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THE CRANK OF PARTITIONS MOD 8, 9 AND 10

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Abstract. Recently new combinatorial interpretations of Ramanujan's partition congruences modulo 5, 7 and 11 were found. These were in terms of the crank. A refinement of the congruence modulo 5 is proved. The number of partitions of $5n+4$ with even crank is congruent to 0 modulo 5. The residue of the even crank modulo 10 divides these partitions into five equal classes. Other relations for the crank modulo 8, 9 and 10 are also proved. The dissections of certain generating functions associated with these results are calculated. All of the results are proved by elementary methods.

1. Introduction.

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan [**Ra1**] discovered and later proved

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Dyson [**D1**] discovered some remarkable combinatorial interpretations of (1.1) and (1.2). Dyson defined the *rank* of a partition as the largest part minus the number of parts. Let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Dyson conjectured that

$$N(k,5,5n+4) = \tfrac{1}{5}p(5n+4) \qquad (0 \le k \le 4), \tag{1.4}$$

and

$$N(k,7,7n+5) = \tfrac{1}{7}p(7n+5) \qquad (0 \le k \le 6). \tag{1.5}$$

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See also [**D3**]. (1.4) and (1.5) were later proved by Atkin and Swinnerton-Dyer [**A-S**]. As pointed out in [**G2**], (1.4) follows from an identity [**G2**; (1.31)] from Ramanujan's "lost" notebook [**Ra2**]. See also [**G3**]. This identity is related to the mock theta conjectures [**A-G1**] which were recently proved by Hickerson [**Hic1**]. Hickerson [**Hic2**] has also found connections between the seventh order mock theta functions and the rank mod 7.

The statement corresponding to (1.4) or (1.5) for the partitions of $11n+6$ is false. However, recently combinatorial interpretations of (1.3) have been found. The first such interpretation is in terms of vector partitions [**G2**].

A *vector partition* is an ordered triple of partitions

$$\vec{\pi} = (\pi_1, \pi_2, \pi_3),$$

where π_1 is a partition into distinct parts and π_2, π_3 are unrestricted partitions. Let $\sharp(\pi_j)$ be the number of parts of π_j and $\sigma(\pi_j)$ be the sum of the parts of π_j. Then for $\vec{\pi}$ the sum of parts, s, a weight, ω, and a crank, r, are defined by

$$s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3), \tag{1.6}$$

$$\omega(\vec{\pi}) = (-1)^{\sharp(\pi_1)}, \tag{1.7}$$

$$r(\vec{\pi}) = \sharp(\pi_2) - \sharp(\pi_3). \tag{1.8}$$

We define the weighted sum

$$N_V(m,n) = \sum_{\vec{\pi}} \omega(\vec{\pi}), \tag{1.9}$$

where the sum is taken over all vector partitions $\vec{\pi}$ with sum n and crank m. $N_V(m,n)$ has the following generating function:

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}. \tag{1.10}$$

Let $N_V(k,t,n)$ denote the sum of $N_V(m,n)$ over all m congruent to k modulo t. The main result of [**G2**] was the following new interpretations of (1.1)-(1.3):

$$N_V(k,5,5n+4) = \tfrac{1}{5} p(5n+4) \qquad (0 \le k \le 4), \tag{1.11}$$

$$N_V(k,7,7n+5) = \tfrac{1}{7} p(7n+5) \qquad (0 \le k \le 6), \tag{1.12}$$

$$N_V(k,11,11n+6) = \tfrac{1}{11} p(11n+6) \qquad (0 \le k \le 10). \tag{1.13}$$

The clue to (1.10) came from another identity [**G2**; (1.30)] from Ramanujan's "lost" notebook.

In [**G3**] we proved that all but one of the coefficients $N_V(m,n)$ are nonnegative. This led us, in [**A-G2**], to interpret (1.11)-(1.13) solely in terms of partitions. For a partition π, let $\lambda(\pi)$ denote the largest part, $\nu(\pi)$ denote the number of ones, and $\mu(\pi)$ denote the number of parts of π larger than $\nu(\pi)$. The crank is given by

$$c(\pi) = \begin{cases} \lambda(\pi), & \text{if } \nu(\pi) = 0, \\ \mu(\pi) - \nu(\pi), & \text{if } \nu(\pi) > 0. \end{cases}$$

Let $M(m, n)$ denote the number of (ordinary) partitions of n with crank m. The main result of [**A-G2**] was

$$M(m, n) = N_V(m, n) \qquad \text{for all } n > 1. \tag{1.14}$$

A combinatorial proof of (1.14) has been found by Dyson [**D4**]. Very recently in [**G-S**], a new, elementary and uniform proof of the congruences (1.1)–(1.3) was found. This has led, in [**G-K-S**], to new combinatorial interpretations.

