from known quadratic transformations.

We now sketch the proof of (2.15abc). In the classical analogue it is well known that each of the theta functions $\theta_2(q)$, $\theta_3(q)$ and $\theta_4(q)$ have infinite product expansions. In the cubic analogue we found that $b(q)$ and $c(q)$ have infinite product expansions. We have the following proposition.

Proposition 2.21.

$$ b(q) = \frac{3}{2}a(q^3) - \frac{1}{2}a(q), \tag{2.22}$$

$$ c(q) = \frac{1}{2}a(q^{\frac{1}{3}}) - \frac{1}{2}a(q), \tag{2.23}$$

$$ b(q) = \prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1-q^{3n})}, \tag{2.24}$$

$$ c(q) = 3q^{\frac{1}{3}}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3}{(1-q^n)}. \tag{2.25}$$

The proofs of (2.22), (2.23) follow easily from (2.12)–(2.14) by series manipulation. See [BBG1, Lemma 2.1]. From (2.22) we obtain

$$a(q^3) = \frac{a(q) + 2b(q)}{3}, \tag{2.26}$$

which is (2.15a). From (2.22) and (2.23) we have

$$c(q^3) = \frac{1}{2}a(q) - \frac{1}{2}a(q^3) = \frac{1}{3}(a(q) - b(q)). \tag{2.27}$$

Now (2.15c) implies

(2.28)

$$\begin{aligned} b(q^3) = \sqrt[3]{a^3(q^3) - b^3(q^3)} &= \sqrt[3]{\left(\frac{a(q) + 2b(q)}{3}\right)^3 - \left(\frac{a(q) - b(q)}{3}\right)^3} \\ &= \sqrt[3]{\frac{b(q)(a^2(q) + a(q)b(q) + b^2(q))}{3}}, \end{aligned}$$

which is (2.15b). We will sketch the proof of (2.15c) below.

It is interesting to note that the infinite product on the right side of (2.25) is the generating function for partitions that are 3-cores [GSK]. A t-core is a partition that has no hook lengths divisible by t. The use of t-cores is important in the study of p-modular representations of the symmetric group S_n. In [GSK] we found combinatorial proofs of two generating function identities for t-cores. Combining these two identities gives a generalization of (2.25). See (2.31) below. There is an analogous generalization of (2.24) although it has no known combinatorial interpretation.

Proposition 2.29. *Let $t > 1$ be an integer and let $\omega = exp(2\pi i/t)$. Then*

(2.30)

$$\sum_{m_0+m_1+\cdots+m_{t-1}=0} \omega^{m_1+2m_2+\cdots+(t-1)m_{t-1}} q^{\frac{1}{2}(m_0^2+\cdots+m_{t-1}^2)} = \prod_{n=1}^{\infty} \frac{(1-q^n)^t}{(1-q^{tn})},$$

(2.31)

$$\sum_{m_0+m_1+\cdots+m_{t-1}=0} q^{\frac{t}{2}(m_0^2+\cdots+m_{t-1}^2)+m_1+2m_2+\cdots+(t-1)m_{t-1}} = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}.$$

Proof. We need the following result due to Euler (which is a corollary of the q-binomial theorem [A, p. 19]).

$$(x;q)_\infty = \prod_{n=1}^{\infty}(1 - xq^{n-1}) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k q^{\binom{k}{2}}}{(q)_k}, \tag{2.32}$$

where as usual $(a)_\infty = (a;q)_\infty := \prod_{n=1}^{\infty}(1 - aq^{n-1})$, $(a)_n = (a;q)_n := (a;q)_\infty / (aq^n;q)_\infty = \prod_{k=1}^{n}(1 - aq^{k-1})$. Observe that

$$(x^t;q^t)_\infty = (x;q)_\infty (x\omega;q)_\infty (x\omega^2;q)_\infty \dots (x\omega^{t-1};q)_\infty,$$

so that (2.32) gives

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{tk} q^{t\binom{k}{2}}}{(q^t;q^t)_k}$$
$$= \sum_{n_0,n_1,\ldots,n_{t-1}\geq 0} \frac{(-x)^{n_0+\cdots+n_{t-1}} \omega^{n_1+2n_2\cdots+(t-1)n_{t-1}} q^{\binom{n_0}{2}+\cdots+\binom{n_{t-1}}{2}}}{(q)_{n_0}\ldots(q)_{n_{t-1}}}.$$

