EXOTIC BAILEY-SLATER SPT-FUNCTIONS II: HECKE-ROGERS-TYPE DOUBLE SUMS AND BAILEY PAIRS FROM GROUPS A, C, E

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Abstract. We continue to investigate spt-type functions that arise from Bailey pairs. We prove simple Ramanujan type congruences for these functions which can be explained by a spt-crank-type function. The spt-crank-type functions are constructed by adding an extra variable z into the generating functions. We find these generating functions to have interesting representations as either infinite products or as Hecke-Rogers-type double series. These series reduce nicely when z is a certain root of unity and allow us to deduce congruences. Additionally we find dissections when z is a certain root of unity to explain the congruences. Our double sum and product formulas require Bailey's Lemma and conjugate Bailey pairs. Our dissection formulas follow from Bailey's Lemma and dissections of known ranks and cranks.

1. Introduction

A partition of n is a non-increasing sequence of positive integers that sum to n. For example, the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. We have a weighted count on partitions given by counting a partition by the number of times the smallest part appears. We let $\mathrm{spt}\,(n)$ denote this weighted count of the partitions of n. From the partitions of 4 we see that $\mathrm{spt}\,(4) = 10$. Andrews introduced the spt function in [2] and there proved that $\mathrm{spt}\,(5n+4) \equiv 0 \pmod{5}$, $\mathrm{spt}\,(7n+5) \equiv 0 \pmod{7}$, and $\mathrm{spt}\,(13n+6) \equiv 0 \pmod{13}$.

We see a generating function for $\mathrm{spt}\,(n)$ is given by

$$S(q) = \sum_{n=1}^{\infty} \mathrm{spt}\,(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty}.$$

Here we use the standard product notation,

$$(z;q)_n = \prod_{j=0}^{n-1}(1-zq^j), \qquad (z;q)_\infty = \prod_{j=0}^{\infty}(1-zq^j), \qquad [z;q]_\infty = (z,q/z;q)_\infty\,,$$
$$(z_1,\dots,z_k;q)_n = (z_1;q)_n \dots (z_k;q)_n\,, \qquad (z_1,\dots,z_k;q)_\infty = (z_1;q)_\infty \dots (z_k;q)_\infty\,,$$
$$[z_1,\dots,z_k;q]_\infty = [z_1;q]_\infty \dots [z_k;q]_\infty\,.$$

In [4] Andrews, the first author, and Liang defined a two variable generalization of the spt function by

$$S(z,q) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_S(m,n) z^m q^n = \sum_{n=1}^{\infty} \frac{q^n \left(q^{n+1};q\right)_\infty}{\left(zq^n, z^{-1}q^n;q\right)_\infty} = \frac{(q;q)_\infty}{\left(z,z^{-1};q\right)_\infty} \sum_{n=1}^{\infty} \frac{\left(z,z^{-1};q\right)_n q^n}{(q;q)_n},$$

so that $S(1,q) = S(q)$. There they reproved the congruences $\mathrm{spt}\,(5n+4) \equiv 0 \pmod{5}$ and $\mathrm{spt}\,(7n+5) \equiv 0 \pmod{7}$ by examining $S(\zeta_5,q)$ and $S(\zeta_7,q)$, where ζ_5 is a primitive fifth root of unity and ζ_7 is a primitive seventh root of unity. Essential to this is the identity

$$(1-z)(1-z^{-1})S(z,q) = R(z,q) - C(z,q), \tag{1.1}$$

where $R(z,q)$ is the generating function of the rank of a partition and $C(z,q)$ is the generating function of the crank of a partition. One can consult [5] for an account of the rank of a partition and for the crank [11] and [3]. To prove (1.1) one applies a limiting case of Bailey's Lemma to a certain Bailey pair.

In [10] the authors used Bailey's Lemma on four different Bailey pairs to study and prove congruences for three spt functions for overpartitions and the spt function for partitions with smallest part even and without

2010 *Mathematics Subject Classification.* Primary 11P83, 11P82, 33D15.

Key words and phrases. Number theory; Andrews' spt-function; congruences; partitions; smallest parts function; Bailey pairs, Bailey's lemma, Hecke-Rogers series.

repeated odd parts. An overpartition is a partition in which the first occurence of a part may be overlined, for example the overpartitions of 3 are 3, $\overline{3}$, $2+1$, $2+\overline{1}$, $\overline{2}+1$, $\overline{2}+\overline{1}$, $1+1+1$, and $\overline{1}+1+1$. We let $\overline{\mathrm{spt}}\,(n)$ denote the total number of appearances of the smallest parts among the overpartitions of n whose smallest part is not overlined, for example $\overline{\mathrm{spt}}\,(3)=6$. The other two spt functions for overpartitions are given by additionally requiring the smallest part to be odd or requiring the smallest part to be even. We let $\mathrm{M2spt}\,(n)$ denote the number of smallest parts in the partitions of n without repeated odd parts and smallest part even. For example the relevant partitions of 6 are 6, $4+2$, and $2+2$ and so $\mathrm{M2spt}\,(n)=4$. In all cases we found the two variable generalizations of the spt functions to be the difference of a known rank function and some sort of crank function and this fact came directly from a Bailey pair. In particular, the generating functions for the respective two variable generalizations for $\overline{\mathrm{spt}}\,(n)$ and $\mathrm{M2spt}\,(n)$ are

$$
\begin{aligned}
\overline{\mathrm{S}}(z,q) &= \sum_{n=1}^{\infty} \frac{q^n\left(-q^{n+1},q^{n+1};q\right)_\infty}{\left(zq^n,z^{-1}q^n;q\right)_\infty} = \frac{\left(-q,q;q\right)_\infty}{\left(z,z^{-1};q\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q\right)_n q^n}{\left(-q,q;q\right)_n},\\
\mathrm{S2}(z,q) &= \sum_{n=1}^{\infty} \frac{q^{2n}\left(-q^{2n+1},q^{2n+2};q^2\right)_\infty}{\left(zq^{2n},z^{-1}q^{2n};q^2\right)_\infty} = \frac{\left(-q,q^2;q^2\right)_\infty}{\left(z,z^{-1};q^2\right)_\infty}\sum_{n=1}^{\infty}\frac{\left(z,z^{-1};q^2\right)_n q^n}{\left(-q,q^2;q^2\right)_n},\\
\overline{\mathrm{S}}(1,q) &= \sum_{n=1}^{\infty} \overline{\mathrm{spt}}\,(n)\,q^n,\\
\mathrm{S2}(1,q) &= \sum_{n=1}^{\infty} \mathrm{M2spt}\,(n)\,q^n.
\end{aligned}
$$

We let $\overline{R}(z,q)$ denote the generating function for the Dyson rank of an overpartition and let $R2(z,q)$ denote the M_2-rank of a partition without repeated odd parts. The Dyson rank of an overpartition is the largest part minus the number of parts. The M_2-rank of a partition without repeated odd parts is the ceiling of half the largest part minus the number of parts. We found that

$$
\begin{aligned}
(1-z)(1-z^{-1})\overline{\mathrm{S}}(z,q) &= \overline{R}(z,q) - (-q;q)_\infty\, C(z,q),\\
(1-z)(1-z^{-1})\mathrm{S2}(z,q) &= R2(z,q) - \left(-q;q^2\right)_\infty C(z,q^2).
\end{aligned}
$$

With this in mind we now consider a spt-crank-type function to be a function of the form

$$
\frac{P(q)}{\left(z,z^{-1};q\right)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q\right)_n q^n \beta_n,
$$

where $P(q)$ is an infinite product and β comes from a Bailey pair relative to $(1,q)$. We consider a spt-type function to be the $z=1$ case of a spt-crank-type function. We recall that a pair of sequences (α,β) is a Bailey pair relative to (a,q) if

$$
\beta_n = \sum_{k=0}^{n}\frac{\alpha_k}{(q;q)_{n-k}\,(aq;q)_{n+k}}.
$$

The authors are in the process of studying all such spt-type functions that arise from the Bailey pairs of Slater from [18] and [19]. In [12] the second author introduced the spt functions corresponding to the pairs $A(1)$, $A(3)$, $A(5)$, and $A(7)$ and proved congruences for these functions by dissecting the spt-crank-type functions when z is a certain root of unity. In this paper we again consider the Bailey pairs from group A as well as the Bailey pairs $C(1)$, $C(5)$, $E(2)$, and $E(4)$. In a coming paper we handle Bailey pairs from groups B, F, G, and J.

Our study of these spt-type functions can be described as follows. We take a bailey pair (α^X,β^X), where each β^X_n has integer coefficients as a series in q, and define the corresponding two variable spt-crank-type function

$$
S_X(z,q) = \frac{P_X(q)}{\left(z,z^{-1};q\right)_\infty}\sum_{n=1}^{\infty}\left(z,z^{-1};q\right)_n \beta^X_n q^n = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M_X(m,n) z^m q^n.
$$

By setting $z = 1$ we get the spt-type function

$$S_X(q) = S_X(1,q) = \sum_{n=1}^{\infty}\left(\sum_{m=-\infty}^{\infty} M_X(m,n)\right) q^n = \sum_{n=1}^{\infty} \mathrm{spt}_X\,(n)\, q^n.$$

We prove simple linear congruences for the $\mathrm{spt}_X\,(n)$ by considering $S_X(\zeta, q)$ where ζ is a primitive root of unity. For t a positive integer we define

$$M_X(k,t,n) = \sum_{m \equiv k \pmod{t}} M_X(m,n).$$

We note that

$$\mathrm{spt}_X\,(n) = \sum_{k=0}^{t-1} M_X(k,t,n).$$

When ζ_t is a t^{th} root of unity, we have

$$S_X(\zeta_t, q) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{t-1} M_X(k,t,n)\zeta_t^k\right) q^n.$$

The last equation is of great importance because if t is prime and ζ_t is a primitive t^{th} root of unity, then the minimal polynomial for ζ_t is $1 + x + x^2 + \cdots + x^{t-1}$. Thus if the coefficient of q^N in $S_X(\zeta_t, q)$ is zero, then $\sum_{k=0}^{t-1} M_X(k,t,N)\zeta_t^k$ is zero and so $M_X(0,t,N) = M_X(1,t,N) = \cdots = M_X(t-1,t,N)$. But then we would have that $\mathrm{spt}_X\,(N) = t \cdot M_X(0,t,N)$ and so clearly $\mathrm{spt}_X\,(N) \equiv 0 \pmod{t}$. That is to say, if the coefficient of q^N in $S_X(\zeta_t, q)$ is zero, then $\mathrm{spt}_X\,(N) \equiv 0 \pmod{t}$. Thus not only do we have the congruence $\mathrm{spt}_X\,(N) \equiv 0 \pmod{t}$, but also the stronger combinatorial result that all of the $M_X(r,t,N)$ are equal.

We return to this idea after defining our new spt-type functions and listing their congruences in the next section. There we introduce new spt-crank-type functions corresponding to the Bailey pairs $C(1)$, $C(5)$, $E(2)$, and $E(4)$ of [18] as well as revisit the Bailey pairs $A(1)$, $A(3)$, $A(5)$, $A(7)$. Not only do we prove congruences for these functions by dissection formulas for $S_X(\zeta, q)$, but we find single series and product identities for $S_{A5}(z,q)$, $S_{A7}(z,q)$, $S_{C5}(z,q)$, and $S_{E2}(z,q)$, and find interesting Hecke-Rogers-type double sum formulas for $S_{A1}(z,q)$, $S_{A3}(z,q)$, $S_{C1}(z,q)$, and $S_{E4}(z,q)$. These series identities can be used to prove most, but not all, of the congruences whereas the dissections prove all of them.

The first author [9] was first to find Hecke-Rogers-type double series for spt-crank-type functions. In particular we have,

$$\left(z, z^{-1}, q; q\right)_\infty S(z,q) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (1 - z^{\frac{n-m}{2}})^2 z^{\frac{m-n}{2}} \left(\frac{-4}{n}\right)\left(\frac{12}{m}\right) q^{\frac{1}{12}\left(\frac{3n^2-m^2}{2}-1\right)},$$

$$(1+z)\left(z, z^{-1}, q; q\right)_\infty \overline{\mathrm{S}}(z,q) = \sum_{n=0}^{\infty}\sum_{m=-[n/2]}^{[n/2]} (-1)^{m+n}(1 - z^{n-2|m|+1})(1 - z^{n-2|m|}) z^{2|m|-n} q^{\frac{n^2-2m^2}{2}+\frac{n}{2}},$$

$$\left(z, z^{-1}, q^2; q^2\right)_\infty \mathrm{S2}(z,-q) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} (-1)^n (1 - z^{n-m})^2 z^{m-n} q^{\frac{2n^2-m^2}{2}+\frac{2n-m}{2}}.$$

Here $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. Such identities are interesting not only because of their use in proving congruences for smallest parts functions, but also because many mock theta functions are special cases of the ranks related to the spt-crank-type functions.

In Section 2 we give preliminaries and state our main results which include congruences for various spt-type functions, single series and Hecke-Rogers-type double series, and product identities for various spt-crank-type functions. In Section 3 we use the machinery of Bailey pairs to prove all the series identities. In Section 4 we evaluate various spt-crank-type fucntions at roots of unity and obtain various results for the $M_X(r,t,n)$ coefficients. In Section 5 we relate our spt-crank-type functions with rank and crank type functions and derive various dissection identities at various roots of unity. In Section 6 we finish with some concluding remarks.

2. Preliminaries and Statement of Results

The following are Bailey pairs relative to $(1,q)$

$$\beta_n^{A1}=\frac{1}{(q;q)_{2n}},\qquad \alpha_n^{A1}=\begin{cases}1 & \text{if } n=0\\ q^{6k^2-k}+q^{6k^2+k} & \text{if } n=3k\\ -q^{6k^2+5k+1} & \text{if } n=3k+1\\ -q^{6k^2-5k+1} & \text{if } n=3k-1\end{cases},$$

$$\beta_n^{A3}=\frac{q^n}{(q;q)_{2n}},\qquad \alpha_n^{A3}=\begin{cases}1 & \text{if } n=0\\ q^{6k^2-2k}+q^{6k^2+2k} & \text{if } n=3k\\ -q^{6k^2+2k} & \text{if } n=3k+1\\ -q^{6k^2-2k} & \text{if } n=3k-1\end{cases},$$

$$\beta_n^{A5}=\frac{q^{n^2}}{(q;q)_{2n}},\qquad \alpha_n^{A5}=\begin{cases}1 & \text{if } n=0\\ q^{3k^2-k}+q^{3k^2+k} & \text{if } n=3k\\ -q^{3k^2+k} & \text{if } n=3k+1\\ -q^{3k^2-k} & \text{if } n=3k-1\end{cases},$$

$$\beta_n^{A7}=\frac{q^{n^2-n}}{(q;q)_{2n}},\qquad \alpha_n^{A7}=\begin{cases}1 & \text{if } n=0\\ q^{3k^2-2k}+q^{3k^2+2k} & \text{if } n=3k\\ -q^{3k^2+4k+1} & \text{if } n=3k+1\\ -q^{3k^2-4k+1} & \text{if } n=3k-1\end{cases},$$

$$\beta_n^{C1}=\frac{1}{(q;q^2)_n\,(q;q)_n},\qquad \alpha_n^{C1}=\begin{cases}1 & \text{if } n=0\\ (-1)^kq^{3k^2-k}(1+q^{2k}) & \text{if } n=2k\\ 0 & \text{if } n=2k+1\end{cases},$$

$$\beta_n^{C5}=\frac{q^{(n^2-n)/2}}{(q;q^2)_n\,(q;q)_n},\qquad \alpha_n^{C5}=\begin{cases}1 & \text{if } n=0\\ (-1)^kq^{k^2-k}(1+q^{2k}) & \text{if } n=2k\\ 0 & \text{if } n=2k+1\end{cases},$$

$$\beta_n^{E2}=\frac{(-1)^n}{(q^2;q^2)_n},\qquad \alpha_n^{E2}=\begin{cases}1 & \text{if } n=0\\ 2(-1)^n & \text{if } n\geq 1\end{cases},$$

$$\beta_n^{E4}=\frac{q^n}{(q^2;q^2)_n},\qquad \alpha_n^{E4}=\begin{cases}1 & \text{if } n=0\\ (-1)^nq^{n^2-n}(1+q^{2n}) & \text{if } n\geq 1\end{cases}.$$