For $n > 1$ we let $M(k, t, n)$ denote the number of partitions of n with crank congruent to k modulo t. Thus, by (1.14), we have

$$M(k, t, n) = N_V(k, t, n) \qquad \text{for all } n > 1. \tag{1.15}$$

To simplify the statement of our results we amend the definition of M so that (1.15) holds for all $n \geq 0$. We prove a refinement of the congruence (1.1):

$$M(k, 2, 5n + 4) \equiv 0 \pmod 5 \qquad \text{for } k = 0,\ 1. \tag{1.16}$$

We also prove combinatorial interpretations of (1.16).

$$M(2k, 10, 5n + 4) = \tfrac{1}{5} M(0, 2, 5n + 4) \qquad (0 \leq k \leq 4), \tag{1.17}$$

$$M(2k + 1, 10, 5n + 4) = \tfrac{1}{5} M(1, 2, 5n + 4) \qquad (0 \leq k \leq 4). \tag{1.18}$$

In other words, the residue of the crank modulo 10 divides both the partitions of $5n+4$ with even crank and those with odd crank each into five equal classes. We note that (1.11) follows from (1.17), (1.18) in view of (1.14). The result analogous to (1.16), (1.17) or (1.18) for the partitions of $7n + 5$ or $11n + 6$ does not hold.

In [**G2**] we proved many other relations besides (1.11)–(1.13) like

$$N_V(0, 5, 5n + 1) + N_V(1, 5, 5n + 1) = 2N_V(2, 5, 5n + 1), \tag{1.19}$$

for the crank of vector partitions mod 5, 7 and 11. Analogous relations for the rank of partitions mod 5 and 7 also hold. These were also observed by Dyson and proved by Atkin and Swinnerton-Dyer. In this paper we prove analogous relations for the crank of partitions (or vector partitions via (1.14)) mod 8, 9 and 10.

Let $t > 1$ be an integer. For a power series in q, $P(q)$, we define the *t-dissection of P* as

$$P(q) = \sum_{k=0}^{t-1} q^k P_k(q^t); \tag{1.20}$$

i.e. $q^k P_k(q^t)$ contains those terms of P in which the exponent of q is congruent to k modulo t. The functions P_k are called the *elements* of the dissection. We define

$$F_t(q) := \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}, \tag{1.21}$$

where $\zeta = exp(2\pi i/t)$. The function F_t is related to the crank in view of (1.10) and (1.14). In fact we have

$$F_t(q) = \sum_{n\geq 0} \left(\sum_{k=0}^{t-1} \zeta^k N_V(k,t,n) \right) q^n, \qquad \text{(cf. [}\mathbf{G2}\text{; (2.5)]).} \tag{1.22}$$

In [**G2**] we calculated the t-dissection of F_t for $t = 5$, 7 and 11. The 5-dissection of F_5 appears in Ramanujan's "lost" notebook. See [**G2**; (1.30)]. In each of the three t-dissections one element is missing. The results (1.11)–(1.13) follow in view of (1.22). For the other elements certain powers of ζ are missing and this leads to relations like (1.19). For $t = 5$ and 7 each element of the t-dissection is a single infinite product. This was also observed by Hirschhorn for $t = 11$ [**Hir3**] and $t = 6$ [**Hir4**].

In §2 we state the nice t'-dissections of F_t where $t' > 1$, $t'|t$ and $t \leq 11$. For the 3-dissection of F_9, the 2-dissection of F_8 and the 5-dissection of F_{10} we find that each element of the dissection is a single infinite product and that certain powers of ζ are missing. This leads to the following relations:

$$M(1,8,2n) = M(3,8,2n), \tag{1.23}$$
$$M(0,8,2n+1) + M(1,8,2n+1) = M(3,8,2n+1) + M(4,8,2n+1), \tag{1.24}$$

$$M(1,9,3n) = M(2,9,3n) = M(4,9,3n), \tag{1.25}$$
$$M(2,9,3n+1) = M(4,9,3n+1), \tag{1.26}$$
$$M(0,9,3n+1) + M(1,9,3n+1) = M(2,9,3n+1) + M(3,9,3n+1), \tag{1.27}$$
$$M(0,9,3n+2) = M(3,9,3n+2), \quad M(1,9,3n+2) = M(4,9,3n+2) \tag{1.28}$$

$$M(1,10,5n) = M(3,10,5n), \quad M(2,10,5n) = M(4,10,5n), \tag{1.29}$$
$$\begin{aligned} M(0,10,5n+1) + M(1,10,5n+1) &= M(2,10,5n+1) + M(3,10,5n+1) \\ &= M(4,10,5n+1) + M(5,10,5n+1), \end{aligned} \tag{1.30}$$
$$M(0,10,5n+2) = M(4,10,5n+2), \quad M(1,10,5n+2) = M(5,10,5n+2) \tag{1.31}$$
$$\begin{aligned} M(0,10,5n+3) - M(3,10,5n+3) &= M(2,10,5n+3) - M(5,10,5n+3) \\ &= M(4,10,5n+3) - M(1,10,5n+3). \end{aligned} \tag{1.32}$$

In §3 we carry out the 3-dissection of F_9 in detail. The proof depends on Macdonald's [**M**] identity for the root system A_2. In §4 we derive the 5-dissection of F_{10}. This depends on Rødseth's [**Rø**] 5-dissection of the generating function for partitions into distinct parts. *Notation.*

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n \qquad \text{where } |q|<1.$$