By picking out the coefficient of x^{2tk} on both sides we find that

$$\frac{1}{(q^t;q^t)_{2k}} = \sum_{n_0+\cdots+n_{t-1}=2tk} \frac{\omega^{n_1+2n_2+\cdots+(t-1)n_{t-1}} q^{\binom{n_0}{2}+\cdots+\binom{n_{t-1}}{2}-t\binom{2k}{2}}}{(q)_{n_0}\ldots(q)_{n_{t-1}}}.$$

By letting $m_i = n_i - 2k$ for $0 \leq i \leq t-1$ we find that

$$\frac{1}{(q^t;q^t)_{2k}} = \sum_{m_0+\cdots+m_{t-1}=0} \frac{\omega^{m_1+2m_2+\cdots+(t-1)m_{t-1}} q^{\frac{1}{2}(m_0^2+\cdots+m_{t-1}^2)}}{(q)_{m_0+k}\ldots(q)_{m_{t-1}+k}}.$$

Letting $k \to \infty$ gives

$$\frac{(q)_\infty^t}{(q^t;q^t)_\infty} = \sum_{m_0+\cdots+m_{t-1}=0} \omega^{m_1+2m_2+\cdots+(t-1)m_{t-1}} q^{\frac{1}{2}(m_0^2+\cdots+m_{t-1}^2)},$$

which is (2.30). The proof of (2.31) is analogous and begins with the observation that

$$(x;q)_\infty = (x;q^t)_\infty (xq;q^t)_\infty (xq^2;q^t)_\infty \ldots (xq^{t-1};q^t)_\infty. \quad \square$$

Putting $t=3$ in (2.30) gives

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{3n})} = \sum_{m_0+m_1+m_2=0} \omega^{m_1-m_2} q^{\frac{1}{2}(m_0^2+m_1^2+m_2^2)}$$
$$= \sum_{m_1,m_2=-\infty}^{\infty} \omega^{m_1-m_2} q^{m_1^2+m_1m_2+m_2^2},$$

which is (2.24) by (2.13). Equation (2.25) can be proved from (2.31) with $t=3$ or by applying Jacobi's imaginary transformation.

We now give a proof of (2.15c) the cubic analogue of Jacobi's identity (1.19c). From (2.22) and (2.23) we have

$$b(q) = a(q^3) - c(q^3). \tag{2.33}$$

The key observation is that, in view of (2.12) and (2.15), the right side of (2.33) gives the 3-dissection of the series expansion of $b(q)$; i.e. the q-series expansion of $b(q)$ has no terms of the form q^{3n+2}, $a(q^3)$ gives all terms of the form q^{3n} and $-c(q^3)$ gives all terms of the form q^{3n+1}. Hence we have

$$\begin{aligned} b(q)b(\omega q)b(\omega^2 q) &= [a(q^3)-c(q^3)][a(q^3)-\omega c(q^3)][a(q^3)-\omega^2 c(q^3)] \\ &= a^3(q^3) - c^3(q^3). \end{aligned} \tag{2.34}$$

However it can be easily shown from (2.24) that

$$b(q)b(\omega q)b(\omega^2 q) = b^3(q^3) \tag{2.35}$$

and (2.15c) follows.

We now discuss the $s = \frac{1}{6}$ case of Corollary 1.45. Using Pfaff's transformation (1.43) and (2.16) we have

$$\begin{aligned} a(q) &= {}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3(q)}{a^3(q)}) \\ &= \frac{a(q)}{b(q)}\, {}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; 1 - \frac{a^3(q)}{b^3(q)}) \end{aligned} \tag{2.36}$$

so that

$${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; 1 - \frac{a^3(q)}{b^3(q)}) = b(q), \tag{2.37}$$

which is a striking analogue of (2.16). It is interesting to note that we may parametrize the hypergeometric function ${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; \cdot)$ neatly in terms of the Dedekind eta-function $\eta(\tau)$. As usual we define

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad (q = \exp(2\pi i \tau)).$$