For each Bailey pair we define a two variable series spt-crank-type series as follows,

$$S_{A1}(z,q)=\frac{(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{q^n\left(z,z^{-1};q\right)_n}{(q;q)_{2n}}=\sum_{n=1}^{\infty}\frac{q^n\left(q^{2n+1};q\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}, \tag{2.1}$$

$$S_{A3}(z,q)=\frac{(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}\left(z,z^{-1};q\right)_n}{(q;q)_{2n}}=\sum_{n=1}^{\infty}\frac{q^{2n}\left(q^{2n+1};q\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}, \tag{2.2}$$

$$S_{A5}(z,q)=\frac{(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n^2+n}\left(z,z^{-1};q\right)_n}{(q;q)_{2n}}=\sum_{n=1}^{\infty}\frac{q^{n^2+n}\left(q^{2n+1};q\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}, \tag{2.3}$$

$$S_{A7}(z,q)=\frac{(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{q^{n^2}\left(z,z^{-1};q\right)_n}{(q;q)_{2n}}=\sum_{n=1}^{\infty}\frac{q^{n^2}\left(q^{2n+1};q\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}, \tag{2.4}$$

$$S_{C1}(z,q)=\frac{\left(q;q^2\right)_\infty(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{q^n\left(z,z^{-1};q\right)_n}{(q;q^2)_n\,(q;q)_n}=\sum_{n=1}^{\infty}\frac{q^n\left(q^{2n+1};q^2\right)_\infty\left(q^{n+1};q\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}, \tag{2.5}$$

$$S_{C5}(z,q)=\frac{\left(q;q^2\right)_\infty(q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=1}^{\infty}\frac{q^{\frac{n^2+n}{2}}\left(z,z^{-1};q\right)_n}{(q;q^2)_n\,(q;q)_n}=\sum_{n=1}^{\infty}\frac{q^{\frac{n^2+n}{2}}\left(q^{2n+1};q^2\right)_\infty\left(q^{n+1};q\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}, \tag{2.6}$$

$$S_{E2}(z,q) = \frac{\left(q^2;q^2\right)_\infty}{\left(z,z^{-1};q\right)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^n \left(z,z^{-1};q\right)_n}{\left(q^2;q^2\right)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n q^n \left(q^{2n+2};q^2\right)_\infty}{\left(zq^n, z^{-1}q^n;q\right)_\infty}, \tag{2.7}$$

$$S_{E4}(z,q) = \frac{\left(q^2;q^2\right)_\infty}{\left(z,z^{-1};q\right)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n} \left(z,z^{-1};q\right)_n}{\left(q^2;q^2\right)_n} = \sum_{n=1}^{\infty} \frac{q^{2n} \left(q^{2n+2};q^2\right)_\infty}{\left(zq^n, z^{-1}q^n;q\right)_\infty}. \tag{2.8}$$

Setting $z=1$ gives the following spt-type functions

$$S_{A1}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{A1}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty \left(q^{n+1};q\right)_n},$$

$$S_{A3}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{A3}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty \left(q^{n+1};q\right)_n},$$

$$S_{A5}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{A5}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty \left(q^{n+1};q\right)_n},$$

$$S_{A7}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{A7}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty \left(q^{n+1};q\right)_n},$$

$$S_{C1}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{C1}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 \left(q^{n+1};q\right)_n \left(q^{2n+2};q^2\right)_\infty},$$

$$S_{C5}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{C5}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{(n^2+n)/2}}{(1-q^n)^2 \left(q^{n+1};q\right)_n \left(q^{2n+2};q^2\right)_\infty},$$

$$S_{E2}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{E2}(n)\, q^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^n \left(-q^{n+1};q\right)_\infty}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty},$$

$$S_{E4}(q) = \sum_{n=1}^{\infty} \mathrm{spt}_{E4}(n)\, q^n = \sum_{n=1}^{\infty} \frac{q^{2n} \left(-q^{n+1};q\right)_\infty}{(1-q^n)^2 \left(q^{n+1};q\right)_\infty}.$$

For group A, the interpretation is a bit more natural in terms of partition pairs rather than a smallest parts function, however we give the smallest parts interpretation for each of these generating functions.

For a partition π we let $s(\pi)$ denote the smallest part, $\ell(\pi)$ the largest part, and $|\pi|$ the sum of parts. We say a pair of partitions (π_1,π_2) is a partition pair of n if $|\pi_1|+|\pi_2|=n$. We let PP denote the set of all partition pairs (π_1,π_2) such that π_1 is non-empty and all parts of π_2 are larger than $s(\pi_1)$ but no more than $2s(\pi_1)$. We let $\widetilde{P}$ denote the set of all partitions π such that all odd parts of π are less than $2s(\pi)$. We let $\overline{P}$ denote the set of all overpartitions π where the smallest part of π is not overlined.

We note that

$$\frac{q^n}{(1-q^n)^2} = q^n + 2q^{2n} + 3q^{3n} + 4q^{4n} + \dots$$

and so in the generating functions we can interpret this as contributing the smallest part and weighted by the number of times the smallest part appears.

We see $\mathrm{spt}_{A1}(n)$ is the number of partition pairs $(\pi_1,\pi_2)\in PP$ of n, counted by the number of times $s(\pi_1)$ occurs. Similarly $\mathrm{spt}_{A3}(n)$ is the number of partition pairs $(\pi_1,\pi_2)\in PP$ of n, counted by the number of times $s(\pi_1)$ occurs past the first. We see $\mathrm{spt}_{A5}(n)$ is the number of partition pairs $(\pi_1,\pi_2)\in PP$ of n, counted by the number of times $s(\pi_1)$ occurs past the first $s(\pi_1)$ times. Lastly, $\mathrm{spt}_{A7}(n)$ is the number of partition pairs $(\pi_1,\pi_2)\in PP$ of n, counted by the number of times $s(\pi_1)$ occurs past the first $s(\pi_1)-1$ times.

For group C, we have that $\mathrm{spt}_{C1}(n)$ is the number of partitions $\pi\in\widetilde{P}$ of n, counted by the number of times $s(\pi)$ occurs. An interpretation for $\mathrm{spt}_{C5}(n)$ is not immediately clear from the generating function. However, in Corollary 2.7 we find that $\mathrm{spt}_{C5}(n)$ is $\mathrm{spt}_{C1}(n)-\mathrm{spt}(n/2)$, where $\mathrm{spt}(n/2)$ is zero if n is odd.

For group E, $\mathrm{spt}_{E4}(n)$ is number of overpartitions $\pi\in\overline{P}$ of n, counted by the number of times $s(\pi)$ occurs past the first. We see $\mathrm{spt}_{E2}(n)$ number of overpartitions $\pi\in\overline{P}$ of n with $s(\pi)$ even, counted by the number of times $s(\pi)$ occurs, minus the number of overpartitions $\pi\in\overline{P}$ of n with $s(\pi)$ odd, counted by the number of times $s(\pi)$ occurs. We note $\operatorname{spt}_{E4}(n) = \overline{\operatorname{spt}}(n) - \frac{\overline{p}(n)}{2}$. Also $\operatorname{spt}_{E2}(n) = \overline{\operatorname{spt}}_2(n) - \overline{\operatorname{spt}}_1(n)$, where $\overline{\operatorname{spt}}_2(n)$ and $\overline{\operatorname{spt}}_1(n)$ are two spt functions studied in [10]; in fact $S_{E2}(z,q) = \overline{\mathrm{S}}_2(z,q) - \overline{\mathrm{S}}_1(z,q)$ where $\overline{\mathrm{S}}_2(z,q)$ and $\overline{\mathrm{S}}_1(z,q)$ are the two variable generalizations of the generating functions for $\overline{\operatorname{spt}}_2(n)$ and $\overline{\operatorname{spt}}_1(n)$ that we studied in [10].

These functions satisfy the following congruences.

Theorem 2.1.

$$
\begin{aligned}
\operatorname{spt}_{A1}(3n) &\equiv 0 \pmod 3,\\
\operatorname{spt}_{A3}(3n+1) &\equiv 0 \pmod 3,\\
\operatorname{spt}_{A3}(5n+1) &\equiv 0 \pmod 5,\\
\operatorname{spt}_{A5}(5n+4) &\equiv 0 \pmod 5,\\
\operatorname{spt}_{A5}(7n+1) &\equiv 0 \pmod 7,\\
\operatorname{spt}_{A7}(5n+4) &\equiv 0 \pmod 5,\\
\operatorname{spt}_{C1}(5n+3) &\equiv 0 \pmod 5,\\
\operatorname{spt}_{C5}(5n+3) &\equiv 0 \pmod 5,\\
\operatorname{spt}_{E2}(3n) &\equiv 0 \pmod 3,\\
\operatorname{spt}_{E4}(3n+1) &\equiv 0 \pmod 3.
\end{aligned}
$$

We use the two variable series to prove these congruences as explained in the introduction. We can prove Theorem 2.1 by showing the following terms are zero: q^{3n} in $S_{A1}(\zeta_3,q)$, q^{3n+1} in $S_{A3}(\zeta_3,q)$, q^{5n+1} in $S_{A3}(\zeta_5,q)$, q^{5n+4} in $S_{A5}(\zeta_5,q)$, q^{7n+1} in $S_{A5}(\zeta_7,q)$, q^{5n+4} in $S_{A7}(\zeta_5,q)$, q^{5n+4} in $S_{C1}(\zeta_5,q)$, q^{5n+3} in $S_{C5}(\zeta_5,q)$, q^{3n} in $S_{E2}(\zeta_3,q)$, and q^{3n+1} in $S_{E4}(\zeta_3,q)$. For convenience, if $F(x)$ is a series in x, then we let $[x^N]F(x)$ denote the coefficient of x^N in $F(x)$.

That these coefficients are zero is a stronger result than just the congruences alone. These coefficients being zero gives a manner in which to split up the numbers $\operatorname{spt}_X(an+b)$ to see the congruences. In each case, this would allow us to define a so called spt-crank based off of the $M_X(m,n)$. An initial interpretation of the $M_X(m,n)$ would be in terms of weighted vector partitions, to find an interpretation in terms of smallest parts would take considerable work. One can find an example of this process in Section 3 of [10], we do not pursue this idea here. We do note, however, that since $S_{E2}(z,q) = \overline{\mathrm{S}}_2(z,q) - \overline{\mathrm{S}}_1(z,q)$, we do have a combinatorial interpretation of $M_{E2}(m,n)$ as being the difference of the corresponding spt-cranks defined in [10]. Also an interpretation in terms of a crank defined on partition pairs for $M_{A1}(m,n)$, $M_{A3}(m,n)$, $M_{A5}(m,n)$, and $M_{A7}(m,n)$ was given by the second author in [12].

We prove the following series representations for the $S_X(z,q)$.

Theorem 2.2.

$$
\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{A1}(z,q)\\
&= \frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}(1-z^{k-1})(1-z^k)z^{1-k}\left((-1)^{k+1}q^{k(k-1)/2} + \sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k(k-1)}{2}+\frac{n(n-3)}{2}+2kn}(1+q^n)\right),
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{A3}(z,q)\\
&= \frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(1-z^{k-1})(1-z^k)z^{1-k}(-1)^{n+k+1}q^{\frac{k(k+1)}{2}+\frac{n(n-3)}{2}+2kn-1}(1-q^{2n+1}),
\end{aligned}
\tag{2.10}
$$

$$
(1+z)\left(z,z^{-1};q\right)_\infty S_{A5}(z,q) = \sum_{k=-\infty}^{\infty}(-1)^k(1-z^k)(1-z^{k+1})z^{-k}q^{\frac{k(3k+1)}{2}},
\tag{2.11}
$$

$$
(1+z)\left(z,z^{-1};q\right)_\infty S_{A7}(z,q) = \sum_{k=-\infty}^{\infty}(-1)^k(1-z^k)(1-z^{k+1})z^{-k}q^{\frac{k(3k-1)}{2}},
\tag{2.12}
$$

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q)$$

$$= \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(1-z^{k-1})(1-z^k)z^{1-k}(-1)^{k+1}q^{\frac{k(k-1)}{2}+\frac{n(3n-1)}{2}+3kn}(1-q^{2k-1})(1-q^{k+n}), \tag{2.13}$$

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{C5}(z,q) = \sum_{k=-\infty}^{\infty}(1-z^{k-1})(1-z^k)z^{1-k}(-1)^k q^{k^2}, \tag{2.14}$$

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{E2}(z,q) = \left(q;q^2\right)_\infty \sum_{k=1}^{\infty}(1-z^k)(1-z^{k-1})z^{1-k}q^{\frac{k(k-1)}{2}}, \tag{2.15}$$

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q)$$
$$= \frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(1-z^{k-1})(1-z^k)z^{1-k}(-1)^{k+n+1}q^{\frac{k(k+1)}{2}+n^2-n+2kn-1}(1-q^{2n+1}). \tag{2.16}$$

By letting z be the appropriate root of unity, we use Theorem 2.2 to prove all of the congruences in Theorem 2.1 except for $\mathrm{spt}_{A3}(5n+3) \equiv 0 \pmod 5$ and $\mathrm{spt}_{C1}(5n+3) \equiv 0 \pmod 5$. From Theorem 2.2 we will also deduce the following Hecke-Rogers-type double series for $S_{A1}(z,q)$, $S_{A3}(z,q)$, $S_{C1}(z,q)$, and $S_{E4}(z,q)$.

Corollary 2.3.

$$(1+z)\left(z,z^{-1},q;q\right)_\infty S_{A1}(z,q) = \sum_{k=0}^{\infty}\sum_{n=-[k/2]}^{[k/2]}(-1)^{n+k}(1-z^{k-2|n|})(1-z^{2|n|-k+1})q^{\frac{k^2-k-3n^2-n}{2}}, \tag{2.17}$$

$$\begin{aligned}(1+z)\left(z,z^{-1},q;q\right)_\infty S_{A3}(z,q) &= \sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(1-z^{k-2n+1})(1-z^{2n-k})q^{\frac{k^2-k-3n^2+n}{2}} \\ &\quad - \sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(1-z^{k-2n})(1-z^{2n-k+1})q^{\frac{k^2+k-3n^2-n}{2}}\end{aligned} \tag{2.18}$$

$$\begin{aligned}(1+z)\left(z,z^{-1},q;q\right)_\infty S_{C1}(z,q) &= \sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1-z^{3n-k+1})(1-z^{k-3n})q^{\frac{k^2-k}{2}-3n^2+n} \\ &\quad - \sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1-z^{3n-k+1})(1-z^{k-3n})q^{\frac{k^2+k}{2}-3n^2-n} \\ &\quad + \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1-z^{3n-k})(1-z^{k-3n+1})q^{\frac{k^2-k}{2}-3n^2+n} \\ &\quad - \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1-z^{3n-k})(1-z^{k-3n+1})q^{\frac{k^2+k}{2}-3n^2-n},\end{aligned} \tag{2.19}$$

$$\begin{aligned}(1+z)\left(z,z^{-1},q;q\right)_\infty S_{E4}(z,q) &= \sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(1-z^{2n-k})(1-z^{k-2n+1})q^{\frac{k^2-k}{2}-n^2} \\ &\quad - \sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(1-z^{2n-k+1})(1-z^{k-2n})q^{\frac{k^2+k}{2}-n^2}.\end{aligned} \tag{2.20}$$

Using the Jacobi Triple Product Identity in the forms

$$\left(zq,z^{-1}q,q^2;q^2\right)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n^2},$$

$$\left(z,z^{-1}q,q;q\right)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n-1)/2},$$

we see Theorem 2.2 gives the remarkable fact that $S_{A1}(z,q)$, $S_{A3}(z,q)$, $S_{C5}(z,q)$, and $S_{E2}(z,q)$ can be written entirely in terms of products.