2. The t-dissections.

In this section we state the "nice" t'-dissections of F_t where $t'>1$, $t'|t$ and $t\le 11$. By nice we mean that the elements are single infinite θ-type products. We delay the proofs of the $t=9$ and $t=10$ cases until later sections. Throughout this section $\zeta_t = exp(2\pi i/t)$. Following [**Hic1**] we give some notation for θ-functions. If $|q|<1$ and $x\ne 0$ then

$$\begin{aligned} j(x,q) &:= \prod_{n=1}^{\infty}(1-q^n)(1-xq^{n-1})(1-x^{-1}q^n) \\ &= \sum_{n=-\infty}^{\infty}(-1)^n x^n q^{\binom{n}{2}}, \end{aligned} \tag{2.1}$$

by Jacobi's triple product identity [**A**; (2.2.10)]. If m is a positive integer and a is an integer we define

$$J_{a,m} := j(q^a, q^m), \tag{2.2}$$
$$\overline{J}_{a,m} := j(-q^a, q^m), \tag{2.3}$$
$$\begin{aligned} J_m &:= j(q^m, q^{3m}) = \prod_{n=1}^{\infty}(1-q^{nm}) \\ &= \sum_{n=-\infty}^{\infty}(-1)^n q^{mn(3n-1)/2}. \end{aligned} \tag{2.4}$$

The t-dissections of F_t for $t=2$, 3, 4 do not appear to be nice. They may be calculated explicitly using (2.1). To calculate the 2-dissection of F_4 we need

Lemma (2.5) (Hirschhorn). *We have*

$$J_1 = J_2 {J_4}^{-1}\left\{\overline{J}_{6,16} - q\overline{J}_{2,16}\right\}. \tag{2.6}$$

Proof.

$$\begin{aligned} J_1 &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{4n})}\prod_{n=1}^{\infty}(1-q^{4n-3})(1-q^{4n-1})(1-q^{4n}) \\ &= J_2 {J_4}^{-1} j(q,q^4), \end{aligned}$$

and the result follows by (2.1). $\square$

Hence

$$F_4 = J_1 J_2 {J_4}^{-1} \tag{2.7}$$
$$= {J_2}^2 {J_4}^{-2}\left\{\overline{J}_{6,16} - q\overline{J}_{2,16}\right\} \qquad \text{(by (2.6))}.$$

This, however, does not yield any relations for the crank mod 4.

The 5-dissection of F_5 appears in Ramanujan's "lost" notebook and is given in [**G2**; (1.30)]:

(2.8)
$$F_5 = J_{25}{J_5}^{-1}\left\{{J_{10,25}}^2 {J_{5,25}}^{-1} + (\zeta_5 - 1 + \zeta_5^{-1})qJ_{10,25} \right.$$
$$\left. -(\zeta_5 + 1 + \zeta_5^{-1})q^2 J_{5,25} - (\zeta_5 + \zeta_5^{-1})q^3 {J_{5,25}}^2 {J_{10,25}}^{-1}\right\}.$$

The 2-, 3- and 6-dissections of F_6 follow easily from (2.4) and Lemma (2.5).

$$F_6 = \left(J_2 {J_6}^{-1}\right) J_3 \tag{2.9}$$
$$= \left(J_2 {J_6}^{-1}\right)\left(J_6 {J_{12}}^{-1}\right)\left(\overline{J}_{18,48} - q^3\overline{J}_{6,48}\right)$$
$$= J_2 {J_{12}}^{-1}\left(\overline{J}_{18,48} - q^3\overline{J}_{6,48}\right).$$

$$F_6 = \left(J_3 {J_6}^{-1}\right) J_2 \tag{2.10}$$
$$= \left(J_3 {J_6}^{-1}\right)\left(J_{24,54} - q^4 J_{6,54} - q^2 J_{12,54}\right).$$

$$F_6 = \left(J_3 {J_6}^{-1}\right) J_2 \tag{2.11}$$
$$= {J_{12}}^{-1}\left(\overline{J}_{18,48} - q^3\overline{J}_{6,48}\right)\left(J_{24,54} - q^4 J_{6,54} - q^2 J_{12,54}\right)$$
$$= {J_{12}}^{-1}\left(\overline{J}_{18,48}J_{24,54} + q^7\overline{J}_{6,48}J_{6,54} - q^2\overline{J}_{18,48}J_{12,54}\right.$$
$$\left. - q^3\overline{J}_{6,48}J_{24,54} - q^4\overline{J}_{18,48}J_{6,54} + q^5\overline{J}_{6,48}J_{12,54}\right).$$

This, however, leads to no relations for the crank mod 6.

The 7-dissection of F_7 does not appear in Ramanujan's "lost" notebook but it is given in [**G2**; Thm (5.1)]:

(2.12)
$$F_7 = {J_7}^{-1}\left({J_{21,49}}^2 + (\zeta_7 - 1 + \zeta_7^{-1})qJ_{14,49}J_{21,49} + (\zeta_7^2 + \zeta_7^{-2})q^2 {J_{14,49}}^2\right.$$
$$- (\zeta_7^2 + \zeta_7 + \zeta_7^{-1} + \zeta_7^{-2})q^3 J_{7,49}J_{21,49}$$
$$\left. - (\zeta_7 + \zeta_7^{-1})q^4 J_{7,49}J_{14,49} - (\zeta_7^2 + 1 + \zeta_7^{-2})q^6 {J_{7,49}}^2\right).$$

The 4-dissection of F_8 follows from the triple product identity (2.1).