Then using (2.15c), (2.24) and (2.25) we may find that (2.37) may be written as

$${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1; -27\frac{\eta^{12}(3\tau)}{\eta^{12}(\tau)}) = \frac{\eta^3(\tau)}{\eta(3\tau)}. \tag{2.38}$$ [1]

3. z-analogues and a cubic analogue of the incomplete elliptic integral of the first kind

The four Jacobian theta functions are

$$\theta_1(z, q) := -i \sum_{n=-\infty}^{\infty} (-1)^n z^{n+\frac{1}{2}} q^{(n+\frac{1}{2})^2}, \tag{3.1}$$

$$\theta_2(z, q) := \sum_{n=-\infty}^{\infty} z^{n+\frac{1}{2}} q^{(n+\frac{1}{2})^2}, \tag{3.2}$$

$$\theta_3(z, q) := \sum_{n=-\infty}^{\infty} z^n q^{n^2}, \tag{3.3}$$

$$\theta_4(z, q) := \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2}. \tag{3.4}$$

[1] Oliver Atkin first alerted me to this identity. This subsequently led to considering its relationship to the cubic analogue of the AGM and to discovering Corollary 1.45.

We observe that $\theta_1(1,q)=0$, $\theta_2(1,q)=\theta_2(q)$, $\theta_3(1,q)=\theta_3(q)$, and $\theta_4(1,q)=\theta_4(q)$. It is well-known that many identities involving the $\theta_i(q)$ have generalizations in terms of the $\theta_i(z,q)$. For instance, a generalization of Jacobi's identity (1.19c), which was important in the analysis of the AGM, is

$$\theta_3^2(q)\,\theta_3^2(z,q)=\theta_2^2(q)\,\theta_2^2(z,q)+\theta_4^2(q)\,\theta_4^2(z,q). \tag{3.5}$$

It would seem natural to look for z-analogues of our functions $a(q)$, $b(q)$, $c(q)$ which were needed to parametrize the cubic analogue of the AGM. In [HGB] we defined

$$a(z,q):=\sum_{m,n=-\infty}^{\infty} z^{m-n}q^{m^2+mn+n^2}, \tag{3.6}$$

$$b(z,q):=\sum_{m,n=-\infty}^{\infty} \omega^{m-n}z^{n}q^{m^2+mn+n^2}, \tag{3.7}$$

$$c(z,q):=\sum_{m,n=-\infty}^{\infty} z^{m-n}q^{(m+\frac13)^2+(m+\frac13)(n+\frac13)+(n+\frac13)^2}. \tag{3.8}$$

Then

$$a(zq,q):=z^{-2}q^{-1}\,a(z,q), \tag{3.9}$$

$$b(zq^3,q):=z^{-2}q^{-3}\,b(z,q), \tag{3.10}$$

$$c(zq,q):=z^{-2}q^{-1}\,c(z,q). \tag{3.11}$$

This means that each of our functions $a(z,q)$, $b(z,q)$, $c(z,q)$ is quasi doubly-periodic [WW, p. 463] when z is replaced by $\exp(2iz)$. This is analogous to the situation for the $\theta_i(z,q)$. Hence we may use the techniques of the theory of elliptic functions to prove identities. As in the $z=1$ case we found that $b(z,q)$ and $c(z,q)$ have product forms.

(3.12)
$$b(z,q):=\prod_{n=1}^{\infty}\frac{(1-zq^n)(1-z^{-1}q^n)(1-q^n)(1-q^{3n})}{(1-zq^{3n})(1-z^{-1}q^{3n})},$$

(3.13)
$$c(z,q):=q^{1/3}(1+z+z^{-1})\prod_{n=1}^{\infty}\frac{(1-z^3q^{3n})(1-z^{-3}q^{3n})(1-q^n)(1-q^{3n})}{(1-zq^n)(1-z^{-1}q^n)}.$$

We found a nice z-analogue of the cubic identity (2.15c).

$$a^3(z,q)=b^2(q)\,b(z^3,q)+c^3(z,q). \tag{3.14}$$

Another nice identity we found was

$$a(z,q)\,a(z^2,q^2)=b(z^3,q)\,b(q^2)+c(z,q)\,c(z^2,q^2), \tag{3.15}$$

which is a z-analogue of the following identity due to Ramanujan:

$$a(q)\,a(q^2)=b(q)\,b(q^2)+c(q)\,c(q^2). \tag{3.16}$$

We amusingly compare this with the cubic modular equation for the theta functions.