Corollary 2.4.

$$
\begin{aligned}
S_{A5}(z,q) &= \frac{z\left(zq^2, z^{-1}q, q^3; q^3\right)_\infty}{(1+z)\left(z,z^{-1};q\right)_\infty} + \frac{\left(zq, z^{-1}q^2, q^3; q^3\right)_\infty}{(1+z)\left(z,z^{-1};q\right)_\infty} - \frac{(q;q)_\infty}{\left(z,z^{-1};q\right)_\infty},\\
S_{A7}(z,q) &= \frac{z\left(zq, z^{-1}q^2, q^3; q^3\right)_\infty}{(1+z)\left(z,z^{-1};q\right)_\infty} + \frac{\left(zq^2, z^{-1}q, q^3; q^3\right)_\infty}{(1+z)\left(z,z^{-1};q\right)_\infty} - \frac{(q;q)_\infty}{\left(z,z^{-1};q\right)_\infty},\\
S_{C5}(z,q) &= \frac{\left(zq, z^{-1}q, q^2; q^2\right)_\infty}{\left(z,z^{-1};q\right)_\infty} - \frac{(q;q)_\infty}{\left(-q,z,z^{-1};q\right)_\infty},\\
S_{E2}(z,q) &= \frac{\left(-zq, -z^{-1}q, q; q\right)_\infty}{\left(z,z^{-1},-q;q\right)_\infty} - \frac{\left(q^2;q^2\right)_\infty}{\left(z,z^{-1};q\right)_\infty}.
\end{aligned}
$$

We prove the following dissections of $S_{C1}(\zeta_5,q)$, $S_{C5}(\zeta_5,q)$, $S_{E2}(\zeta_3,q)$, and $S_{E4}(\zeta_3,q)$. The dissections for $S_{A1}(\zeta_3,q)$, $S_{A3}(\zeta_3,q)$, $S_{A3}(\zeta_5,q)$, $S_{A5}(\zeta_5,q)$, $S_{A5}(\zeta_7,q)$, and $S_{A7}(\zeta_5,q)$ can be found in [12].

Theorem 2.5.

$$
\begin{aligned}
&(1-\zeta_5)(1-\zeta_5^{-1})S_{C1}(\zeta_5,q)\\
&= \frac{\left(q^{50};q^{50}\right)_\infty\left[q^{20};q^{50}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty^2} - \frac{\left(q^{50};q^{50}\right)_\infty\left[q^{25};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty} - 2(\zeta_5+\zeta_5^{-1})q^5\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{5};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\\
&\quad + q^{10}\frac{\zeta_5+\zeta_5^{-1}-2}{\left(q^{50};q^{50}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)}}{1-q^{50n+10}} - q^6(\zeta_5+\zeta_5^{-1})\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{10};q^{50}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty^2}\\
&\quad + 2q\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty} - (\zeta_5+\zeta_5^{-1})q\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{25};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\\
&\quad - q^{16}\frac{2\zeta_5+2\zeta_5^{-1}+1}{\left(q^{50};q^{50}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)}}{1-q^{50n+20}} + q^2\frac{\left(q^{50};q^{50}\right)_\infty}{\left[q^{10};q^{50}\right]_\infty} + 2(\zeta_5+\zeta_5^{-1})q^2\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\\
&\quad + q^4(\zeta_5+\zeta_5^{-1})\frac{\left(q^{50};q^{50}\right)_\infty}{\left[q^{20};q^{50}\right]_\infty} - 2q^4\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{5};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty},\\
&(1-\zeta_5)(1-\zeta_5^{-1})S_{C5}(\zeta_5,q)\\
&= \frac{\left(q^{50};q^{50}\right)_\infty\left[q^{20};q^{50}\right]_\infty}{\left[q^{10};q^{50}\right]_\infty^2} - \frac{\left(q^{50};q^{50}\right)_\infty\left[q^{25};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty} - 2(\zeta_5+\zeta_5^{-1})q^5\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{5};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\\
&\quad - (\zeta_5+\zeta_5^4)q^6\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{10};q^{50}\right]_\infty}{\left[q^{20};q^{50}\right]_\infty^2} + 2q\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty} - (\zeta_5+\zeta_5^{-1})q\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{25};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\\
&\quad + (\zeta_5+\zeta_5^4-1)q^2\frac{\left(q^{50};q^{50}\right)_\infty}{\left[q^{10};q^{50}\right]_\infty} - 2(\zeta_5+\zeta_5^{-1})q^2\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{15};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\\
&\quad - (\zeta_5+\zeta_5^4+1)q^4\frac{\left(q^{50};q^{50}\right)_\infty}{\left[q^{20};q^{50}\right]_\infty} - 2q^4\frac{\left(q^{50};q^{50}\right)_\infty\left[q^{5};q^{50}\right]_\infty}{\left[q^{5};q^{25}\right]_\infty},\\
&S_{E2}(\zeta_3,q) = -q\frac{\left(q^{18};q^{18}\right)_\infty\left(q^{9};q^{9}\right)_\infty}{\left(q^{3};q^{3}\right)_\infty} + q^2\frac{\left(q^{18};q^{18}\right)_\infty^4}{\left(q^{9};q^{9}\right)_\infty^2\left(q^{6};q^{6}\right)_\infty},\\
&S_{E4}(\zeta_3,q) = \frac{1}{2} - \frac{\left(q^{9};q^{9}\right)_\infty^4\left(q^{6};q^{6}\right)_\infty}{2\left(q^{18};q^{18}\right)_\infty^2\left(q^{3};q^{3}\right)_\infty^2} + q^2\frac{\left(q^{18};q^{18}\right)_\infty}{\left(q^{9};q^{9}\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n}}{1-q^{9n+3}}.
\end{aligned}
$$

From these identities we see $S_{C1}(\zeta_5,q)$ has no q^{5n+3} terms, $S_{C5}(\zeta_5,q)$ has no q^{5n+3} terms, $S_{E2}(\zeta_3,q)$ has no q^{3n} terms, and $S_{E4}(\zeta_3,q)$ has no q^{3n+1} terms. This gives another proof of the congruences for $\mathrm{spt}_{C5}(5n+3)$,

$\operatorname{spt}_{E2}(3n)$, and $\operatorname{spt}_{E4}(3n+1)$, and the only proof of the congruence for $\operatorname{spt}_{C1}(5n+3)$. The dissection from [12] gives the only proof of the congruence for $\operatorname{spt}_{A3}(5n+1)$.

We note that Theorems 2.2 and 2.5 both prove the congruences, but the identities are inherently different. In Theorem 2.2 we have identities that are valid for all z, but it may be difficult to nicely identify all terms in the ℓ-dissection when $z=\zeta_\ell$. In Theorem 2.5, the identities are for z being a fixed root of unity and all terms in the dissection are explicitely determined. The dissection formulas come as a consequence of each $S_X(z,q)$ being the difference of a rank-type function and a crank-type function. A consequence of the proofs of Theorem 2.2 and Corollary 2.3 are the following product identities.

Corollary 2.6.

$$(q;q)_\infty^2=\sum_{k=1}^{\infty}\left((-1)^{k+1}q^{k(k-1)/2}+\sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k(k-1)}{2}+\frac{n(n-3)}{2}+2kn}(1+q^n)\right), \tag{2.21}$$

$$(q;q)_\infty^2=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+k+1}q^{\frac{k(k+1)}{2}+\frac{n(n-3)}{2}+2kn-1}(1-q^{2n+1}), \tag{2.22}$$

$$(q;q)_\infty^2\left(q;q^2\right)_\infty=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{k+1}q^{\frac{k(k-1)}{2}+\frac{n(3n-1)}{2}+3kn}(1-q^{2k-1})(1-q^{k+n}), \tag{2.23}$$

$$(q;q)_\infty\left(q^2;q^2\right)_\infty=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{k+n+1}q^{\frac{k(k+1)}{2}+n^2-n+2kn-1}(1-q^{2n+1}), \tag{2.24}$$

$$(q;q)_\infty^2\left(q;q^2\right)_\infty=\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{[k/3]}(-1)^{n+k+1}q^{\frac{k^2-k}{2}-3n^2+n}(1-q^k), \tag{2.25}$$

$$(q;q)_\infty\left(q^2;q^2\right)_\infty=\sum_{k=0}^{\infty}\sum_{n=-[k/2]}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2}. \tag{2.26}$$

We note that (2.26) is known. It was derived by Andrews in [1], Bressoud in [7], and recently reproved by the first author in [9]. Some consequences of the proof of Theorem 2.5 are the following relations between $S_{C1}(z,q)$ and $S_{C5}(z,q)$.

Corollary 2.7. *For all n and m,*

$$\begin{aligned}N_S(m,n)&=M_{C1}(m,2n)-M_{C5}(m,2n),\\ M_{C1}(m,2n+1)&=M_{C5}(m,2n+1),\\ \operatorname{spt}(n)&=spt_{C1}(2n)-spt_{C5}(2n),\\ spt_{C1}(2n+1)&=spt_{C5}(2n+1).\end{aligned}$$

In Section 3 we prove the series identities of Theorem 2.2 as well as Corollaries 2.3, 2.4, and 2.6. In Section 4 we use Theorem 2.2 to prove the appropriate terms are zero to deduce the congruences in Theorem 2.1. In Section 5 we prove the dissection formulas of Theorem 2.5.

3. Proof of the Series Identities

To prove the identities for $S_{A1}(z,q)$, $S_{A3}(z,q)$, $S_{C1}(z,q)$, and $S_{E4}(z,q)$ we need the following preliminary result.

Proposition 3.1.

$$\begin{aligned}&(1+z)\left(z,z^{-1};q\right)_\infty S_{A1}(z,q)\\ &=-(1+z)(q;q)_\infty+\frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}(z^k+z^{-k+1})\Big((-1)^{k+1}q^{k(k-1)/2}\\ &\quad+\sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k(k-1)}{2}+\frac{n(n-3)}{2}+2kn}(1+q^n)\Big),\end{aligned} \tag{3.1}$$

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{A3}(z,q)$$
$$= -(1+z)\,(q;q)_\infty + \frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(z^k+z^{-k+1})(-1)^{n+k+1}q^{\frac{k(k+1)}{2}+\frac{n(n-3)}{2}+2kn-1}(1-q^{2n+1}), \tag{3.2}$$
$$(1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q)$$
$$= -(1+z)\,(q;q)_\infty\left(q;q^2\right)_\infty$$
$$+\frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(z^k+z^{1-k})(-1)^{k+1}q^{\frac{k(k-1)}{2}+\frac{n(3n-1)}{2}+3kn}(1-q^{2k-1})(1-q^{k+n}), \tag{3.3}$$
$$(1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q)$$
$$= -(1+z)\left(q^2;q^2\right)_\infty + \frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(z^k+z^{1-k})(-1)^{k+n+1}q^{\frac{k(k+1)}{2}+n^2-n+2kn-1}(1-q^{2n+1}). \tag{3.4}$$

From this we deduce equations (2.21), (2.22), (2.23), and (2.24) of Corollary 2.6. These product identities along with Proposition 3.1 imply equations (2.9), (2.10), (2.13), and (2.16) of Theorem 2.2. Equations (2.11), (2.12), (2.14), and (2.15) do not require this additional step.

The proofs of the identities in Theorem 2.2 and Proposition 3.1 are to verify the coefficients of each power of z match. We do this by rearranging $S_X(z,q)$, extracting the coefficient of z^k, which is a series in q, and then using one of two general Bailey pairs with either a limiting case of Bailey's Lemma or an identity from Bailey's Transform with a suitable conjugate Bailey pair. We recall that (α,β) form a Bailey pair relative to (a,q) if

$$\beta_n = \sum_{k=0}^{n}\frac{\alpha_k}{(aq;q)_{n+k}\,(q;q)_{n-k}}$$

and Bailey's Lemma, which can be found in [6], gives if (α,β) is a Bailey pair relative to (a,q) then

$$\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n = \frac{(aq/\rho_1,aq/\rho_2;q)_\infty}{(aq,aq/\rho_1\rho_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n}{(aq/\rho_1,aq/\rho_2;q)_n}.$$

Lemma 3.2. *If α and β are a Bailey pair relative to (a,q) then*

$$\sum_{n=0}^{\infty}a^nq^{n^2}\beta_n = \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}a^nq^{n^2}\alpha_n, \tag{3.5}$$
$$\sum_{n=0}^{\infty}(-\sqrt{aq};q)_n\,a^{n/2}q^{n^2/2}\beta_n = \frac{(-\sqrt{aq};q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}a^{n/2}q^{n^2/2}\alpha_n, \tag{3.6}$$
$$\sum_{n=0}^{\infty}\left(a;q^2\right)_n(-1)^nq^n\beta_n = \frac{\left(aq^2;q^2\right)_\infty}{(aq,-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-a)}{(1-aq^{2n})}(-1)^nq^n\alpha_n, \tag{3.7}$$
$$\sum_{n=0}^{\infty}q^n\beta_n = \frac{1}{(aq,q;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-a)^nq^{n(n+1)/2+2nr+r}\alpha_r, \tag{3.8}$$
$$\sum_{n=0}^{\infty}q^{2n}\beta_n = \frac{1}{(aq,q;q)_\infty}\left(\sum_{r=0}^{\infty}q^{2r}\alpha_r+\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}(-1)^na^{n-1}q^{n(n+1)/2+2nr}(1+aq^{2r})\alpha_r\right), \tag{3.9}$$
$$\sum_{n=0}^{\infty}\left(aq;q^2\right)_nq^{2n}\beta_n = \frac{1}{(q;q)_\infty\left(aq^2;q^2\right)_\infty(1+q)}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-a)^nq^{n^2+n+2nr+2r}(1-q^{2n+2})\alpha_r. \tag{3.10}$$

If α and β are a Bailey pair relative to (a^2q,q) then

$$\sum_{n=0}^{\infty}(-aq;q)_n\,q^n\beta_n = \frac{(-aq;q)_\infty}{(q,a^2q^2;q)_\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}a^{3n}q^{n(3n+5)/2+3nr+r}(1-aq^{n+r+1})\alpha_r. \tag{3.11}$$

Proof. Equation (3.5) is the well known identity obtained by letting $\rho_1, \rho_2 \to \infty$ in Bailey's lemma. For (3.6) we let $\rho_1 \to \infty$ and $\rho_2 = -\sqrt{aq}$ in Bailey's Lemma, simplifying then gives the result. For (3.7) we let $\rho_1 = \sqrt{a}$ and $\rho_2 = -\sqrt{a}$ in Bailey's Lemma, simplifying then gives the result. Equations (3.8), (3.9), (3.10), and (3.11) are parts 1, 11, 12, and 13 of Theorem 1 from [14]. These identities are from conjugate Bailey pairs rather than specializations of ρ_1 and ρ_2. □

The following are two Bailey pairs relative to (a, q), both of which follow immediately from the definition of a Bailey pair:

$$\beta_n(a) = \frac{1}{(aq, q; q)_n}, \tag{3.12}$$

$$\alpha_n(a) = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}, \tag{3.13}$$

and

$$\beta_n^*(a) = \frac{1}{(aq^2, q; q)_n}, \tag{3.14}$$

$$\alpha_n^*(a) = \begin{cases} 1 & n = 0 \\ -aq & n = 1 \\ 0 & n \geq 2 \end{cases}. \tag{3.15}$$

In rearranging each $S_X(z, q)$, we use Proposition 4.1 of [9], which is

$$(1+z)\left(z, z^{-1}; q\right)_n = \sum_{j=-n}^{n+1} (-1)^{j+1} \frac{(q;q)_{2n}}{(q;q)_{n+j}\,(q;q)_{n-j+1}} (1 - q^{2j-1}) z^j q^{j(j-3)/2+1}. \tag{3.16}$$

Proof of (3.1). By (3.16) we have

$$\begin{aligned}(1+z)\left(z, z^{-1}; q\right)_\infty S_{A1}(z, q) &= (q;q)_\infty \sum_{n=1}^{\infty} \frac{(1+z)\left(z, z^{-1}; q\right)_n q^n}{(q;q)_{2n}} \\ &= (q;q)_\infty \sum_{n=1}^{\infty} \sum_{j=-n}^{n+1} \frac{(-1)^{j+1} z^j (1 - q^{2j-1}) q^{n + j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \end{aligned} \tag{3.17}$$

Since $S(z^{-1}, q) = S(z, q)$, we find that the coefficient of z^{-j} in $(1+z)\left(z, z^{-1}; q\right)_\infty S_{A1}(z, q)$ is the same as the coefficient of z^{j+1}. We next find these coefficients for $j \geq 1$. We will use the Bailey pair α and β from (3.12) and (3.13) and apply (3.8).