(2.13)
$$F_8 = \prod_{n=1}^{\infty}(1-q^n)^2(1-iq^n)(1+iq^n)(1-\zeta_8^3 q^n)(1-\zeta_8^{-3}q^n)(1+q^n)(1-q^{8n})^{-1}$$
$$= J_4 {J_8}^{-1}\prod_{n=1}^{\infty}(1-\zeta_8^3 q^n)(1-\zeta_8^{-3}q^n)(1-q^n)$$
$$= J_4 {J_8}^{-1}(1-\zeta_8^3)^{-1} j(\zeta_8^3, q)$$
$$= J_4 {J_8}^{-1}\left(\overline{J}_{28,64} - (\zeta_8^3 + 1 + \zeta_8^{-3})q\overline{J}_{20,64}\right.$$
$$\left. - q^6\overline{J}_{4,64} + (\zeta_8^3 + 1 + \zeta_8^{-3})q^3\overline{J}_{12,64}\right).$$

The 2-dissection of F_8 follows from (2.13) and (2.1).

$$F_8 = J_4 {J_8}^{-1} \left(J_{6,16} - (\zeta_8^3 + 1 + \zeta_8^{-3}) q J_{2,16} \right). \tag{2.14}$$

The two relations (1.23), (1.24) for the crank of partitions mod 8 follow by the argument of [**G2**; p. 61].

The 3-dissection of F_9 is

$$\begin{aligned} F_9 = J_3 {J_{27}}^2 {J_9}^{-1} \big(&{J_{3,27}}^{-1} - (1 - \zeta_9 + \zeta_9^2 + \zeta_9^5) q {J_{6,27}}^{-1} \\ &+ (\zeta_9^2 - \zeta_9 - \zeta_9^4) q^2 {J_{12,27}}^{-1} \big). \end{aligned} \tag{2.15}$$

The proof is given in §3 and depends on Macdonald's [**M**] identity for the root system A_2. The relations (1.25)–(1.28) for the crank of partitions mod 9 follow by the argument of [**G2**; p. 61].

The 5-dissection of F_{10} is

(2.16)

$$\begin{aligned} F_{10} = {} & J_{25} J_{15,50} {J_{50}}^{-1} + (\zeta_{10}^2 - \zeta_{10}^3) q J_5 {J_{10}}^{-1} \overline{J}_{10,25} J_{50} {J_{25}}^{-1} \\ & + (\zeta_{10}^2 - \zeta_{10}^3) q^2 J_5 {J_{10}}^{-1} \overline{J}_{5,25} J_{50} {J_{25}}^{-1} + (1 - \zeta_{10}^2 + \zeta_{10}^3) q^3 J_{25} J_{5,50} {J_{50}}^{-1}. \end{aligned}$$

This identity is derived in §4. The relations (1.17), (1.18) and (1.29)–(1.32) for the crank of partitions mod 10 then follow from (2.16) and the relations for the crank mod 5 which follow from (2.8) (namely [**G2**; (1.27), (1.40)–(1.43)]). More details are given in §4.

The 11-dissection of F_{11} was calculated in [**G2**; §6]. This was subsequently simplified by Hirschhorn [**Hir3**] using Winquist's [**Wi**] identity. Winquist's identity is the B_2 case of the Macdonald identities [**M**]. See [**Hir2**] for a different extension of Winquist's identity.

(2.17)

$$\begin{aligned} F_{11} = {J_{121}}^2 \big(& {J_{11,121}}^{-1} + (\zeta_{11} - 1 + \zeta_{11}^{-1}) q {J_{22,121}}^{-1} {J_{33,121}}^{-1} J_{55,121} \\ & + (\zeta_{11}^2 + \zeta_{11}^{-2}) q^2 {J_{11,121}}^{-1} J_{33,121} {J_{44,121}}^{-1} \\ & + (\zeta_{11}^3 + 1 + \zeta_{11}^{-3}) q^3 {J_{11,121}}^{-1} J_{22,121} {J_{33,121}}^{-1} \\ & + (\zeta_{11}^4 + \zeta_{11}^2 + 1 + \zeta_{11}^{-2} + \zeta_{11}^{-4}) q^4 {J_{22,121}}^{-1} \\ & - (\zeta_{11}^4 + \zeta_{11}^2 + \zeta_{11}^{-2} + \zeta_{11}^{-4}) q^5 {J_{22,121}}^{-1} J_{44,121} {J_{55,121}}^{-1} \\ & + (\zeta_{11}^4 + \zeta_{11} + \zeta_{11}^{-1} + \zeta_{11}^{-4}) q^7 {J_{33,121}}^{-1} \\ & + (\zeta_{11}^4 + \zeta_{11}^3 + \zeta_{11} + \zeta_{11}^{-1} + \zeta_{11}^{-3} + \zeta_{11}^{-4}) q^{19} J_{11,121} {J_{44,121}}^{-1} {J_{55,121}}^{-1} \\ & - (\zeta_{11}^4 + 1 + \zeta_{11}^{-4}) q^9 {J_{44,121}}^{-1} - (\zeta_{11}^3 + \zeta_{11}^{-3}) q^{10} {J_{55,121}}^{-1} \big). \end{aligned}$$

3. The 3-dissection of F_9.

In this section we derive the 3-dissection of F_9 given in (2.15). Throughout this section $\zeta = \zeta_9 = exp(2\pi i/9)$. We write

(3.1)
$$\begin{aligned} F_9 &= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-\zeta q^n)(1-\zeta^{-1}q^n)} \\ &= \frac{\prod_{n=1}^{\infty}(1-q^n)^2(1-\zeta^2q^{n-1})(1-\zeta^{-2}q^n)(1-\zeta^4q^{n-1})(1-\zeta^{-4}q^n)(1-\zeta^6q^{n-1})(1-\zeta^{-6}q^n)}{(1-\zeta^2)(1-\zeta^4)(1-\zeta^6)\prod_{n=1}^{\infty}(1-q^{9n})} \end{aligned}$$

We outline our line of attack. The numerator of this last expression may be written as a double series by Theorem (3.5) below. By utilising the symmetry of the quadratic form that appears in the right side of (3.6) we may simplify our series. We find that each element of the 3-dissection of this simplified series is a special case of (3.6) and (2.15) will follow.