$$\theta_3(q)\,\theta_3(q^3) = \theta_2(q)\,\theta_2(q^3) + \theta_4(q)\,\theta_4(q^3). \tag{3.17}$$

It is well-known that the inverse function of the incomplete elliptic integral of the first kind [WW, Chap. XXII] can be identified in terms of the Jacobian theta functions, and that this inverse function has a nice Fourier series expansion. On the bottom of page 257 of Ramanujan's second notebook [R] there is an identity (see (3.20) below) which gives the Fourier series of the inverse function of a cubic analogue of the incomplete elliptic integral of the first kind. Our proof of this identity depends crucially on identities for $b(z,q)$ and certain other z-analogues.

Theorem 3.18. *([BeBhG]) For $0 \le q < 1$, $0 \le \varphi \le \pi/2$, let $a = a(q)$, $c = c(q)$ and $h = \frac{c^3}{a^3}$ and define*

$$a\,z = \int_0^{\varphi} {}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; h\sin^2 t)\,dt. \tag{3.19}$$

Then

$$\varphi = z + 3\sum_{n=1}^{\infty} \frac{\sin(2nz)\,q^n}{n(1+q^n+q^{2n})}. \tag{3.20}$$

Before giving the idea of the proof we consider a classical analogue. Equation (3.20) is reminiscent of the Fourier series expansion of the Jacobian elliptic function $\operatorname{am} u$ [WW, p. 511]. Indeed, we have the following classical analogue.

Theorem 3.21. *([S2]) Let $a = \theta_3^2(q)$, $c = \theta_2^2(q)$ and $k^2 = \frac{c^2}{a^2}$ and define*

$$a\,z = \int_0^{\varphi} {}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; k^2\sin^2 t)\,dt. \tag{3.22}$$

Then

$$\varphi = z + 2\sum_{n=1}^{\infty} \frac{\sin(2nz)\,q^n}{n(1+q^n)}. \tag{3.23}$$

Proof. We have

$${}_2F_1(\frac{1}{2}+\frac{1}{2}a, \frac{1}{2}-\frac{1}{2}a; \frac{1}{2}; \sin^2 z) = \frac{\cos az}{\cos z} \qquad ([\text{E, p. 101, Eq(11)}]), \tag{3.24}$$

so that

$${}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; x) = \frac{1}{\sqrt{1-x}} \qquad (|x| < 1).$$

Hence,

$$\begin{aligned} a\,z = z\,\vartheta_3^2 &= \int_0^{\varphi} {}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; k^2\sin^2\theta)\,d\theta \qquad (\text{where } \vartheta_3 = \theta_3(q)) \\ &= \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \end{aligned}$$

by letting $t = \sin\theta$. This last integral is the incomplete elliptic integral of the first kind. Thus, from the definition of $\mathrm{sn}\,(z,k)$ ([WW, p. 492]) we have

$$\sin\varphi = \mathrm{sn}\,(\vartheta_3^2 z).$$

Now,

$$\begin{aligned}\frac{d\varphi}{dz} &= \varphi_3^2 \frac{\mathrm{cn}\,(\varphi_3^2 z)\,\mathrm{dn}\,(\varphi_3^2 z)}{\cos\varphi} \qquad \text{(by [WW, Eq(I) p. 492])}\\ &= \varphi_3^2\,\mathrm{dn}\,(\varphi_3^2 z) \qquad \text{(by [WW, Eq(II) p. 493])}\\ &= 1 + 4\sum_{n=1}^{\infty} \frac{\cos(2nz)\,q^n}{1+q^n} \qquad \text{([WW, p. 511]).}\end{aligned}$$