For $j = 1$, we take $j = 1$ in (3.17) and so $[z](1+z)\left(z, z^{-1}; q\right)_\infty S_{A1}(z, q)$ is given by

$$\begin{aligned}(q;q)_\infty \sum_{n=1}^{\infty} \frac{(1-q)q^n}{(q;q)_{n+1}\,(q;q)_n} &= (q;q)_\infty \sum_{n=1}^{\infty} \frac{q^n}{(q^2;q)_n\,(q;q)_n} \\ &= (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q^2;q)_n\,(q;q)_n} - (q;q)_\infty \\ &= (q;q)_\infty \sum_{n=0}^{\infty} q^n \beta_n(q) - (q;q)_\infty \\ &= \frac{(q;q)_\infty}{(q^2, q; q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n + \frac{n(n+1)}{2}} - (q;q)_\infty \\ &= \frac{1}{(q^2;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n + \frac{n(n+1)}{2}} - (q;q)_\infty. \end{aligned}$$

For $j \geq 2$, the calculations are similar. We have $[z^j](1+z)\left(z, z^{-1}; q\right)_\infty S_{A1}(z,q)$ is given by

$$
\begin{aligned}
&(q;q)_\infty \sum_{n=j-1}^{\infty} \frac{(-1)^{j+1}(1-q^{2j-1})q^{n+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}} \\
&= (-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{n+2j-1}\,(q;q)_n} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} \frac{q^n}{(q^{2j},q;q)_n} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} q^n \beta_n(q^{2j-1}) \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty}{(q;q)_{2j-1}\,(q^{2j},q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2+2jn} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2+2jn}. \qquad (3.18)
\end{aligned}
$$

We note this formula agrees at $j=1$ with the coefficient of z, except for the missing $-(q;q)_\infty$.

We rearrange these terms slightly. We have

$$
\begin{aligned}
&(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2} \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2+2jn} \\
&= \sum_{n=0}^{\infty} (-1)^{n+j+1} q^{j(j-1)/2+n(n-1)/2+2jn} + \sum_{n=0}^{\infty} (-1)^{n+j} q^{j(j+3)/2+n(n-1)/2+2jn-1} \\
&= \sum_{n=0}^{\infty} (-1)^{n+j+1} q^{j(j-1)/2+n(n-1)/2+2jn} + \sum_{n=1}^{\infty} (-1)^{n+j+1} q^{j(j-1)/2+n(n-3)/2+2jn} \\
&= (-1)^{j+1} q^{j(j-1)/2} + \sum_{n=1}^{\infty} (-1)^{n+j+1} q^{j(j-1)/2+n(n-3)/2+2jn}(1+q^n).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(1+z)\left(z, z^{-1}; q\right)_\infty S_{A1}(z,q) &= -(1+z)\,(q;q)_\infty + \frac{1}{(q;q)_\infty} \sum_{j=1}^{\infty} (z^j + z^{1-j})(-1)^{j+1} q^{j(j-1)/2} \\
&\quad + \frac{1}{(q;q)_\infty} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (z^j + z^{1-j})(-1)^{n+j+1} q^{j(j-1)/2+n(n-3)/2+2jn}(1+q^n),
\end{aligned}
$$

which completes the proof of (3.1). □

Proof of (3.2). By (3.16) we have

$$
(1+z)\left(z, z^{-1}; q\right)_\infty S_{A3}(z,q) = (q;q)_\infty \sum_{n=1}^{\infty} \sum_{j=-n}^{n+1} \frac{(-1)^{j+1} z^j (1-q^{2j-1}) q^{2n+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \qquad (3.19)
$$

Similar to before we find that the coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z, z^{-1}; q\right)_\infty S_{A3}(z,q)$. We again use the Bailey pair α and β from (3.12) and (3.13) but this time apply (3.9). Starting with $j \geq 2$, we have

$$
\begin{aligned}
&[z^j](1+z)\left(z, z^{-1}; q\right)_\infty S_{A3}(z,q) \\
&= (q;q)_\infty \sum_{n=j-1}^{\infty} \frac{(-1)^{j+1}(1-q^{2j-1})q^{2n+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\,(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{n+2j-1}\,(q;q)_n} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\,(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^{2j},q;q)_n} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\,(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} q^{2n}\beta_n(q^{2j-1}) \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\,(q;q)_\infty}{(q;q)_{2j-1}\,(q^{2j},q;q)_\infty} \left(1+\sum_{n=1}^{\infty}(-1)^n q^{(2j-1)(n-1)+n(n+1)/2}(1+q^{2j-1})\right) \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}}{(q;q)_\infty} \left(1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2+2jn-2j+1}(1+q^{2j-1})\right).
\end{aligned}
$$

For $j=1$, we instead have

$$
\begin{aligned}
&[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{A3}(z,q) \\
&= -(q;q)_\infty + \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}}{(q;q)_\infty} \left(1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2+2jn-2j+1}(1+q^{2j-1})\right).
\end{aligned}
$$

We rearrange these terms by

$$
\begin{aligned}
&(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2+2jn-2j+1}(1+q^{2j-1})\right) \\
&= (-1)^{j+1}q^{j(j+1)/2-1} + (-1)^j q^{j(j+5)/2-2} + \sum_{n=1}^{\infty}(-1)^{n+j+1}q^{j(j-3)/2+n(n-1)/2+2jn}(1-q^{4j-2}) \\
&= (-1)^{j+1}q^{j(j+1)/2-1} + (-1)^j q^{j(j+5)/2-2} + \sum_{n=1}^{\infty}(-1)^{n+j+1}q^{j(j-3)/2+n(n-1)/2+2jn} \\
&\quad + \sum_{n=1}^{\infty}(-1)^{n+j}q^{j(j+5)/2+n(n-1)/2+2jn-2} \\
&= \sum_{n=1}^{\infty}(-1)^{n+j+1}q^{j(j-3)/2+n(n-1)/2+2jn} + \sum_{n=-1}^{\infty}(-1)^{n+j}q^{j(j+5)/2+n(n-1)/2+2jn-2} \\
&= \sum_{n=0}^{\infty}(-1)^{n+j+1}q^{j(j+1)/2+n(n-3)/2+2jn-1}(1-q^{2n+1}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(1+z)\left(z,z^{-1};q\right)_\infty S_{A3}(z,q) &= -(1+z)\,(q;q)_\infty \\
&\quad + \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(z^j+z^{1-j})(-1)^{n+j+1}q^{j(j+1)/2+n(n-3)/2+2jn-1}(1-q^{2n+1}),
\end{aligned}
$$

which is (3.2). □

Proof of (2.11). We have

$$
(1+z)\left(z,z^{-1};q\right)_\infty S_{A5}(z,q) = (q;q)_\infty \sum_{n=1}^{\infty}\sum_{j=-n}^{n+1} \frac{(-1)^{j+1}z^j(1-q^{2j-1})q^{n^2+n+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \tag{3.20}
$$

The coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{A5}(z,q)$. We again use the same Bailey pair α and β as before but this time apply (3.5).

Starting with $j \geq 2$, we have

$$\begin{aligned}
&[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{A5}(z,q)\\
&=(q;q)_\infty \sum_{n=j-1}^{\infty} \frac{(-1)^{j+1}(1-q^{2j-1})q^{n^2+n+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}\\
&=(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}\,(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-n+2nj}}{(q;q)_{n+2j-1}\,(q;q)_n}\\
&=\frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}\,(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} \frac{q^{n^2-n+2nj}}{(q^{2j},q;q)_n}\\
&=\frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}\,(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} q^{n^2}q^{(2j-1)n}\beta_n(q^{2j-1})\\
&=\frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}\,(q;q)_\infty}{(q;q)_{2j-1}\,(q^{2j};q)_\infty}\\
&=(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}.
\end{aligned}$$

For $j=1$ we instead have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{A5}(z,q) = -\,(q;q)_\infty + (-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}.$$

We note that $(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}$ is invariant under $j\mapsto -j+1$. Thus we have

$$\begin{aligned}
(1+z)\left(z,z^{-1};q\right)_\infty S_{A5}(z,q) &= -(1+z)\,(q;q)_\infty + \sum_{j=-\infty}^{\infty} z^j(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-5j}{2}+1}\\
&= -(1+z)\,(q;q)_\infty + \sum_{j=-\infty}^{\infty} z^j(-1)^{j+1}q^{\frac{3j^2-5j}{2}+1} + \sum_{j=-\infty}^{\infty} z^j(-1)^j q^{\frac{3j^2-j}{2}}\\
&= -(1+z)\,(q;q)_\infty + \sum_{j=-\infty}^{\infty} z^{j+1}(-1)^j q^{\frac{3j^2+j}{2}} + \sum_{j=-\infty}^{\infty} z^{-j}(-1)^j q^{\frac{3j^2+j}{2}}\\
&= \sum_{j=-\infty}^{\infty} (z^{j+1}+z^{-j}-1-z)(-1)^j q^{\frac{3j^2+j}{2}}\\
&= \sum_{j=-\infty}^{\infty} (1-z^j)(1-z^{j+1})z^{-j}(-1)^j q^{\frac{3j^2+j}{2}}.
\end{aligned}$$

This proves (2.11).

□

Proof of (2.12). We have

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{A7}(z,q) = (q;q)_\infty \sum_{n=1}^{\infty}\sum_{j=-n}^{n+1} \frac{(-1)^{j+1}z^j(1-q^{2j-1})q^{n^2+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \tag{3.21}$$

The coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{A7}(z,q)$. This time we use the Bailey pair α^* and β^* from (3.14) and (3.15) and apply (3.5).

Starting with $j \geq 2$, we have

$$\begin{aligned}
&[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{A7}(z,q)\\
&=(q;q)_\infty \sum_{n=j-1}^{\infty} \frac{(-1)^{j+1}(1-q^{2j-1})q^{n^2+j(j-3)/2+1}}{(q;q)_{n+j}\,(q;q)_{n-j+1}}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-7j}{2}+2}(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-2n+2nj}}{(q;q)_{n+2j-1}(q;q)_n} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-7j}{2}+2}(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} \frac{q^{n^2-2n+2nj}}{(q^{2j},q;q)_n} \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-7j}{2}+2}(q;q)_\infty}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} q^{n^2}q^{(2j-2)n}\beta_n^*(q^{2j-2}) \\
&= \frac{(-1)^{j+1}(1-q^{2j-1})q^{\frac{3j^2-7j}{2}+2}(q;q)_\infty}{(q;q)_{2j-1}(q^{2j-1};q)_\infty}(1-q^{4j-2}) \\
&= (-1)^{j+1}q^{\frac{3j^2-7j}{2}+2}(1-q^{4j-2}).
\end{aligned}
$$

For $j=1$ we instead have

$$
[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{A7}(z,q) = -(q;q)_\infty + (-1)^{j+1}q^{\frac{3j^2-7j}{2}+2}(1-q^{4j-2}).
$$

We note that $(-1)^{j+1}q^{\frac{3j^2-7j}{2}+2}(1-q^{4j-2})$ is invariant under $j \mapsto -j+1$. Thus we have

$$
\begin{aligned}
(1+z)\left(z,z^{-1};q\right)_\infty S_{A7}(z,q) &= -(1+z)(q;q)_\infty + \sum_{j=-\infty}^{\infty} z^j(-1)^{j+1}q^{\frac{3j^2-7j}{2}+2}(1-q^{4j-2}) \\
&= -(1+z)(q;q)_\infty + \sum_{j=-\infty}^{\infty} z^j(-1)^{j+1}q^{\frac{3j^2-7j}{2}+2} + \sum_{j=-\infty}^{\infty} z^j(-1)^j q^{\frac{3j^2+j}{2}} \\
&= -(1+z)(q;q)_\infty + \sum_{j=-\infty}^{\infty} z^{j+1}(-1)^j q^{\frac{3j^2-j}{2}} + \sum_{j=-\infty}^{\infty} z^{-j}(-1)^j q^{\frac{3j^2-j}{2}} \\
&= \sum_{j=-\infty}^{\infty} (z^{j+1}+z^{-j}-1-z)(-1)^j q^{\frac{3j^2-j}{2}} \\
&= \sum_{j=-\infty}^{\infty} (1-z^j)(1-z^{j+1})z^{-j}(-1)^j q^{\frac{3j^2-j}{2}}.
\end{aligned}
$$

This proves (2.12).

□

Proof of (3.3). We use the fact that

$$
\frac{1}{(q;q^2)_n(q;q)_n} = \frac{\left(q^2;q^2\right)_n}{(q;q)_{2n}(q;q)_n} = \frac{(q,-q;q)_n}{(q;q)_{2n}(q;q)_n} = \frac{(-q;q)_n}{(q;q)_{2n}}.
$$

By (3.16) we then have

$$
\begin{aligned}
(1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q) &= \left(q;q^2\right)_\infty (q;q)_\infty \sum_{n=1}^{\infty} \frac{q^n(1+z)\left(z,z^{-1};q\right)_n(-q;q)_n}{(q;q)_{2n}} && (3.22) \\
&= \left(q;q^2\right)_\infty (q;q)_\infty \sum_{n=1}^{\infty}\sum_{j=-n}^{n+1} \frac{(-1)^{j+1}z^j(1-q^{2j-1})q^{n+j(j-3)/2+1}(-q;q)_n}{(q;q)_{n+j}(q;q)_{n-j+1}}. && (3.23)
\end{aligned}
$$

Again we find that the coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q)$. We again use the Bailey pair α and β from (3.12) and (3.13) but this time apply (3.11). Starting with $j \geq 2$, we have

$$
\begin{aligned}
&[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q) \\
&= \left(q;q^2\right)_\infty (q;q)_\infty \sum_{n=j-1}^{\infty} \frac{(-1)^{j+1}(1-q^{2j-1})q^{n+j(j-3)/2+1}(-q;q)_n}{(q;q)_{n+j}(q;q)_{n-j+1}}
\end{aligned}
$$

$$= (-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty\left(q;q^2\right)_\infty\sum_{n=0}^{\infty}\frac{q^n\,(-q;q)_{n+j-1}}{(q;q)_{n+2j-1}\,(q;q)_n}$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty\left(q;q^2\right)_\infty(-q;q)_{j-1}}{(q;q)_{2j-1}}\sum_{n=0}^{\infty}\frac{q^n\left(-q^j;q\right)_n}{(q^{2j},q;q)_n}$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty\left(q;q^2\right)_\infty(-q;q)_{j-1}}{(q;q)_{2j-1}}\sum_{n=0}^{\infty}\left(-q^j;q\right)_n q^n\beta_n(q^{2j-1})$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}\,(q;q)_\infty\left(q;q^2\right)_\infty(-q;q)_{j-1}\left(-q^j;q\right)_\infty}{(q;q)_{2j-1}\,(q^{2j},q;q)_\infty}\sum_{n=0}^{\infty}q^{3n(j-1)+n(3n+5)/2}(1-q^{j+n})$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{n(3n-1)/2+3nj}(1-q^{j+n}).$$

For $j=1$, we instead have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q)$$
$$= -\,(q;q)_\infty\left(q;q^2\right)_\infty + \frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j-1)/2}}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{n(3n-1)/2+3nj}(1-q^{j+n}).$$

Thus

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q)$$
$$= -(1+z)\,(q;q)_\infty\left(q;q^2\right)_\infty + \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(z^j+z^{1-j})(-1)^{j+1}q^{\frac{j(j-1)}{2}+\frac{n(3n-1)}{2}+3nj}(1-q^{j+n})(1-q^{2j-1}),\tag{3.24}$$

which is (3.3). □

Proof of (2.14). Again using $\frac{1}{(q;q^2)_n(q;q)_n}=\frac{(-q;q)_n}{(q;q)_{2n}}$ and (3.16) we have

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{C5}(z,q) = \left(q;q^2\right)_\infty(q;q)_\infty\sum_{n=1}^{\infty}\sum_{j=-n}^{n+1}\frac{(-1)^{j+1}z^j(1-q^{2j-1})q^{n(n+1)/2+j(j-3)/2+1}\,(-q;q)_n}{(q;q)_{n+j}\,(q;q)_{n-j+1}}.\tag{3.25}$$

Again we find that the coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{C5}(z,q)$. We again use the Bailey pair α and β from (3.12) and (3.13) but this time apply (3.6). Starting with $j\geq 2$, we have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{C5}(z,q)$$
$$= \left(q;q^2\right)_\infty(q;q)_\infty\sum_{n=j-1}^{\infty}\frac{(-1)^{j+1}(1-q^{2j-1})q^{n(n+1)/2+j(j-3)/2+1}\,(-q;q)_n}{(q;q)_{n+j}\,(q;q)_{n-j+1}}$$
$$= (-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}\,(q;q)_\infty\left(q;q^2\right)_\infty\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2+nj}\,(-q;q)_{n+j-1}}{(q;q)_{n+2j-1}\,(q;q)_n}$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}\,(q;q)_\infty\left(q;q^2\right)_\infty(-q;q)_{j-1}}{(q;q)_{2j-1}}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2+nj}\left(-q^j;q\right)_n}{(q^{2j},q;q)_n}$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}\,(q;q)_\infty\left(q;q^2\right)_\infty(-q;q)_{j-1}}{(q;q)_{2j-1}}\sum_{n=0}^{\infty}\left(-q^j;q\right)_n q^{n(2j-1)/2}q^{n^2/2}\beta_n(q^{2j-1})$$
$$= \frac{(-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}\,(q;q)_\infty\left(q;q^2\right)_\infty(-q;q)_{j-1}\left(-q^j;q\right)_\infty}{(q;q)_{2j-1}\,(q^{2j};q)_\infty}$$
$$= (-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}.$$