We need the A_2 case of the Macdonald identities [**M**] given below in Theorem (3.5). All dissections of F_t for $t \le 8$ needed only Jacobi's triple product identity (2.1) which is the A_1 case of the Macdonald identities. The 11-dissection of F_{11} depended on Winquist's identity which is the B_2 case of the Macdonald identities. See [**D2**] for some early history of these identities and see [**St1**], [**St2**] for an elementary treatment. A limiting case of the A_2 case goes back to Klein and Fricke [**K-F**; p. 373]. The proof we give is due to Richard Askey. We need a little lemma.

If we define

$$\epsilon_k := \begin{cases} 1 & \text{if } k \equiv 0 \pmod 3, \\ -1 & \text{if } k \equiv 1 \pmod 3, \\ 0 & \text{if } k \equiv 2 \pmod 3, \end{cases} \tag{3.2}$$

then we have

Lemma (3.3) ([A-S; p. 99]). *For $|q| < 1$, we have*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2-kn} = \epsilon_k q^{-k(k-1)/6} \prod_{n=1}^{\infty} (1-q^n). \tag{3.4}$$

Proof. By considering (2.4), the result follows by applying a suitable change of variables to the sum. □

Theorem (3.5). *For $|q| < 1$, x, $y \neq 0$ we have*

(3.6)
$$\begin{aligned} &\prod_{n=1}^{\infty}(1-q^n)^2(1-xq^{n-1})(1-x^{-1}q^n)(1-yq^{n-1})(1-y^{-1}q^n)(1-xyq^{n-1})(1-x^{-1}y^{-1}q^n) \\ &\quad = \sum_{a,k=-\infty}^{\infty} \epsilon_{k+2} x^{a+1} y^{k-a+1} q^{a^2-ak+\frac{a^2-1}{3}}. \end{aligned}$$

Proof (R. Askey). We denote the left side of (3.6) by $f(x,y)$. Then by (2.1) and Lemma (3.3) we have

(3.7)
$$\begin{aligned}
(q)_\infty f(x,y) &= j(x,q)j(y,q)j(xy,q)\\
&= \sum_{k,m,n=-\infty}^{\infty} (-1)^{k+m+n} x^{k+n} y^{m+n} q^{\binom{k}{2}+\binom{m}{2}+\binom{n}{2}}\\
&= \sum_{a,b=-\infty}^{\infty} (-1)^{a+b} x^a y^b q^{\binom{a}{2}+\binom{b}{2}} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2-(a+b)n}\\
&= \sum_{a,k=-\infty}^{\infty} \epsilon_{k+2} x^{a+1} y^{k-a+1} q^{a^2-ak+\frac{a^2-1}{3}} (q)_\infty,
\end{aligned}$$

by replacing a by $a+1$ and b by $k-a+1$ in the previous sum, and (3.6) follows. □

By (3.1), (3.6) we have

$$F_9 = \frac{1}{(1-\zeta^2)(1-\zeta^4)(1-\zeta^6)J_9} \sum_{\substack{a,k=-\infty\\ 3\nmid k}}^{\infty} \epsilon_{k+2} \zeta^{4k-2a+6} q^{a^2-ak+\frac{k^2-1}{3}}. \tag{3.8}$$

Let $T : \mathbb{Z}^2 \to \mathbb{Z}^2$, be defined by $T((a,k)) = (2a-k, 3a-k)$. Then T preserves the form in the exponent of q in (3.8) and the lattice $\mathbb{Z}^2$. Also, T has order 6; the orbit of (a,k) is given below:

$$\begin{array}{ccccc}
(a,k) & \longrightarrow & (2a-k, 3a-k) & \longrightarrow & (a-k, 3a-2k)\\
\uparrow & & & & \downarrow\\
(k-a,\ 2k-3a) & \longleftarrow & (k-2a, k-3a) & \longleftarrow & (-a, -k)
\end{array} \tag{3.9}$$

See [**G-S**; §4] for a computational method for computing the symmetry group of a form. For $3\nmid k$ we note that each element (a',k') of the orbit satisfies $3\nmid k'$. In fact, by considering the three cases $a \equiv 0, \pm k \pmod 3$ we see that each orbit (with $3\nmid k$) has a unique element (a',k') with $a' \equiv 0$ and $k' \equiv 1 \pmod 3$. It follows that the sum on the right side of (3.8) may be written as