The identity (3.23) follows by integrating with respect to z. $\square$

The idea of the proof of Theorem 3.18. Putting $a = \frac{1}{3}$ and $x = \sin z$ in (3.24) we find

$$(3.25)\qquad {}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x^2) = (1-x^2)^{-1/2}\cos(\frac{1}{3}\sin^{-1} x),$$

which is an algebraic function. Indeed, if we let $S = S(x) := {}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x^2)$ then we find that

$$(3.26)\qquad 4S^3(1-x^2) - 3S - 1 = 0.$$

Thus, if we define

$$(3.27)\qquad \Phi(\theta) := \theta + 3\sum_{n=1}^{\infty} \frac{\sin(2n\theta)\,q^n}{n(1+q^n+q^{2n})},$$

then in order to prove (3.20) we need to show that

$$(3.28)\qquad \sin^2\Phi = \frac{1}{4h}\left[4 - \frac{1}{a^3}\left(\frac{d\Phi}{d\theta}\right)^3 - \frac{3}{a^2}\left(\frac{d\Phi}{d\theta}\right)^2\right].$$

Letting $\Psi = \Psi(\theta)$ be the right side of (3.28) we find that (3.28) will follow from

$$(3.29)\qquad \left[\frac{d\Psi}{d\theta}\right]^2 = 4\Psi(1-\Psi)\left[\frac{d\Phi}{d\theta}\right]^2.$$

It turns out that $\frac{d\Phi}{d\theta}$ has an analytic continuation to an elliptic function (with $q = \exp(2\pi i\tau)$ as usual). Hence (3.29) can be proved using the techniques of elliptic functions. See [BeBhG] for the details.

Li-Chien Shen [S1] has found a proof of Theorem 3.18 that involves only the classical elliptic functions. In [S2], he has also found a partial analogue for the hypergeometric function ${}_2F_1(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \alpha\sin^2 t)$.

Theorem 3.30. *(*[S2]*) Let $a = \sqrt{\frac{\theta_3^4(q)+\theta_4^4(q)}{2}}$, $c = \frac{\theta_2^2(q)}{\sqrt{2}}$ and $\alpha = \frac{c^4}{a^4}$ and define*

$$(3.31)\qquad a\,z = \int_0^{\varphi} {}_2F_1(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \alpha\sin^2 t)\,dt.$$

Then

$$(3.32)\qquad \cos\varphi = cn\,(u,k)\,dn\,(u,k),$$

where $u = \theta_3^2(q^2)z$ and $k = \frac{\theta_2^2(q^2)}{\theta_3^2(q^2)}$.

4. Hypergeometric analogues of the AGM

In this section we make explicit the means M_1 and M_2 of Theorem 1.39. Full details and proofs appear in [BB2], [BBG2].

Quadratic $s = 0$ **(Gauss).** *If* $M_1(a,b) := \frac{a+b}{2}$ *and* $M_2(a,b) := \sqrt{ab}$ *then*

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2)}.$$

Cubic $s = 0$ [BBG2]**.** *If*

$$\begin{aligned} U = U(a,b) &:= \sqrt{a^2 - \sqrt[3]{4a^2b^2(a^2-b^2)}} \\ M_1(a,b) &:= \frac{U + \sqrt{3a^2 - U^2 + (4ab^2 - 2a^3)/U}}{3} =: A \\ M_2(a,b) &:= M_1(b,a) = \frac{bA + ab}{3A - a}, \end{aligned}$$

then

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2)}.$$

Quadratic $s = \frac{1}{6}$ [BBG2]**.** *If*

$$\begin{aligned} M_1(a,b) &:= \frac{\sqrt[3]{2b^3 - a^3 + 2\sqrt{b^6 - a^3b^3}} + \sqrt[3]{2b^3 - a^3 - 2\sqrt{b^6 - a^3b^3}}}{2}, \\ M_2(a,b) &:= \frac{\sqrt[3]{b^3 + \sqrt{b^6 - a^3b^3}} + \sqrt[3]{b^3 - \sqrt{b^6 - a^3b^3}}}{2}, \end{aligned}$$

then

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3)}.$$

Cubic $s = \frac{1}{6}$ [BB3]**,**[BB4]**,**[BBG1]**,**[BBG2]**.** *If*

$$\begin{aligned} M_1(a,b) &:= \frac{a+2b}{3}, \\ M_2(a,b) &:= \sqrt[3]{\frac{b(a^2+ab+b^2)}{3}}, \end{aligned}$$

then

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3)}.$$

Quadratic $s = \frac{1}{4}$ **[BB2].** *If*

$$\begin{aligned} M_1(a,b) &:= \frac{a+3b}{4}, \\ M_2(a,b) &:= \sqrt{\frac{b(a+b)}{2}}, \end{aligned}$$

then

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1^2(\frac{1}{4}, \frac{3}{4}; 1; 1-x^2)}.$$