For $j = 1$, we instead have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q)$$
$$= -\left(q;q\right)_\infty \left(q;q^2\right)_\infty + (-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}.$$

We note that $(-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}$ is invariant under $j \mapsto 1-j$ and so

$$\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{C5}(z,q)\\
&= -(1+z)\left(q;q\right)_\infty \left(q;q^2\right)_\infty + \sum_{j=-\infty}^{\infty} z^j(-1)^{j+1}(1-q^{2j-1})q^{j^2-2j+1}\\
&= -(1+z)\left(q;q\right)_\infty \left(q;q^2\right)_\infty + \sum_{j=-\infty}^{\infty} z^j(-1)^{j+1}q^{j^2-2j+1} + \sum_{j=-\infty}^{\infty} z^j(-1)^j q^{j^2}\\
&= -(1+z)\left(q;q\right)_\infty \left(q;q^2\right)_\infty + \sum_{j=-\infty}^{\infty} z^{1-j}(-1)^j q^{j^2} + \sum_{j=-\infty}^{\infty} z^j(-1)^j q^{j^2}.
\end{aligned}$$

By Gauss we have

$$\left(q;q\right)_\infty \left(q;q^2\right)_\infty = \sum_{j=-\infty}^{\infty}(-1)^j q^{j^2},$$

and $z^{1-j} + z^j - 1 - z = (1-z^{j-1})(1-z^j)z^{1-j}$ which then proves (2.14). □

Proof of (2.15). Using $\frac{1}{(q^2;q^2)_n} = \frac{(q;q^2)_n}{(q;q)_{2n}}$ and (3.16) we have

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{E2}(z,q) = \left(q^2;q^2\right)_\infty \sum_{n=1}^{\infty} \frac{(-1)^n q^n (1+z)\left(z,z^{-1};q\right)_n \left(q;q^2\right)_n}{(q;q)_{2n}} \tag{3.26}$$

$$= \left(q^2;q^2\right)_\infty \sum_{n=1}^{\infty}\sum_{j=-n}^{n+1} \frac{(-1)^{n+j+1} z^j (1-q^{2j-1}) q^{n+j(j-3)/2+1}\left(q;q^2\right)_n}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \tag{3.27}$$

Again we find that the coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{E2}(z,q)$. We again use the Bailey pair α and β from (3.12) and (3.13) but this time apply (3.7). Starting with $j \geq 2$, we have

$$\begin{aligned}
&[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{E2}(z,q)\\
&= \left(q^2;q^2\right)_\infty \sum_{n=j-1}^{\infty} \frac{(-1)^{n+j+1}(1-q^{2j-1})q^{n+j(j-3)/2+1}\left(q;q^2\right)_n}{(q;q)_{n+j}\,(q;q)_{n-j+1}}\\
&= (1-q^{2j-1})q^{j(j-1)/2}\left(q^2;q^2\right)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^n \left(q;q^2\right)_{n+j-1}}{(q;q)_{n+2j-1}\,(q;q)_n}\\
&= \frac{(1-q^{2j-1})q^{j(j-1)/2}\left(q^2;q^2\right)_\infty \left(q;q^2\right)_{j-1}}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} \frac{(-1)^n q^n \left(q^{2j-1};q^2\right)_n}{(q^{2j},q;q)_n}\\
&= \frac{(1-q^{2j-1})q^{j(j-1)/2}\left(q^2;q^2\right)_\infty \left(q;q^2\right)_{j-1}}{(q;q)_{2j-1}} \sum_{n=0}^{\infty} \left(q^{2j-1};q^2\right)_n (-1)^n q^n \beta_n(q^{2j-1})\\
&= \frac{(1-q^{2j-1})q^{j(j-1)/2}\left(q^2;q^2\right)_\infty \left(q;q^2\right)_{j-1}\left(q^{2j+1};q^2\right)_\infty}{(q;q)_{2j-1}\,(q^{2j},-q;q)_\infty}\\
&= q^{j(j-1)/2}\left(q;q^2\right)_\infty.
\end{aligned}$$

For $j = 1$, we instead have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{E2}(z,q) = -\left(q^2;q^2\right)_\infty + q^{j(j-1)/2}\left(q;q^2\right)_\infty.$$

By Gauss we have

$$\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}=\sum_{j=1}^{\infty}q^{j(j-1)/2},$$

and so

$$\begin{aligned}
&(1+z)\left(z,z^{-1};q\right)_\infty S_{E2}(z,q)\\
&=-(1+z)\left(q^2;q^2\right)_\infty+\left(q;q^2\right)_\infty\sum_{j=1}^{\infty}(z^j+z^{1-j})q^{j(j-1)/2}\\
&=\left(q;q^2\right)_\infty\sum_{j=1}^{\infty}(z^j+z^{1-j}-1-z)q^{j(j-1)/2}\\
&=\left(q;q^2\right)_\infty\sum_{j=1}^{\infty}(1-z^{j-1})(1-z^j)z^{1-j}q^{j(j-1)/2}.
\end{aligned}$$

This proves (2.15). ☐

Proof of (3.4). Again using $\frac{1}{(q^2;q^2)_n}=\frac{(q;q^2)_n}{(q;q)_{2n}}$, and (3.16) we have

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q)=\left(q^2;q^2\right)_\infty\sum_{n=1}^{\infty}\sum_{j=-n}^{n+1}\frac{(-1)^{j+1}z^j(1-q^{2j-1})q^{2n+j(j-3)/2+1}\left(q;q^2\right)_n}{(q;q)_{n+j}\,(q;q)_{n-j+1}}. \tag{3.28}$$

Again we find that the coefficients of z^{-j} and z^{1+j} are the same in $(1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q)$. Here we use the Bailey pair α^* and β^* from (3.14) and (3.15) and apply (3.10). Starting with $j\geq 2$, we have

$$\begin{aligned}
&[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q)\\
&=\left(q^2;q^2\right)_\infty\sum_{n=j-1}^{\infty}\frac{(-1)^{j+1}(1-q^{2j-1})q^{2n+j(j-3)/2+1}\left(q;q^2\right)_n}{(q;q)_{n+j}\,(q;q)_{n-j+1}}\\
&=(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\left(q^2;q^2\right)_\infty\sum_{n=0}^{\infty}\frac{q^{2n}\left(q;q^2\right)_{n+j-1}}{(q;q)_{n+2j-1}\,(q;q)_n}\\
&=\frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\left(q^2;q^2\right)_\infty\left(q;q^2\right)_{j-1}}{(q;q)_{2j-1}}\sum_{n=0}^{\infty}\frac{q^{2n}\left(q^{2j-1};q^2\right)_n}{\left(q^{2j},q;q\right)_n}\\
&=\frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\left(q^2;q^2\right)_\infty\left(q;q^2\right)_{j-1}}{(q;q)_{2j-1}}\sum_{n=0}^{\infty}\left(q^{2j-1};q^2\right)_n q^{2n}\beta^*_n(q^{2j-2})\\
&=\frac{(-1)^{j+1}(1-q^{2j-1})q^{j(j+1)/2-1}\left(q^2;q^2\right)_\infty\left(q;q^2\right)_{j-1}}{(q;q)_{2j-1}\,(q;q)_\infty\left(q^{2j};q^2\right)_\infty(1+q)}\left(\sum_{n=0}^{\infty}(-1)^nq^{n(2j-2)+n^2+n}(1-q^{2n+2})(1-q^{2j+2n+1})\right)\\
&=\frac{(-1)^{j+1}q^{j(j+1)/2-1}}{(q;q)_\infty(1+q)}\left(\sum_{n=0}^{\infty}(-1)^nq^{n^2-n+2nj}(1-q^{2n+2})(1-q^{2j+2n+1})\right).
\end{aligned}$$

The $(1+q)$ in the denominator will cancel by elementary manipulations. We have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^nq^{n^2-n+2nj}(1-q^{2n+2})(1-q^{2j+2n+1})\\
&=\sum_{n=0}^{\infty}(-1)^nq^{n^2-n+2nj}+\sum_{n=0}^{\infty}(-1)^{n+1}q^{n^2+n+2nj+2}+\sum_{n=0}^{\infty}(-1)^{n+1}q^{n^2+n+2nj+2j+1}+\sum_{n=0}^{\infty}(-1)^nq^{n^2+3n+2nj+2j+3}\\
&=\sum_{n=0}^{\infty}(-1)^nq^{n^2-n+2nj}+\sum_{n=0}^{\infty}(-1)^{n+1}q^{n^2+n+2nj+2}+\sum_{n=1}^{\infty}(-1)^nq^{n^2-n+2nj+1}+\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+n+2nj+1}
\end{aligned}$$

$$= 1 - q^2 + \sum_{n=1}^{\infty}(-1)^n q^{n^2-n+2nj}(1+q) + \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+n+2nj+1}(1+q)$$
$$= (1+q)(1-q) + (1+q)\sum_{n=1}^{\infty}(-1)^n q^{n^2-n+2nj}(1-q^{2n+1})$$
$$= (1+q)\sum_{n=0}^{\infty}(-1)^n q^{n^2-n+2nj}(1-q^{2n+1}).$$

And so we have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q) = \frac{(-1)^{j+1}q^{j(j+1)/2-1}}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n^2-n+2nj}(1-q^{2n+1}).$$

For $j=1$, we instead have

$$[z^j](1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q) = -\left(q^2;q^2\right)_\infty + \frac{(-1)^{j+1}q^{j(j+1)/2-1}}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n^2-n+2nj}(1-q^{2n+1}).$$

Thus

$$\begin{aligned}(1+z)\left(z,z^{-1};q\right)_\infty S_{E4}(z,q) &= -(1+z)\left(q^2;q^2\right)_\infty \\ &+ \frac{1}{(q;q)_\infty}\sum_{j=1}^{\infty}(z^j+z^{1-j})\sum_{n=0}^{\infty}(-1)^{j+n+1}q^{j(j+1)/2+n^2-n+2nj-1}(1-q^{2n+1}),\end{aligned} \tag{3.29}$$

which is (3.4). □

Proof of (2.21), (2.22), (2.23), and (2.24). As noted in (3.17),

$$(1+z)\left(z,z^{-1};q\right)_\infty S_{A1}(z,q) = (q;q)_\infty \sum_{n=1}^{\infty}\frac{(1+z)\left(z,z^{-1};q\right)_n q^n}{(q;q)_{2n}}.$$

Thus setting $z=1$ in (3.1) gives

$$0 = -2\,(q;q)_\infty + \frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}2\left((-1)^{k+1}q^{k(k-1)/2} + \sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k(k-1)}{2}+\frac{n(n-3)}{2}+2kn}(1+q^n)\right).$$

This proves (2.21). Similarly (2.22), (2.23), and (2.24), follow by setting $z=1$ in (3.2), (3.3), and (3.4) respectively. □

Proof of (2.9), (2.10), (2.13), and (2.16) . We see (2.9) follows directly from (3.1) and (2.21), noting that $z^k + z^{1-k} - 1 - z = (1-z^{k-1})(1-z^k)z^{1-k} = (z^{1-k}-1)(1-z^k)z^{1-k}$. The remaining three identities of Theorem 2.2 follow as well with the corresponding product identities product identities of Corollary 2.6. □

Proof for Corollary 2.3 . The proofs only require elementary rearrangements of series and Theorem 2.2. For (2.17) we have

$$\sum_{k=1}^{\infty}\sum_{n=-[k/2]}^{[k/2]}(-1)^{n+k}(1-z^{k-2|n|})(1-z^{2|n|-k+1})q^{\frac{k^2-k-3n^2-n}{2}}$$
$$= \sum_{k=0}^{\infty}\sum_{n=-[k/2]}^{[k/2]}(-1)^{n+k}(1-z^{k-2|n|})(1-z^{2|n|-k+1})q^{\frac{k^2-k-3n^2-n}{2}}$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k\geq 2|n|}(-1)^{n+k}(1-z^{k-2|n|})(1-z^{2|n|-k+1})q^{\frac{k^2-k-3n^2-n}{2}}$$
$$= \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}(1-z^k)(1-z^{-k+1})q^{\frac{k^2-k+n^2-n}{2}+2k|n|-|n|}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}(1-z^k)(1-z^{-k+1})q^{\frac{k^2-k+n^2-n}{2}+2k|n|-|n|}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k+1}(1-z^k)(1-z^{k-1})z^{1-k}q^{\frac{k^2-k+n^2-n}{2}+2k|n|-|n|}$$

$$= \sum_{k=1}^{\infty}(1-z^k)(1-z^{k-1})z^{1-k}\left((-1)^{k+1}q^{\frac{k^2-k}{2}}+\sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k^2-k+n^2-n}{2}+2kn-n}\right.$$
$$\left.+\sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k^2-k+n^2+n}{2}+2kn-n}\right)$$
$$= \sum_{k=1}^{\infty}(1-z^k)(1-z^{k-1})z^{1-k}\left((-1)^{k+1}q^{\frac{k^2-k}{2}}+\sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k^2-k+n^2-3n}{2}+2kn}(1+q^n)\right)$$
$$= (1+z)\left(q,z,z^{-1};q\right)_\infty S_{A1}(z,q).$$

For (2.18) we have

$$\sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(1-z^{k-2n+1})(1-z^{2n-k})q^{\frac{k^2-k-3n^2+n}{2}}$$
$$-\sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(1-z^{k-2n})(1-z^{2n-k+1})q^{\frac{k^2+k-3n^2-n}{2}}$$
$$= \sum_{n=1}^{\infty}\sum_{k=2n}^{\infty}(-1)^{n+k}(1-z^{k-2n+1})(1-z^{2n-k})q^{\frac{k^2-k-3n^2+n}{2}}$$
$$-\sum_{n=0}^{\infty}\sum_{k=2n}^{\infty}(-1)^{n+k}(1-z^{k-2n})(1-z^{2n-k+1})q^{\frac{k^2+k-3n^2-n}{2}}$$
$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k+1}(1-z^k)(1-z^{-k+1})q^{\frac{k^2-3k+n^2-5n}{2}+2kn+1}$$
$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}(1-z^k)(1-z^{-k+1})q^{\frac{k^2+k+n^2+n}{2}+2kn}$$
$$= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}(1-z^k)(1-z^{-k+1})q^{\frac{k^2+k+n^2-3n}{2}+2kn-1}$$
$$-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}(1-z^k)(1-z^{-k+1})q^{\frac{k^2+k+n^2+n}{2}+2kn}$$
$$= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(1-z^k)(1-z^{k-1})z^{1-k}(-1)^{n+k+1}q^{\frac{k^2+k+n^2-3n}{2}+2kn-1}(1-q^{2n+1})$$
$$= (1+z)\left(q,z,z^{-1};q\right)_\infty S_{A3}(z,q).$$