(3.10)
$$\begin{aligned}
&\sum_{\substack{a\equiv 0(3)\\ k\equiv 1(3)}} \zeta^6(\zeta^{4k-2a} - \zeta^{8a-2k} + \zeta^{10a-6k} - \zeta^{2a-4k} + \zeta^{2k-8a} - \zeta^{6k-10a}) q^{a^2-ak+\frac{a^2-1}{3}}\\
&= \sum_{n,k=-\infty}^{\infty} \zeta^6(\zeta^{3n+3k+4} - \zeta^{3n+6k+4} + \zeta^{3k} - \zeta^{6n+6k+5} + \zeta^{6n+3k+5} - \zeta^{6k}) q^{3n^2-9nk+9k^2-n-3k+2}
\end{aligned}$$

(by replacing a by $3k-3$ and k by $3n-5$ in the first sum)

$$= \sum_{n,k=-\infty}^{\infty} \zeta^6(\zeta^{3k} - \zeta^{6k})(1 + \zeta^{3n+4} + \zeta^{6n+5})q^{3n^2-9nk+9k^2-n-3k+2}$$

$$= (1 - \zeta^3) \sum_{n,k=-\infty}^{\infty} \epsilon_{k+2}(1 + \zeta^{3n+4} + \zeta^{6n+5})q^{3n^2-9nk+9k^2-n-3k+2}.$$

We may now find the 3-dissection easily by replacing n in the sum by $3a+i$ (for $i = 2$, 1, 0). This gives three series each of which is a special case of (3.6). Our sum (3.10) is

(3.11)

$$\begin{aligned}
&(1 - \zeta^3) \Bigg((1 + \zeta + \zeta^8) \sum_{a,k=-\infty}^{\infty} \epsilon_{k+2} q^{27a^2-27ka+9k^2-21k+33a+12} \\
&\qquad + (1 + \zeta^2 + \zeta^7) \sum_{a,k=-\infty}^{\infty} \epsilon_{k+2} q^{27a^2-27ka+9k^2-12k+15a+4} \\
&\qquad + (1 + \zeta^4 + \zeta^5) \sum_{a,k=-\infty}^{\infty} \epsilon_{k+2} q^{27a^2-27ka+9k^2-3k-3a+2} \Bigg) \\
&= (1 - \zeta^3) \big((1 + \zeta + \zeta^8) q^{30} J_{27}^{-1} j(q^{12}, q^{27}) j(q^{-21}, q^{27}) j(q^{-9}, q^{27}) \\
&\qquad + (1 + \zeta^2 + \zeta^7) q^{22} J_{27}^{-1} j(q^{3}, q^{27}) j(q^{-12}, q^{27}) j(q^{-9}, q^{27}) \\
&\qquad + (1 + \zeta^4 + \zeta^5) q^{20} J_{27}^{-1} j(q^{-6}, q^{27}) j(q^{-3}, q^{27}) j(q^{-9}, q^{27}) \big) .
\end{aligned}$$

In the last equation we have applied (3.6) with (q, x, y) replaced by $(q^{27}, q^{12}, q^{-21})$, (q^{27}, q^{3}, q^{-12}), and (q^{27}, q^{-6}, q^{-3}). Since $j(x^{-1}, q) = -x^{-1}j(x, q)$, we obtain

(3.12)

$$\begin{aligned}
F_9 = \frac{1}{(1 - \zeta^2)(1 - \zeta^4)(1 - \zeta^6)J_9}(1 - \zeta^3) \Bigg(&(1 + \zeta + \zeta^8) \frac{J_{12,27} J_{21,27} J_9}{J_{27}} \\
&+ (1 + \zeta^2 + \zeta^7) q \frac{J_{3,27} J_{12,27} J_9}{J_{27}} \\
&- (1 + \zeta^4 + \zeta^5) q^2 \frac{J_{3,27} J_{6,27} J_9}{J_{27}} \Bigg) ,
\end{aligned}$$

from which (2.15) follows easily.

4. The 5-dissection of F_{10} and the crank mod 10.

In this section we derive (2.16), the 5-dissection of F_{10}. The proof depends on Rødseth's [**Rø**] 5-dissection of the generating function for partitions into distinct parts (see (4.3) below). Throughout this section $\zeta = \zeta_{10} = exp(\pi i/5)$. Now,

$$\begin{aligned} F_{10} &= \frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty} \\ &= (-q;q)_\infty(q)_\infty(\zeta^3 q;q)_\infty(\zeta^{-3}q;q)_\infty/(-q^5;q^5)_\infty \\ &= \frac{(-q;q)_\infty j(\zeta^{-3},q)}{(1+\zeta^2)(-q^5;q^5)_\infty} \qquad \text{(by (2.1))}. \end{aligned} \tag{4.1}$$

We need the 5-dissection of $j(\zeta^{-3},q)$ and $(-q;q)_\infty$. The 5-dissection of $j(\zeta^{-3},q)$ follows easily from (2.1).

$$\begin{aligned} j(\zeta^{-3},q) &= (1+\zeta^2)\overline{J}_{10,25} + (\zeta^4-\zeta^3)q\overline{J}_{5,25} - \zeta q^3\overline{J}_{0,25} \\ &= (1+\zeta^2)\overline{J}_{10,25} + (\zeta^4-\zeta^3)q\overline{J}_{5,25} - 2\zeta q^3 J_{50}^2 J_{25}^{-1}, \end{aligned} \tag{4.2}$$

by (2.1), (2.2). Rødseth [**Rø**; Thm 1, p. 9] has found the 5-dissection of $(-q;q)_\infty$.