Cubic $s = \frac{1}{4}$ **[BBG2].** *If*

$$\begin{aligned} M_1(a,b) &:= \sqrt{\frac{a^2b + 3B(b^2 - B^2)}{b}}, \\ B := M_2(a,b) &:= \frac{U + \sqrt{3b^2 - U^2 - (2a^2b)/U}}{3}, \end{aligned}$$

where
$$U := \sqrt{b^2 + \sqrt[3]{b^2(a^4 - b^4)}},$$

then

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; 1-x^4)}.$$

Quadratic $s = \frac{1}{3}$ **[BBG2].** *If*

$$\begin{aligned} A &:= M_1(a,b) \\ &:= \Bigg(\frac{5}{16}\big(a^6 + 8b^6 - 8a^3b^3 + 4b^{3/2}(b^3 - a^3)^{1/2}a^3 - 8b^{9/2}(b^3 - a^3)^{1/2}\big)^{1/3} \\ &\quad + \frac{5}{16}\big(a^6 + 8b^6 - 8a^3b^3 - 4b^{3/2}(b^3 - a^3)^{1/2}a^3 + 8b^{9/2}(b^3 - a^3)^{1/2}\big)^{1/3} \\ &\quad + \frac{3}{8}a^2\Bigg)^{1/2}, \\ M_2(a,b) &:= \left(\frac{22b^3A^2 - 2a^2b^3 - 11a^3A^2 - 22a^2A^3 + a^5 + 32A^5}{4(16A^2 - 11a^2)}\right)^{1/3}, \end{aligned}$$

then

$$M_1 \otimes M_2(1,x) = \frac{1}{{}_2F_1^2(\frac{1}{6}, \frac{5}{6}; 1; 1-x^3)}.$$

We now sketch why the four values $s = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$ occur in Theorem 1.39 and how to discover symbolically the modular forms that are involved in the parametrization of the corresponding hypergeometric function. The approach we take is very classical. It is well-known that modular functions arise by inverting the ratio of solutions to certain hypergeometric differential equations. See [Gr]. Accordingly, we let

(4.1)
$$u_1(x) := {}_2F_1(\frac{1}{2} - s, \frac{1}{2} + s; 1; x)$$

and

(4.2)
$$\begin{aligned} u_2(x) &:= \left(-\frac{\cos(\pi s)}{\pi}\right) \ln x \, u_1(x) \\ &+ \frac{\cos(\pi s)}{\pi} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-s)_n(\frac{1}{2}+s)_n}{(n!)^2} \left(2\psi(n+1) - \psi(\tfrac{1}{2}+s+n) - \psi(\tfrac{1}{2}-s+n)\right) x^n \\ &= {}_2F_1(\frac{1}{2} - s, \frac{1}{2} + s; 1; 1-x) \end{aligned}$$

for $0 < x < 1$. These form a fundamental set of solutions to the hypergeometric differential equation

$$x(1-x)\,y'' + (1-2x)\,y' - (\tfrac{1}{2}-s)(\tfrac{1}{2}+s)\,y = 0.$$

We let

$$\begin{aligned} q(x) &:= \exp\left(-\frac{\pi}{\cos(\pi s)} \frac{u_2(x)}{u_1(x)}\right) \\ &= x\,\exp(-G(x)), \end{aligned} \tag{4.3}$$

where

(4.4)
$$G(x) := \frac{\sum_{n=0}^{\infty} \frac{(\frac{1}{2}-s)_n(\frac{1}{2}+s)_n}{(n!)^2} \left(2\psi(n+1) - \psi(\frac{1}{2}+s+n) - \psi(\frac{1}{2}-s+n)\right) x^n}{u_1(x)}.$$