For (2.19) we have

$$\sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1-z^{3n-k+1})(1-z^{k-3n})q^{\frac{k^2-k}{2}-3n^2+n}$$
$$-\sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1-z^{3n-k+1})(1-z^{k-3n})q^{\frac{k^2+k}{2}-3n^2-n}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1-z^{3n-k})(1-z^{k-3n+1})q^{\frac{k^2-k}{2}-3n^2+n}\\
&-\sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1-z^{3n-k})(1-z^{k-3n+1})q^{\frac{k^2+k}{2}-3n^2-n}\\
&=\sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1-z^{3n-k+1})(1-z^{k-3n})q^{\frac{k^2-k}{2}-3n^2-n}(q^{2n}-q^k)\\
&+\sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1-z^{3n-k})(1-z^{k-3n+1})q^{\frac{k^2-k}{2}-3n^2-n}(q^{2n}-q^k)\\
&=\sum_{n=0}^{\infty}\sum_{k=3n}^{\infty}(-1)^{n+k}(1-z^{3n-k+1})(1-z^{k-3n})q^{\frac{k^2-k}{2}-3n^2-n}(q^{2n}-q^k)\\
&+\sum_{n=1}^{\infty}\sum_{k=3n}^{\infty}(-1)^{n+k}(1-z^{3n-k})(1-z^{k-3n+1})q^{\frac{k^2-k}{2}-3n^2-n}(q^{2n}-q^k)\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{k}(1-z^{1-k})(1-z^{k})q^{\frac{k^2-k+3n^2-5n}{2}+3kn}(q^{2n}-q^{k+3n})\\
&-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k}(1-z^{1-k})(1-z^{k})q^{\frac{k^2-3k+3n^2-11n}{2}+3kn+1}(q^{2n}-q^{k+3n-1})\\
&=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{k}(1-z^{1-k})(1-z^{k})q^{\frac{k^2-k+3n^2-n}{2}+3kn}(1-q^{k+n})\\
&-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{k}(1-z^{1-k})(1-z^{k})q^{\frac{k^2+3k+3n^2-n}{2}+3kn-1}(1-q^{k+n})\\
&=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{k}(1-z^{1-k})(1-z^{k})q^{\frac{k^2-k+3n^2-n}{2}+3kn}(1-q^{k+n})(1-q^{2k-1})\\
&=(1+z)\left(q,z,z^{-1};q\right)_\infty S_{C1}(z,q).
\end{aligned}
$$

For (2.20) we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(1-z^{2n-k})(1-z^{k-2n+1})q^{\frac{k^2-k}{2}-n^2}\\
&\quad-\sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(1-z^{2n-k+1})(1-z^{k-2n})q^{\frac{k^2+k}{2}-n^2}\\
&=\sum_{n=1}^{\infty}\sum_{k=2n}^{\infty}(-1)^{n+k}(1-z^{2n-k})(1-z^{k-2n+1})q^{\frac{k^2-k}{2}-n^2}\\
&\quad-\sum_{n=0}^{\infty}\sum_{k=2n}^{\infty}(-1)^{n+k}(1-z^{2n-k+1})(1-z^{k-2n})q^{\frac{k^2+k}{2}-n^2}\\
&=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k+1}(1-z^{1-k})(1-z^{k})q^{\frac{k^2-3k}{2}+n^2-3n+2kn+1}\\
&\quad-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}(1-z^{1-k})(1-z^{k})q^{\frac{k^2+k}{2}+n^2+n+2kn}
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}(1-z^{1-k})(1-z^k)q^{\frac{k^2+k}{2}+n^2-n+2kn-1}$$
$$- \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}(1-z^{1-k})(1-z^k)q^{\frac{k^2+k}{2}+n^2+n+2kn}$$
$$= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}(1-z^{1-k})(1-z^k)q^{\frac{k^2+k}{2}+n^2-n+2kn-1}(1-q^{2n+1})$$
$$= (1+z)\left(q,z,z^{-1};q\right)_\infty S_{E4}(z,q).$$

□

Proof of (2.25) and (2.26). Using essentially the same series rearrangements as in the proof of (2.19), we find that

$$\sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(-z^{3n-k+1}-z^{k-3n})q^{\frac{k^2-k}{2}-3n^2+n}$$
$$- \sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(-z^{3n-k+1}-z^{k-3n})q^{\frac{k^2+k}{2}-3n^2-n}$$
$$+ \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(-z^{3n-k}-z^{k-3n+1})q^{\frac{k^2-k}{2}-3n^2+n}$$
$$- \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(-z^{3n-k}-z^{k-3n+1})q^{\frac{k^2+k}{2}-3n^2-n}$$
$$= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{k}(-z^{1-k}-z^k)q^{\frac{k^2-k+3n^2-n}{2}+3kn}(1-q^{k+n})(1-q^{2k-1}) - (1+z)\sum_{n=0}^{\infty}q^{\frac{3n^2-n}{2}}(1-q^n).$$

With (3.24) we then have

$$- (1+z)\left(q;q\right)_\infty^2\left(q;q^2\right)_\infty$$
$$= (1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q) - \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(z^j+z^{1-j})(-1)^{j+1}q^{\frac{j(j-1)}{2}+\frac{n(3n-1)}{2}+3nj}(1-q^{j+n})(1-q^{2j-1})$$
$$= (1+z)\left(z,z^{-1};q\right)_\infty S_{C1}(z,q) - \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{k}(-z^{1-k}-z^k)q^{\frac{k^2-k+3n^2-n}{2}+3kn}(1-q^{k+n})(1-q^{2k-1})$$
$$- (1+z)\sum_{n=0}^{\infty}q^{\frac{3n^2-n}{2}}(1-q^n)$$
$$= \sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1+z)q^{\frac{k^2-k}{2}-3n^2+n} - \sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}(1+z)q^{\frac{k^2+k}{2}-3n^2-n}$$
$$+ \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1+z)q^{\frac{k^2-k}{2}-3n^2+n} - \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}(1+z)q^{\frac{k^2+k}{2}-3n^2-n} - (1+z)\sum_{n=0}^{\infty}q^{\frac{3n^2-n}{2}}(1-q^n),$$

so that

$$\left(q;q\right)_\infty^2\left(q;q^2\right)_\infty$$
$$= -\sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}q^{\frac{k^2-k}{2}-3n^2+n} + \sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2-n} - \sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}q^{\frac{k^2-k}{2}-3n^2+n}$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2-n}+\sum_{n=0}^{\infty}q^{\frac{3n^2-n}{2}}(1-q^n)$$

$$=\sum_{k=1}^{\infty}\sum_{n=0}^{[k/3]}(-1)^{n+k+1}q^{\frac{k^2-k}{2}-3n^2+n}+\sum_{k=1}^{\infty}\sum_{n=-[k/3]}^{0}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}-\sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}q^{\frac{k^2-k}{2}-3n^2+n}$$

$$+\sum_{k=1}^{\infty}\sum_{n=-[k/3]}^{-1}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}$$

$$+\sum_{n=0}^{\infty}q^{\frac{3n^2-n}{2}}(1-q^n)$$

We use that

$$\sum_{k=1}^{\infty}\sum_{n=-[k/3]}^{-1}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}=\sum_{k=0}^{\infty}\sum_{n=-[k/3]}^{-1}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}$$

$$=\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{-1}(-1)^{n+k+1}q^{\frac{k^2-k}{2}-3n^2+n}$$

and

$$-\sum_{k=1}^{\infty}\sum_{n=1}^{[k/3]}(-1)^{n+k}q^{\frac{k^2-k}{2}-3n^2+n}=\sum_{k=0}^{\infty}\sum_{n=1}^{[(k+1)/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}=\sum_{k=1}^{\infty}\sum_{n=1}^{[(k+1)/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n},$$

to find that

$$(q;q)_\infty^2\left(q;q^2\right)_\infty$$

$$=\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{[k/3]}(-1)^{n+k+1}q^{\frac{k^2-k}{2}-3n^2+n}+\sum_{k=1}^{\infty}\sum_{n=-[k/3]}^{[(k+1)/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}+\sum_{n=0}^{\infty}q^{\frac{3n^2-n}{2}}(1-q^n).$$

But $[(k+1)/3]=[k/3]$ when $k\not\equiv 2 \pmod 3$ and $[k/3]=-[(k-1)/3]$ when $k\not\equiv 3 \pmod 3$, so we find

$$\sum_{k=1}^{\infty}\sum_{n=-[k/3]}^{[(k+1)/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}=\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{[k/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}-\sum_{k=1}^{\infty}q^{\frac{3k^2-k}{2}}+\sum_{k=1}^{\infty}q^{\frac{3k^2+k}{2}}.$$

Thus

$$(q;q)_\infty^2\left(q;q^2\right)_\infty=\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{[k/3]}(-1)^{n+k+1}q^{\frac{k^2-k}{2}-3n^2+n}+\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{[k/3]}(-1)^{n+k}q^{\frac{k^2+k}{2}-3n^2+n}$$

$$=\sum_{k=1}^{\infty}\sum_{n=-[(k-1)/3]}^{[k/3]}(-1)^{n+k+1}q^{\frac{k^2-k}{2}-3n^2+n}(1-q^k).$$

Using essentially the same series rearrangements as in the proof of (2.20), we find that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(-z^{2n-k}-z^{k-2n+1})q^{\frac{k^2-k}{2}-n^2}-\sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(-z^{2n-k+1}-z^{k-2n})q^{\frac{k^2+k}{2}-n^2}$$

$$=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k+1}(z^{1-k}+z^k)q^{\frac{k^2+k}{2}+n^2-n+2kn-1}(1-q^{2n+1})-\sum_{n=1}^{\infty}(-1)^{n+1}(1+z)q^{n^2+n}.$$

With (3.29) we then have

$$-\left(1+z\right)\left(q;q\right)_\infty\left(q^2;q^2\right)_\infty$$

$$= (1+z)\left(q,z,z^{-1};q\right)_\infty S_{E4}(z,q) - \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k+1}(z^{1-k}+z^k)q^{\frac{k^2+k}{2}+n^2-n+2kn-1}(1-q^{2n+1})$$

$$= (1+z)\left(q,z,z^{-1};q\right)_\infty S_{E4}(z,q) - \sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(-z^{2n-k}-z^{k-2n+1})q^{\frac{k^2-k}{2}-n^2}$$

$$+\sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(-z^{2n-k+1}-z^{k-2n})q^{\frac{k^2+k}{2}-n^2} - \sum_{n=1}^{\infty}(-1)^{n+1}(1+z)q^{n^2+n}$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}(1+z)q^{\frac{k^2-k}{2}-n^2} - \sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}(1+z)q^{\frac{k^2+k}{2}-n^2} - \sum_{n=1}^{\infty}(-1)^{n+1}(1+z)q^{n^2+n},$$

so that

$$\begin{aligned}(q;q)_\infty\left(q^2;q^2\right)_\infty &= -\sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}q^{\frac{k^2-k}{2}-n^2} + \sum_{k=1}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} + \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2+n}\\ &= -\sum_{k=1}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}q^{\frac{k^2-k}{2}-n^2} + \sum_{k=0}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} + \sum_{n=0}^{\infty}(-1)^{n+1}q^{n^2+n}\\ &= \sum_{k=0}^{\infty}\sum_{n=1}^{[(k+1)/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} + \sum_{k=0}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} + \sum_{n=0}^{\infty}(-1)^{n+1}q^{n^2+n}.\end{aligned}$$

We now note $[(k+1)/2]=[k/2]$ when k is even and so we find that

$$\sum_{k=0}^{\infty}\sum_{n=1}^{[(k+1)/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} = \sum_{k=0}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} + \sum_{k=0}^{\infty}(-1)^k q^{k^2+k}.$$

Thus

$$(q;q)_\infty\left(q^2;q^2\right)_\infty = \sum_{k=0}^{\infty}\sum_{n=1}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} + \sum_{k=0}^{\infty}\sum_{n=0}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2} = \sum_{k=0}^{\infty}\sum_{n=-[k/2]}^{[k/2]}(-1)^{n+k}q^{\frac{k^2+k}{2}-n^2}.$$

□

4. Proofs of Congruences by Theorem 2.2

We recall to prove the congruences in Theorem 2.1 we are to show the following terms are zero: q^{3n} in $S_{A1}(\zeta_3,q)$, q^{3n+1} in $S_{A3}(\zeta_3,q)$, q^{5n+4} in $S_{A5}(\zeta_5,q)$, q^{7n+1} in $S_{A5}(\zeta_7,q)$, q^{5n+4} in $S_{A7}(\zeta_5,q)$, q^{5n+3} in $S_{C5}(\zeta_5,q)$, q^{3n} in $S_{E2}(\zeta_3,q)$, and q^{3n+1} in $S_{E4}(\zeta_3,q)$. The double series do not appear to easily give that the terms q^{5n+1} in $S_{A3}(\zeta_5,q)$ and q^{5n+3} in $S_{C1}(\zeta_5,q)$ are zero.

Corollary 4.1. *For $n\ge 0$, $M_{A1}(0,3,3n)=M_{A1}(1,3,3n)=M_{A1}(2,3,3n)=\frac{1}{3}\mathrm{spt}_{A1}(3n)$.*

Proof. We are to show that $[q^{3n}]S_{A1}(\zeta_3,q)=0$ for $n\ge 0$. We note that

$$\frac{1}{\left(\zeta_3 q,\zeta_3^{-1}q;q\right)_\infty} = \frac{(q;q)_\infty}{(q^3;q^3)_\infty}.$$

By (2.9) we have that

$$\begin{aligned}&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^2)S_{A1}(\zeta_3,q)\\ &= \frac{1}{(q^3;q^3)_\infty}\sum_{k=1}^{\infty}(1-\zeta_3^{k-1})(1-\zeta_3^k)\zeta_3^{1-k}\left((-1)^{k+1}q^{k(k-1)/2} + \sum_{n=1}^{\infty}(-1)^{n+k+1}q^{\frac{k(k-1)}{2}+\frac{n(n-3)}{2}+2kn}(1+q^n)\right).\end{aligned}$$

Upon inspection we find that when the q^{3N} terms occur, we have either $k\equiv 0 \pmod 3$ or $k\equiv 1 \pmod 3$. However for such values of k we have either $(1-\zeta_3^k)=0$ or $(1-\zeta_3^{k-1})=0$. Thus the coefficient of q^{3N} in $S_{A1}(\zeta_3,q)$ must be zero. □

Corollary 4.2. *For $n \geq 0$, $M_{A3}(0,3,3n+1) = M_{A3}(1,3,3n+1) = M_{A3}(2,3,3n+1) = \frac{1}{3}\mathrm{spt}_{A3}(3n+1)$.*

Proof. We are to show for $N \geq 0$ that $[q^{3N+1}]S_{A3}(\zeta_3, q) = 0$. By (2.10) we have that

$$\begin{aligned}
&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^2)S_{A3}(z,q) \\
&= \frac{1}{(q^3;q^3)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(1-\zeta_3^{k-1})(1-\zeta_3^k)\zeta_3^{1-k}(-1)^{n+k+1}q^{\frac{k(k+1)}{2}+\frac{n(n-3)}{2}+2kn-1}(1-q^{2n+1}).
\end{aligned}$$

Similar to before we find that when the q^{3N+1} terms occur, we have either $k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$. Again for such values of k we have either $(1-\zeta_3^k) = 0$ or $(1-\zeta_3^{k-1}) = 0$. Thus the coefficient of q^{3N+1} in $S_{A3}(\zeta_3, q)$ must be zero.

□

Corollary 4.3. *For $n \geq 0$, $M_{E2}(0,3,3n) = M_{E2}(1,3,3n) = M_{E2}(2,3,3n) = \frac{1}{3}\mathrm{spt}_{E2}(3n)$.*

Proof. We are to show for $N \geq 0$ that $[q^{3N}]S_{E2}(\zeta_3, q) = 0$. By (2.15) we have that

$$\begin{aligned}
&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^2)S_{E2}(z,q) \\
&= \frac{(q;q)_\infty (q;q^2)_\infty}{(q^3;q^3)_\infty}\sum_{k=1}^{\infty}(1-\zeta_3^k)(1-\zeta_3^{k-1})\zeta_3^{1-k}q^{\frac{k(k-1)}{2}} \\
&= \frac{1}{(q^3;q^3)_\infty}\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}(1-\zeta_3^k)(1-\zeta_3^{k-1})\zeta_3^{1-k}(-1)^n q^{\frac{k(k-1)}{2}+n^2}.
\end{aligned}$$

When the q^{3N} terms occur, we have either $k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$. Thus the coefficient of q^{3N} in $S_{E2}(\zeta_3, q)$ must be zero.