$$\begin{aligned} (-q;q)_\infty &= J_{10}J_5^{-3}\overline{J}_{10,25}^2 + qJ_{10}^2J_{25}^3J_5^{-4}J_{50}^{-1} + q^2J_{10}J_5^{-3}\overline{J}_{5,25}^2 \\ &\quad + 2q^3J_{10}J_{50}^2J_5^{-3}J_{25}^{-1}\overline{J}_{10,25} + 2q^4J_{10}J_{50}^2J_5^{-3}J_{25}^{-1}\overline{J}_{5,25}. \end{aligned} \tag{4.3}$$

Rødseth's proof depends on the theory of modular functions. We note that an elementary proof of (4.3) can be obtained from dissecting the following identity due to Ramanujan which was proved by Watson [**Wa**; (2) p. 60].

$$\psi^2(q) - q\psi^2(q^5) = \frac{(q^5;q^5)_\infty^2(-q;q)_\infty}{(-q^5;q^5)_\infty}, \tag{4.4}$$

where

$$\psi(q) = \sum_{n\ge 0} q^{\binom{n+1}{2}}. \tag{4.5}$$

We will need the companion of (4.4) [**Wa**; (1) p. 60]:

$$\phi^2(-q^5) - \phi^2(-q) = 4q\frac{(q^{10};q^{10})_\infty^2(q;q^2)_\infty}{(q^5;q^{10})_\infty}, \tag{4.6}$$

where

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{4.7}$$

We should point out that (4.3), (4.4) and (4.6) follow from an identity of Hirschhorn [**Hir1**; (2.1)]. In fact, (4.3), (4.4) and (4.6) (resp.) are the special cases $(q,a,b)\mapsto(q^{5/2},q^{-3/2},q^{-1/2})$, $(q^{1/2},q^{-1/2},q^{-1/2})$, $(q,1,1)$ (resp.) of Hirschhorn's identity. The identities, (4.4) and (4.6), are originally from a list of 40 identities due to Ramanujan. See [**Bir**]. The proof of these 40 identities has been completed recently by Biagioli [**Bia**]. Using the triple product identity (2.1) we may write (4.6) in J-notation.

$$\frac{J_5^4}{J_{10}^2}-\frac{J_1^4}{J_2^2}=4q\frac{J_1J_{10}^3}{J_2J_5}. \tag{4.8}$$

We will also need the following two identities:

$$\overline{J}_{2,5}^3-2qJ_5^{-1}J_{10}^2\overline{J}_{1,5}^2=J_1^2J_5J_{10}^{-1}J_{3,10} \tag{4.9}$$

and

$$2J_5^{-1}J_{10}^2\overline{J}_{2,5}^2-\overline{J}_{1,5}^3=J_1^2J_5J_{10}^{-1}J_{1,10}. \tag{4.10}$$

These two identities may be proved using Rødseth's [**Rø**] method of modular functions, however, they are simply special cases of an identity for θ-functions due to Atkin and Swinnerton-Dyer [**A-S**; (3.7)]. In fact, if we let $(w,z,\zeta,t)=(q^5,-q^2,q^2,-q)$ in this identity and multiply the result by $J_5^4/\overline{J}_{1,5}$ we obtain

$$\overline{J}_{2,5}^3-\frac{J_{2,5}^3J_{1,5}}{\overline{J}_{1,5}}-q\overline{J}_{1,5}^2\overline{J}_{0,5}=0, \tag{4.11}$$

which is equivalent to (4.9) since

$$J_{1,5}J_{2,5}=J_1J_5, \tag{4.12}$$

$$\frac{J_{2,5}^2}{\overline{J}_{1,5}}=\frac{J_1J_{3,10}}{J_{10}}, \tag{4.13}$$

and

$$\overline{J}_{0,5}=2\frac{J_{10}^2}{J_5}. \tag{4.14}$$

Similarly, (4.10) follows by letting $(w,z,\zeta,t)=(q^5,-q^2,q,-q)$ in [**A-S**; (3.7)] and multiplying the result by $J_5^4/\overline{J}_{2,5}$. An elementary proof of [**A-S**; (3.7)] is given in [**G1**; pp.103-4].

We are now ready to carry out the 5-dissection. By (4.1), (4.2) and (4.3) we have

(4.15)
$$\begin{aligned}F_{10}=(1+\zeta^2)^{-1}J_5J_{10}^{-1}\Big\{&(1+\zeta^2)J_5^{-3}J_{10}\left(\overline{J}_{10,25}^3-2q^5J_{25}^{-1}\overline{J}_{5,25}^2J_{50}^2\right)\\&+\zeta qJ_5^{-4}J_{10}^2J_{25}^{-1}\overline{J}_{10,25}J_{50}\left(J_{25}^4J_{50}^{-2}-4q^5J_5J_{10}^{-1}J_{25}^{-1}J_{50}^3\right)\\&+\zeta q^2J_5^{-4}J_{10}^2J_{25}^{-1}\overline{J}_{5,25}J_{50}\left(J_{25}^4J_{50}^{-2}-4q^5J_5J_{10}^{-1}J_{25}^{-1}J_{50}^3\right)\\&+(\zeta^4-\zeta^3)q^3J_5^{-3}J_{10}\left(\overline{J}_{5,25}^3-2J_{25}^{-1}\overline{J}_{10,25}^2J_{50}^2\right)\\&+2(1-\zeta+\zeta^2-\zeta^3+\zeta^4)q^4J_5^{-4}J_{10}^2J_{25}^2J_{50}\Big\}.\end{aligned}$$