Now $q(x)$ is analytic and one-to-one in a neighbourhood of $x = 0$. Hence we let $X_s(q)$ denote the inverse function which is analytic in a neighbourhood of $q = 0$. If we let

$$q = \exp\left(\frac{i\pi\tau}{\cos(\pi s)}\right) \tag{4.5}$$

and consider X_s as a function of τ then we find that

$$X_s(\tau + \lambda) = X_s(\tau) \tag{4.6}$$

where

$$\lambda := 2\,\cos(\pi s), \tag{4.7}$$

and

$$X_s\left(\frac{-1}{\tau}\right) = 1 - X_s(\tau). \tag{4.8}$$

Now we consider the Hecke group $\mathfrak{G}(\lambda)$ ([O, p. *xiii*]) which is the group generated by

$$\tau \longmapsto \tau + \lambda, \qquad \tau \longmapsto -1/\tau.$$

It follows that $g(\tau) := X_s(\tau)(1 - X_s(\tau))$ is a $\mathfrak{G}(\lambda)$-invariant function. Now we define

$$a(\tau) = a_s(\tau) := F_s(X_s(\tau)). \tag{4.9}$$

We have

$$a(\tau + \lambda) = a(\tau), \tag{4.10}$$

and by using (4.1), (4.2), (4.3) and (4.5) it can be shown that

$$a\left(\frac{-1}{\tau}\right) = \left(\frac{\tau}{i}\right) a(\tau). \tag{4.11}$$

Hence $a(\tau)$ is a modular form of dimension -1 (or weight $k = 1$) and multiplier $C = 1$ for $\mathfrak{G}(\lambda)$. Following [O, p. *xiii*] we denote the space of such forms by $\mathfrak{M}(\lambda, k, C)$. From [O, Chapter I], $\mathfrak{M}(\lambda, k, C)$ is finite dimensional when $0 \le \lambda \le 2$ and in addition $\mathfrak{M}(\lambda, k, C) \ne 0$ when $\lambda = 2\cos(\pi s)$ and $s = 0$ or $s = \frac{1}{n}$ and n is an integer greater than 2. The problem is to find algebraic relations between the functions $a(\tau)$, $a(p\tau)$, $X_s(\tau)$ and $X_s(p\tau)$ for fixed $p > 1$. Such relations with luck lead to p-th order mean iterations. For $s = 0$, $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$ it turns out that $a(\tau)$ or some integral power of $a(\tau)$ corresponds to a modular form on a certain congruence subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We define

$$b(\tau) := b_s(\tau) = [1 - X_s(\tau)]^{(\frac{1}{2}-s)}\, a_s(\tau) \tag{4.12}$$

so that

$$X_s(\tau) = 1 - \left(\frac{b_s(\tau)}{a_s(\tau)}\right)^{1/(\frac{1}{2}-s)}. \tag{4.13}$$

Now suppose $s = 0$, $\frac{1}{6}$, $\frac{1}{4}$, or $\frac{1}{3}$. The functions $a_s(\tau)$, $b_s(\tau)$, $a_s(p\tau)$, $b_s(p\tau)$ or some integral power of these functions will be modular forms on some congruence subgroup for fixed $p > 1$. Now writing as functions of q, the forms $a_s(q)$, $b_s(q)$ parametrize the hypergeometric function $F_s(\cdot)$ by construction. To find quadratic and cubic means we find for $p = 2, 3$ homogeneous relations (or modular equations) of the form

$$P(a_s(q), b_s(q), a_s(q^p), b_s(q^p)) \equiv 0.$$

Solving these relations gives explicit means. Such relations always exist. This is seen as follows. Let P have degree k. Then it is well-known that

The dimension of the space of modular forms (above) of weight $k \sim c_1\, k$
(for some nonzero constant c_1, see [CO])

and

the number of monomials $a_s^{k_1}(q)\, a_s^{k_2}(q^p) b_s^{k-k1-k2}(q) \sim \dfrac{1}{2}k^2$.

Hence there will always be a relation for large enough k. The relations may be proved by identifying the functions $a_s(q)$, $b_s(q)$ from their q-series expansions and then using the theory of modular forms. Alternatively, each relation may be proved by rewriting it as a hypergeometric function identity and using the techniques of differential equations to prove it. All of this can be done symbolically, and has been carried out in [BBG2] where more details are given.

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