□

Corollary 4.4. *For $n \geq 0$, $M_{E4}(0,3,3n+1) = M_{E4}(1,3,3n+1) = M_{E4}(2,3,3n+1) = \frac{1}{3}\mathrm{spt}_{E4}(3n+1)$.*

Proof. We are to show for $N \geq 0$ that $[q^{3N+1}]S_{E4}(\zeta_3, q) = 0$. By (2.16) we have that

$$\begin{aligned}
&(1+\zeta_3)(1-\zeta_3)(1-\zeta_3^2)S_{E4}(z,q) \\
&= \frac{1}{(q^3;q^3)_\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(1-\zeta_3^{k-1})(1-\zeta_3^k)\zeta_3^{1-k}(-1)^{k+n+1}q^{\frac{k(k+1)}{2}+n^2-n+2kn-1}(1-q^{2n+1}).
\end{aligned}$$

Again when the q^{3N+1} terms occur we must have either $k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$. Thus the coefficient of q^{3N+1} in $S_{E4}(z,q)$ is zero. □

Corollary 4.5. *For $n \geq 0$, $M_{A5}(0,5,5n+4) = M_{A5}(1,5,5n+4) = M_{A5}(2,5,5n+4) = M_{A5}(3,5,5n+4) = M_{A5}(4,5,5n+4) = \frac{1}{5}\mathrm{spt}_{A5}(5n+4)$.*

Proof. We are to show for $N \geq 0$ that $[q^{5N+4}]S_{A5}(\zeta_5, q) = 0$. By Lemma 3.9 of [11] we have

$$\frac{1}{(\zeta_5 q, \zeta_5^{-1}q; q)_\infty} = \frac{1}{[q^5;q^{25}]_\infty} + q\frac{(\zeta_5+\zeta_5^{-1})}{[q^{10};q^{25}]_\infty}.$$

By (2.11) we then have

$$\begin{aligned}
&(1+\zeta_5)(1-\zeta_5)(1-\zeta_5^{-1})S_{A5}(\zeta_5,q) \\
&= \frac{1}{[q^5;q^{25}]_\infty}\sum_{k=-\infty}^{\infty}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k+1})\zeta_5^{-k}q^{\frac{k(3k+1)}{2}} \\
&\quad + \frac{(\zeta_5+\zeta_5^{-2})q}{[q^{10};q^{25}]_\infty}\sum_{k=-\infty}^{\infty}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k+1})\zeta_5^{-k}q^{\frac{k(3k+1)}{2}}.
\end{aligned}$$

However we find that $\frac{k(3k+1)}{2}$ is never congruent to 3 or 4 modulo 5, so we see the coefficient of q^{5N+4} in $S_{A5}(\zeta_5, q)$ must be zero.

□

Corollary 4.6. *For $n \geq 0$, $M_{A7}(0,5,5n+4) = M_{A7}(1,5,5n+4) = M_{A7}(2,5,5n+4) = M_{A7}(3,5,5n+4) = M_{A7}(4,5,5n+4) = \frac{1}{5}\mathrm{spt}_{A7}(5n+4)$.*

Proof. We are to show for $N \geq 0$ that $[q^{5N+4}]S_{A7}(\zeta_5,q) = 0$. By (2.12) we then have

$$\begin{aligned}&(1+\zeta_5)(1-\zeta_5)(1-\zeta_5^{-1})S_{A7}(\zeta_5,q)\\&= \frac{1}{[q^5;q^{25}]_\infty}\sum_{k=-\infty}^{\infty}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k+1})\zeta_5^{-k}q^{\frac{k(3k-1)}{2}}\\&\quad+\frac{(\zeta_5+\zeta_5^{-2})q}{[q^{10};q^{25}]_\infty}\sum_{k=-\infty}^{\infty}(-1)^k(1-\zeta_5^k)(1-\zeta_5^{k+1})\zeta_5^{-k}q^{\frac{k(3k-1)}{2}}.\end{aligned}$$

However we find that $\frac{k(3k-1)}{2}$ is never congruent to 3 or 4 modulo 5, so we see the coefficient of q^{5N+4} in $S_{A7}(\zeta_5,q)$ must be zero. ☐

Corollary 4.7. *For $n \geq 0$, $M_{C5}(0,5,5n+3) = M_{C5}(1,5,5n+3) = M_{C5}(2,5,5n+3) = M_{C5}(3,5,5n+3) = M_{C5}(4,5,5n+3) = \frac{1}{5}\mathrm{spt}_{C5}(5n+3)$.*

Proof. We are to show for $N \geq 0$ that $[q^{5N+3}]S_{C5}(\zeta_5,q) = 0$. By (2.14) we have

$$\begin{aligned}&(1+\zeta_5)(1-\zeta_5)(1-\zeta_5^{-1})S_{C5}(\zeta_5,q)\\&= \frac{1}{[q^5;q^{25}]_\infty}\sum_{k=-\infty}^{\infty}(1-\zeta_5^{k-1})(1-\zeta_5^k)\zeta_5^{1-k}(-1)^kq^{k^2}+\frac{(\zeta_5+\zeta_5^{-2})q}{[q^{10};q^{25}]_\infty}\sum_{k=-\infty}^{\infty}(1-\zeta_5^{k-1})(1-\zeta_5^k)\zeta_5^{1-k}(-1)^kq^{k^2}.\end{aligned}$$

However we find that k^2 is never congruent to 2 or 3 modulo 5, so we see the coefficient of q^{5N+3} in $S_{C5}(\zeta_5,q)$ must be zero. ☐

Corollary 4.8. *For $n \geq 0$, $M_{A5}(0,7,7n+1) = M_{A5}(1,7,7n+1) = M_{A5}(2,7,7n+1) = M_{A5}(3,7,7n+1) = M_{A5}(4,7,7n+1) = M_{A5}(5,7,7n+1) = M_{A5}(6,7,7n+1) = \frac{1}{7}\mathrm{spt}_{A5}(7n+1)$.*

Proof. We are to show for $N \geq 0$ that $[q^{7N+1}]S_{A5}(\zeta_7,q) = 0$. To do this we actually find the 7-dissection of $S_{A5}(\zeta_7,q)$ and see there are no q^{7N+1} terms. We claim that

$$\begin{aligned}S_{A5}(\zeta_7,q) = \left(q^{49};q^{49}\right)_\infty\Bigg(&-(1+\zeta+\zeta^6)\frac{q^{14}}{[q^{42},q^{49},q^{56};q^{147}]_\infty}+q^2\frac{[q^{14};q^{49}]_\infty}{[q^7,q^{21};q^{49}]_\infty}\\&-(1+\zeta^2+\zeta^5)\frac{q^9}{[q^{21},q^{49},q^{70};q^{147}]_\infty}+(\zeta_7+\zeta_7^6)\frac{q^3}{[q^{14};q^{49}]_\infty}+(1+\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6)\frac{q^4}{[q^{21};q^{49}]_\infty}\\&+(\zeta+\zeta^6)q^5\frac{[q^{35};q^{147}]_\infty}{[q^{21},q^{28},q^{49},q^{49};q^{147}]_\infty}+(\zeta+\zeta^6)q^{19}\frac{[q^{14};q^{147}]_\infty}{[q^{21},q^{49},q^{49},q^{70};q^{147}]_\infty}\\&+(2+\zeta_7^2+\zeta_7^5)\frac{q^6}{[q^{14},q^{49},q^{63};q^{147}]_\infty}\Bigg),\end{aligned}\tag{4.1}$$

which we see has no terms of the form q^{7N+1}. To begin we have by (2.11) that

$$S_{A5}(\zeta_7,q) = \frac{1}{\left(\zeta_7q,\zeta_7^{-1}q;q\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-\zeta_7^n)(1-\zeta_7^{n+1})}{(1+\zeta_7)(1-\zeta_7)(1-\zeta_7^{-1})}\zeta_7^{-n}q^{\frac{n(3n+1)}{2}},$$

so we can instead verify

$$\begin{aligned}&\frac{(q;q)_\infty}{\left(\zeta_7q,\zeta_7^{-1}q;q\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-\zeta_7^n)(1-\zeta_7^{n+1})}{(1+\zeta_7)(1-\zeta_7)(1-\zeta_7^{-1})}\zeta_7^{-n}q^{\frac{n(3n+1)}{2}}\\&= (q;q)_\infty\left(q^{49};q^{49}\right)_\infty\Bigg(-(1+\zeta+\zeta^6)\frac{q^{14}}{[q^{42},q^{49},q^{56};q^{147}]_\infty}+q^2\frac{[q^{14};q^{49}]_\infty}{[q^7,q^{21};q^{49}]_\infty}\\&\quad-(1+\zeta^2+\zeta^5)\frac{q^9}{[q^{21},q^{49},q^{70};q^{147}]_\infty}+(\zeta_7+\zeta_7^6)\frac{q^3}{[q^{14};q^{49}]_\infty}+(1+\zeta_7+\zeta_7^2+\zeta_7^5+\zeta_7^6)\frac{q^4}{[q^{21};q^{49}]_\infty}\end{aligned}$$

$$+(\zeta+\zeta^6)q^5\frac{[q^{35};q^{147}]_\infty}{[q^{21},q^{28},q^{49},q^{49};q^{147}]_\infty}+(\zeta+\zeta^6)q^{19}\frac{[q^{14};q^{147}]_\infty}{[q^{21},q^{49},q^{49},q^{70};q^{147}]_\infty}$$
$$+(2+\zeta_7^2+\zeta_7^5)\frac{q^6}{[q^{14},q^{49},q^{63};q^{147}]_\infty}\Bigg). \tag{4.2}$$

For the series we have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-\zeta_7^n)(1-\zeta_7^{n+1})}{(1+\zeta_7)(1-\zeta_7)(1-\zeta_7^{-1})}\zeta_7^{-n}q^{\frac{n(3n+1)}{2}}$$
$$=\sum_{k=-3}^{3}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+k}(1-\zeta_7^k)(1-\zeta_7^{k+1})\zeta_7^{-k}}{(1+\zeta_7)(1-\zeta_7)(1-\zeta_7^{-1})}q^{\frac{(7n+k)(3(7n+k)+1)}{2}}$$
$$=\sum_{k=-3}^{3}\frac{(-1)^k(1-\zeta_7^k)(1-\zeta_7^{k+1})\zeta_7^{-k}}{(1+\zeta_7)(1-\zeta_7)(1-\zeta_7^{-1})}q^{\frac{k(3k+1)}{2}}\sum_{n=-\infty}^{\infty}(-1)^nq^{21nk+77}q^{\frac{147n(n-1)}{2}}$$
$$=(1+\zeta_7+\zeta_7^6)q^{12}\left[q^{14};q^{147}\right]_\infty\left(q^{147};q^{147}\right)_\infty-q^5\left[q^{35};q^{147}\right]_\infty\left(q^{147};q^{147}\right)_\infty+q^2\left(q^{49};q^{49}\right)_\infty$$
$$-(1+\zeta_7+\zeta_7^6)q^7\left[q^{119};q^{147}\right]_\infty\left(q^{147};q^{147}\right)_\infty+(1-\zeta_7^3-\zeta_7^4)q^{15}\left[q^{140};q^{147}\right]_\infty\left(q^{147};q^{147}\right)_\infty. \tag{4.3}$$

The last line of the above equality follows from the Jacobi Triple Product Identity.

By Theorem 5.1 of [11] we have

$$\frac{(q;q)_\infty}{(\zeta_7q,\zeta_7^{-1}q;q)_\infty}=\left(q^{49};q^{49}\right)_\infty\Bigg(\frac{[q^{21};q^{49}]_\infty}{[q^7,q^{14};q^{49}]_\infty}+(\zeta_7+\zeta_7^6-1)q\frac{1}{[q^7;q^{49}]_\infty}+(\zeta_7^2+\zeta_7^5)q^2\frac{[q^{14};q^{49}]_\infty}{[q^7,q^{21};q^{49}]_\infty}$$
$$+(\zeta_7^3+\zeta_7^4+1)q^3\frac{1}{[q^{14};q^{49}]_\infty}-(\zeta_7+\zeta_7^6)q^4\frac{1}{[q^{21};q^{49}]_\infty}-(\zeta_7^2+\zeta_7^5+1)q^6\frac{[q^7;q^{49}]_\infty}{[q^{14},q^{21};q^{49}]_\infty}\Bigg). \tag{4.4}$$

By Euler's Pentagonal Numbers Theorem and the Jacobi Triple Product Identity we have

$$(q;q)_\infty=\left(q^{49};q^{49}\right)_\infty\left(\frac{[q^{14};q^{49}]_\infty}{[q^7;q^{49}]_\infty}-q\frac{[q^{21};q^{49}]_\infty}{[q^{14};q^{49}]_\infty}-q^2+q^5\frac{[q^7;q^{49}]_\infty}{[q^{21};q^{49}]_\infty}\right). \tag{4.5}$$

To verify (4.2), we multiply the right hand side of (4.1) by (4.5) collect the q^{7N+k} terms, for $0\le k\le 6$, and verify each of those is equal to the corresponding term from multiplying (4.3) by (4.4). We do not include the full details, but as an example, in verifying the q^{7N} terms match, we find we are to prove the following identity:

$$\left(q^{49};q^{49}\right)_\infty^2\Bigg(-(1+\zeta+\zeta^6)q^{14}\frac{[q^{14};q^{49}]_\infty}{[q^7;q^{49}]_\infty[q^{42},q^{49},q^{56};q^{147}]_\infty}-(2+\zeta_7^2+\zeta_7^5)q^7\frac{[q^{21};q^{49}]_\infty}{[q^{14};q^{49}]_\infty[q^{14},q^{49},q^{63};q^{147}]_\infty}$$
$$-(\zeta+\zeta^6)q^7\frac{[q^{35};q^{147}]_\infty}{[q^{21},q^{28},q^{49},q^{49};q^{147}]_\infty}-(\zeta+\zeta^6)q^{21}\frac{[q^{14};q^{147}]_\infty}{[q^{21},q^{49},q^{49},q^{70};q^{147}]_\infty}+q^7\frac{[q^7,q^{14};q^{49}]_\infty}{[q^7,q^{21},q^{21};q^{49}]_\infty}$$
$$-(1+\zeta^2+\zeta^5)q^{14}\frac{[q^7;q^{49}]_\infty}{[q^{21};q^{49}]_\infty[q^{21},q^{49},q^{70};q^{147}]_\infty}\Bigg)$$
$$=\left(q^{49};q^{49}\right)_\infty\left(q^{147};q^{147}\right)_\infty\Bigg(-(1+\zeta_7+\zeta_7^6)q^7\frac{[q^{21};q^{49}]_\infty[q^{28};q^{147}]_\infty}{[q^7,q^{14};q^{49}]_\infty}-q^{14}\frac{[q^{14};q^{49}]_\infty[q^{14};q^{147}]_\infty}{[q^7,q^{21};q^{49}]_\infty}$$
$$-(\zeta_7^2+\zeta_7^5)q^7\frac{[q^{14};q^{49}]_\infty[q^{35};q^{147}]_\infty}{[q^7,q^{21};q^{49}]_\infty}-(\zeta_7^2+\zeta_7^5+2)q^{21}\frac{[q^7;q^{49}]_\infty[q^7;q^{147}]_\infty}{[q^{14},q^{21};q^{49}]_\infty}\Bigg). \tag{4.6}$$

Dividing both sides by $-\left(q^{49};q^{49}\right)_\infty^2 q^{14}\frac{\left[q^{14};q^{49}\right]_\infty}{\left[q^7;q^{49}\right]_\infty\left[q^{42},q^{49},q^{56};q^{147}\right]_\infty}$ and then replacing q^7 by q, we find (4.6) to be equivalent to

$$
\begin{aligned}
&(1+\zeta+\zeta^6)+(2+\zeta_7^2+\zeta_7^5)q^{-1}\frac{\left[q,q^3;q^7\right]_\infty\left[q^6,q^8;q^{21}\right]_\infty}{\left[q^2,q^2;q^7\right]_\infty\left[q^2,q^9;q^{21}\right]_\infty}+(\zeta+\zeta^6)q^{-1}\frac{\left[q;q^7\right]_\infty\left[q^5,q^6,q^8;q^{21}\right]_\infty}{\left[q^2;q^7\right]_\infty\left[q^3,q^4,q^7;q^{21}\right]_\infty}\\
&\quad+(\zeta+\zeta^6)q\frac{\left[q;q^7\right]_\infty\left[q^2,q^6,q^8;q^{21}\right]_\infty}{\left[q^2;q^7\right]_\infty\left[q^3,q^7,q^{10};q^{21}\right]_\infty}-q^{-1}\frac{\left[q,q;q^7\right]_\infty\left[q^6,q^7,q^8;q^{21}\right]_\infty}{\left[q,q^3,q^3;q^7\right]_\infty}\\
&\quad+(1+\zeta^2+\zeta^5)\frac{\left[q,q;q^7\right]_\infty\left[q^6,q^8;q^{21}\right]_\infty}{\left[q^2,q^3;q^7\right]_\infty\left[q^3,q^{10};q^{21}\right]_\infty}\\
&=(1+\zeta_7+\zeta_7^6)q^{-1}\frac{\left[q,q^3;q^7\right]_\infty\left[q^4,q^6,q^8;q^{21}\right]_\infty}{\left[q,q^2,q^2;q^7\right]_\infty}+\frac{\left[q;q^7\right]_\infty\left[q^2,q^6,q^8;q^{21}\right]_\infty}{\left[q,q^3;q^7\right]_\infty}\\
&\quad+(\zeta_7^2+\zeta_7^5)q^{-1}\frac{\left[q;q^7\right]_\infty\left[q^5,q^6,q^8;q^{21}\right]_\infty}{\left[q,q^3;q^7\right]_\infty}+(\zeta_7^2+\zeta_7^5+2)q\frac{\left[q,q;q^7\right]_\infty\left[q,q^6,q^8;q^{21}\right]_\infty}{\left[q^2,q^2,q^3;q^7\right]_\infty}.
\end{aligned}
$$

In this form each of the ten terms is a modular function with respect to $\Gamma_1(21)$ by Theorem 3 of [16]. By moving all ten terms to one side of the equation, we are to verify a modular function is identically zero. We do this by checking that otherwise this modular function would violate the valence formula. Using Theorem 4 of [16] we compute the order at each cusp of $\Gamma_1(21)$, not equivalent to infinity, of each generalized eta quotient. At each cusp we taking the minimal order at that cusp among the generalized eta quotients. Summing these minimal orders gives -13. By the valence formula, if the order at ∞ is 14 or larger, the sum of generalized eta quotients is identically zero. This is easily verified with Maple.