Here we have used

$$\overline{J}_{1,5}\overline{J}_{2,5} = J_1^{-1} J_2 J_5^3 J_{10}^{-1}. \tag{4.16}$$

By using (4.8), (4.9) and (4.10) we find the 5-dissection simplifies, and we have

$$\begin{aligned} F_{10} &= J_{25}J_{50}^{-1}J_{15,50} + q(\zeta^2-\zeta^3)J_5J_{10}^{-1}\overline{J}_{10,25}J_{25}^{-1}J_{50} \\ &\quad + q^2(\zeta^2-\zeta^3)J_5J_{10}^{-1}\overline{J}_{5,25}J_{25}^{-1}J_{50} + q^3(1-\zeta^2+\zeta^3)J_{25}J_{50}^{-1}J_{5,50}, \end{aligned} \tag{4.17}$$

which is (2.16).

We now turn to the relations for the crank mod 10 (1.16)–(1.18) and (1.29)–(1.32). We will need the relations for the crank mod 5 from [**G2**; (1.27), (1.40)–(1.43)].

$$M(1,5,5n) = M(2,5,5n), \tag{4.18}$$
$$M(0,5,5n+1) + M(1,5,5n+1) = 2M(2,5,5n+1), \tag{4.19}$$
$$M(0,5,5n+2) = M(1,5,5n+2), \tag{4.20}$$
$$M(0,5,5n+3) = M(2,5,5n+3), \tag{4.21}$$
$$M(0,5,5n+4) = M(1,5,5n+4) = M(2,5,5n+4). \tag{4.22}$$

By using the fact that

$$M(k,t,n) = M(t-k,k,n) \qquad \text{(by [}\mathbf{G2}\text{; (1.10)] and (1.14))}, \tag{4.23}$$

and $\zeta^4 = -1+\zeta-\zeta^2+\zeta^3$ we find that (1.22) may be written as

$$\begin{aligned} F_{10} = \sum_{n\ge 0} &\big(M(0,10,n)+M(1,10,n)-M(4,10,n)-M(5,10,n) \\ &+ (\zeta^2-\zeta^3)\left(M(1,10,n)+M(2,10,n)-M(3,10,n)-M(4,10,n)\right)\big)\, q^n. \end{aligned} \tag{4.24}$$

Since the coefficients of the J-functions are rational integers and $[\mathbb{Q}(\zeta):\mathbb{Q}] = 4$ we may equate the coefficients of ζ^k on the right side of (4.17) with those on the right side of (4.24). In this way we find that

$$M(1,10,5n) + M(2,10,5n) = M(3,10,5n) + M(4,10,5n), \tag{4.25}$$

$$M(0,10,5n+1) + M(1,10,5n+1) = M(4,10,5n+1) + M(5,10,5n+1), \tag{4.26}$$

$$M(0,10,5n+2) + M(1,10,5n+2) = M(4,10,5n+2) + M(5,10,5n+2), \tag{4.27}$$

$$\begin{aligned} &M(0,10,5n+3) + 2M(1,10,5n+3) + M(2,10,5n+3) \\ &\quad = M(3,10,5n+3) + 2M(4,10,5n+3) + M(5,10,5n+3), \end{aligned} \tag{4.28}$$

(4.29)
$$M(0,10,5n+4)+M(1,10,5n+4)=M(4,10,5n+4)+M(5,10,5n+4),$$
(4.30)
$$M(1,10,5n+4)+M(2,10,5n+4)=M(3,10,5n+4)+M(4,10,5n+4).$$

From (4.22) we have

(4.31)
$$\begin{aligned}M(0,10,5n+4)+M(5,10,5n+4)&=M(1,10,5n+4)+M(4,10,5n+4)\\&=M(2,10,5n+4)+M(3,10,5n+4),\end{aligned}$$

which together with (4.29) and (4.30) implies

$$M(0,10,5n+4)=M(2,10,5n+4)=M(4,10,5n+4), \tag{4.32}$$

and

$$M(1,10,5n+4)=M(3,10,5n+4)=M(5,10,5n+4). \tag{4.33}$$

These are (1.17), (1.18) and (1.16) follows. Similarly (1.29)–(1.32) follow from (4.25)–(4.28) and (4.18)–(4.21).

5. Remarks.

The identity, (4.6), which was needed in the proof of our main result has some strange connections with self-conjugate partitions. By replacing q by $-q$ and using the triple product identity (2.1) we find that this identity may be written as

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} - \sum_{m,n=-\infty}^{\infty} q^{5m^2+5n^2} = 4q \sum_{m,n=-\infty}^{\infty} q^{5m^2+2m+5n^2+4n}. \tag{5.1}$$

The series on the right side of (5.1) turns out to be the generating function for the number of self-conjugate partitions whose 5-cores are themselves. See [**G-K-S**]. Using Jacobi's formula [**H-W**; Thm 278] for the number of representations of a number as a sum of two squares it is possible to give exact formulae for the number of such self-conjugate partitions.

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