We then repeat this process for the other six values of k. In each case we have to prove an identity between several infinite products. Dividing by one of the terms yields an identity between modular functions with respect to $\Gamma_1(21)$. We examine the orders of each modular function at the cusps to and find we must show the order at ∞ is larger than some number. In all cases this number is rather small, with 13 being the largest.

□

5. Dissection Formulas

We recall we are to find the 3-dissections of $S_{E2}(\zeta_3,q)$ and $S_{E4}(\zeta_3,q)$, and the 5-dissections of $S_{C1}(\zeta_5,q)$ and $S_{C5}(\zeta_5,q)$. We prove Theorem 2.5 by relating each series $S_X(z,q)$ to an appropriate rank-like and crank-like function, both of which have dissections that are either known or easy to deduce.

We recall $R(z,q)$ is the generating function for the rank of ordinary partitions. By [20], one form of the generating function of the rank is

$$R(z,q)=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}N(m,n)z^mq^n=\frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{\frac{n(3n+1)}{2}}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right). \quad (5.1)$$

We recall $C(z,q)$ is the generating function for the crank of ordinary partitions. By [11], two forms of the generating function are given by

$$C(z,q)=\frac{(q;q)_\infty}{(zq,z^{-1}q;q)_\infty}=\frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{\frac{n(n+1)}{2}}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

We also define two series of a similar form to that of $R(z,q)$ and $C(z,q)$,

$$
\begin{aligned}
R_1(z,q)&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^n}{(1-zq^n)(1-z^{-1}q^n)}\right),\\
R_2(z,q)&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2}(1+q^{2n})}{(1-zq^n)(1-z^{-1}q^n)}\right).
\end{aligned}
$$

We used $R_1(z,q)$ in [10] in proving dissections similar to what we are doing here, and as we will see shortly $R_2(z,q)$ is essentially the rank of an overpartition.

Proposition 5.1.

$$S_{C1}(z,q) = \frac{1}{(1-z)(1-z^{-1})}\left(R(z,q^2) - \left(q;q^2\right)_\infty C(z,q)\right), \tag{5.2}$$

$$S_{C5}(z,q) = \frac{1}{(1-z)(1-z^{-1})}\left(C(z,q^2) - \left(q;q^2\right)_\infty C(z,q)\right), \tag{5.3}$$

$$S_{E2}(z,q) = \frac{1}{(1-z)(1-z^{-1})}\left(R_1(z,q) - \left(-q;q\right)_\infty C(z,q)\right), \tag{5.4}$$

$$S_{E4}(z,q) = \frac{1}{(1-z)(1-z^{-1})}\left(R_2(z,q) - \left(-q;q\right)_\infty C(z,q)\right). \tag{5.5}$$

We note that $(-q;q)_\infty C(z,q) = \frac{(-q,q;q)_\infty}{(zq,z^{-1}q;q)_\infty}$ is a residual crank studied in [8] and [10]. Most of these functions have known dissections when z is either a primitive third or fifth root of unity.

By taking subtracting the expressions for $S_{C1}(z,q)$ and $S_{C5}(z,q)$ in Proposition 5.1 and using (1.1), we have Corollary 2.7.

Proposition 5.2.

$$\begin{aligned} R(\zeta_5,q) &= \frac{\left(q^{25};q^{25}\right)_\infty \left[q^{10};q^{25}\right]_\infty}{\left[q^5;q^{25}\right]_\infty^2} + q^5\frac{\zeta_5+\zeta_5^{-1}-2}{\left(q^{25};q^{25}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2}}{1-q^{25n+5}} + q\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^5;q^{25}\right]_\infty} \\ &\quad + q^2(\zeta_5+\zeta_5^{-1})\frac{\left(q^{25};q^{25}\right)_\infty}{\left[q^{10};q^{25}\right]_\infty} - q^3(\zeta_5+\zeta_5^{-1})\frac{\left(q^{25};q^{25}\right)_\infty\left[q^5;q^{25}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty^2} \\ &\quad - q^8\frac{2\zeta_5+2\zeta_5^{-1}+1}{\left(q^{25};q^{25}\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2}}{1-q^{25n+10}}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} C(\zeta_5,q) &= \left(q^{25};q^{25}\right)_\infty\left(\frac{\left[q^{10};q^{25}\right]_\infty}{\left[q^5;q^{25}\right]_\infty^2} + (\zeta_5+\zeta_5^4-1)q\frac{1}{\left[q^5;q^{25}\right]_\infty} - (\zeta_5+\zeta_5^4+1)q^2\frac{1}{\left[q^{10};q^{25}\right]_\infty}\right. \\ &\quad \left. -(\zeta_5+\zeta_5^4)q^3\frac{\left[q^5;q^{25}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty^2}\right), \end{aligned} \tag{5.7}$$

$$R_1(\zeta_3,q) = \frac{\left(q^9;q^9\right)_\infty^4\left(q^6;q^6\right)_\infty}{\left(q^3;q^3\right)_\infty^2\left(q^{18};q^{18}\right)_\infty^2} - 4q\frac{\left(q^{18};q^{18}\right)_\infty\left(q^9;q^9\right)_\infty}{\left(q^3;q^3\right)_\infty} + q^2\frac{\left(q^{18};q^{18}\right)_\infty^4}{\left(q^9;q^9\right)_\infty^2\left(q^6;q^6\right)_\infty}, \tag{5.8}$$

$$\begin{aligned} R_2(\zeta_3,q) &= \frac{3}{2} - \frac{\left(q^9;q^9\right)_\infty^4\left(q^6;q^6\right)_\infty}{2\left(q^{18};q^{18}\right)_\infty^2\left(q^3;q^3\right)_\infty^2} - q\frac{\left(q^9;q^9\right)_\infty\left(q^{18};q^{18}\right)_\infty}{\left(q^3;q^3\right)_\infty} - 2q^2\frac{\left(q^{18};q^{18}\right)_\infty^4}{\left(q^9;q^9\right)_\infty^2\left(q^6;q^6\right)_\infty} \\ &\quad + 3q^2\frac{\left(q^{18};q^{18}\right)_\infty}{\left(q^9;q^9\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n}}{1-q^{9n+3}}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} \left(q;q^2\right)_\infty C(\zeta_5,q) &= \left(q^{50};q^{50}\right)_\infty\left(\frac{\left[q^{25};q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty} + 2(\zeta_5+\zeta_5^{-1})q^5\frac{\left[q^5;q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty}\right. \\ &\quad \left. -2q\frac{\left[q^{15};q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty} + (\zeta_5+\zeta_5^{-1})q\frac{\left[q^{25};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty} - 2(\zeta_5+\zeta_5^{-1})q^2\frac{\left[q^{15};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty} + 2q^4\frac{\left[q^5;q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty}\right), \end{aligned} \tag{5.10}$$

$$(-q;q)_\infty C(\zeta_3,q) = \frac{\left(q^9;q^9\right)_\infty^4\left(q^6;q^6\right)_\infty}{\left(q^3;q^3\right)_\infty^2\left(q^{18};q^{18}\right)_\infty^2} - q\frac{\left(q^{18};q^{18}\right)_\infty\left(q^9;q^9\right)_\infty}{\left(q^3;q^3\right)_\infty} - 2q^2\frac{\left(q^{18};q^{18}\right)_\infty^4}{\left(q^9;q^9\right)_\infty^2\left(q^6;q^6\right)_\infty}. \tag{5.11}$$

We find that Theorem 2.5 follows from Propositions 5.1 and 5.2.

Proof of Proposition 5.1. We apply Bailey's Lemma, with $\rho_1 = z$ and $\rho_2 = z^{-1}$, to each $S_X(z,q)$ with the corresponding Bailey pair, the identities then quickly follow. We write

$$\begin{aligned}
S_{C1}(z,q) &= \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \frac{\left(z,z^{-1};q\right)_n q^n}{(q;q^2)_n (q;q)_n} - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \left(z,z^{-1};q\right)_n q^n \beta_n^{C1} - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)(1-z^{-1})\left(q^2;q^2\right)_\infty} \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^n \alpha_n^{C1}}{(1-zq^n)(1-z^{-1}q^n)} - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)(1-z^{-1})\left(q^2;q^2\right)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{3n^2+n}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)((1-z^{-1}))} \left(R(z,q^2) - \left(q;q^2\right)_\infty C(z,q)\right).
\end{aligned}$$

This proves (5.2).

Next we have

$$\begin{aligned}
S_{C5}(z,q) &= \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \frac{\left(z,z^{-1};q\right)_n q^{n(n+1)/2}}{(q;q^2)_n (q;q)_n} - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \left(z,z^{-1};q\right)_n q^n \beta_n^{C5} - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)(1-z^{-1})\left(q^2;q^2\right)_\infty} \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^n \alpha_n^{C5}}{(1-zq^n)(1-z^{-1}q^n)} - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)(1-z^{-1})\left(q^2;q^2\right)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{\left(q;q^2\right)_\infty (q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)((1-z^{-1}))} \left(C(z,q^2) - \left(q;q^2\right)_\infty C(z,q)\right).
\end{aligned}$$

This proves (5.3).

For $S_{E2}(z,q)$ we have

$$\begin{aligned}
S_{E2}(z,q) &= \frac{\left(q^2;q^2\right)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \frac{\left(z,z^{-1};q\right)_n (-1)^n q^n}{\left(q^2;q^2\right)_n} - \frac{\left(q^2;q^2\right)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \left(z,z^{-1};q\right)_n q^n \beta_n^{E2} - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^n \alpha_n^{E2}}{(1-zq^n)(1-z^{-1}q^n)} - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^n}{(1-zq^n)(1-z^{-1}q^n)}\right) - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)((1-z^{-1}))} \left(R_1(z,q) - (-q;q)_\infty C(z,q)\right).
\end{aligned}$$

This proves (5.4).

Lastly, for $S_{E4}(z,q)$ we have

$$S_{E4}(z,q) = \frac{\left(q^2;q^2\right)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \frac{\left(z,z^{-1};q\right)_n q^{2n}}{\left(q^2;q^2\right)_n} - \frac{\left(q^2;q^2\right)_\infty}{(z,z^{-1};q)_\infty}$$

$$
\begin{aligned}
&= \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=0}^{\infty}\left(z,z^{-1};q\right)_n q^n\beta_n^{E4} - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})\,(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(1-z)(1-z^{-1})q^n\alpha_n^{E4}}{(1-zq^n)(1-z^{-1}q^n)} - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})\,(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2}(1+q^{2n})}{(1-zq^n)(1-z^{-1}q^n)}\right) - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)((1-z^{-1}))}\left(R_2(z,q)-(-q;q)_\infty\, C(z,q)\right).
\end{aligned}
$$

This proves (5.5).

□

Proof of Proposition 5.2. Equation (5.6) follows from Theorem 4 of [5], (5.7) is (3.8) of [11], (5.8) is Theorem 2.13 of [10], and (5.11) is Theorem 2.9 of [10].

We first prove (5.10). By Gauss and the Jacobi Triple Product identity we have

$$
\begin{aligned}
\left(q;q^2\right)_\infty (q;q)_\infty &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \sum_{k=-2}^{2}(-1)^k q^{k^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{10nk}q^{25n^2} \\
&= \left(q^{50};q^{50}\right)_\infty \sum_{k=-2}^{2}(-1)^k q^{k^2}\left[q^{10k+25};q^{50}\right]_\infty \\
&= \left(q^{50};q^{50}\right)_\infty\left(\left[q^{25};q^{50}\right]_\infty - 2q\left[q^{15};q^{50}\right]_\infty + 2q^4\left[q^5;q^{50}\right]_\infty\right). \qquad (5.12)
\end{aligned}
$$

By Lemma 3.9 of [11] we have

$$
\frac{1}{\left(\zeta_5 q,\zeta_5^{-1}q;q\right)_\infty} = \frac{1}{\left[q^5;q^{25}\right]_\infty} + (\zeta_5+\zeta_5^{-1})q\frac{1}{\left[q^{10};q^{25}\right]_\infty}. \qquad (5.13)
$$

Multiplying (5.12) and (5.13) yields

$$
\begin{aligned}
\frac{\left(q;q^2\right)_\infty (q;q)_\infty}{\left(\zeta_5 q,\zeta_5^{-1}q;q\right)_\infty} = \left(q^{50};q^{50}\right)_\infty\Bigg(&\frac{\left[q^{25};q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty} + 2(\zeta_5+\zeta_5^{-1})q^5\frac{\left[q^5;q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty} \\
&-2q\frac{\left[q^{15};q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty} + (\zeta_5+\zeta_5^{-1})q\frac{\left[q^{25};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty} - 2(\zeta_5+\zeta_5^{-1})q^2\frac{\left[q^{15};q^{50}\right]_\infty}{\left[q^{10};q^{25}\right]_\infty} + 2q^4\frac{\left[q^5;q^{50}\right]_\infty}{\left[q^5;q^{25}\right]_\infty}\Bigg),
\end{aligned}
$$

which is (5.10).

We note we could also use the 5-dissection of Rødseth [17] for $\left(-q;q^2\right)_\infty$ and replace q with $-q$ to get

$$
\begin{aligned}
\left(q;q^2\right)_\infty = &\frac{\left(q^5;q^5\right)_\infty\left(q^{25};q^{25}\right)_\infty^2\left[q^{15};q^{50}\right]_\infty}{\left(q^{10};q^{10}\right)_\infty^3} - q\frac{\left(q^5;q^5\right)_\infty\left(q^{50};q^{50}\right)_\infty^2\left[q^{15};q^{50}\right]_\infty^2}{\left(q^{10};q^{10}\right)_\infty^3} \\
&- q^7\frac{\left(q^5;q^5\right)_\infty\left(q^{50};q^{50}\right)_\infty^2\left[q^5;q^{50}\right]_\infty^2}{\left(q^{10};q^{10}\right)_\infty^3} - q^3\frac{\left(q^5;q^5\right)_\infty\left(q^{25};q^{25}\right)_\infty^2\left[q^5;q^{50}\right]_\infty}{\left(q^{10};q^{10}\right)_\infty^3} \\
&+ q^4\frac{\left(q^5;q^5\right)_\infty^2\left(q^{50};q^{50}\right)_\infty^3}{\left(q^{10};q^{10}\right)_\infty^4\left(q^{25};q^{25}\right)_\infty}.
\end{aligned}
$$

However multiplying this with the 5-dissection for $C(\zeta_5,q)$ would be a much longer calculation.

For (5.9) we recall $\overline{R}(z,q)$ is the generating function for the Dyson rank of an overpartition. By [13] the generating function for the Dyson rank of an overpartition is given by

$$
\overline{R}(z,q) = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-zq^n)}\right).
$$

By Theorem 1.1 of [15] we have

$$\overline{R}(\zeta_3,q)=\frac{\left(q^9;q^9\right)_\infty^4\left(q^6;q^6\right)_\infty}{\left(q^{18};q^{18}\right)_\infty^2\left(q^3;q^3\right)_\infty^2}+2q\frac{\left(q^9;q^9\right)_\infty\left(q^{18};q^{18}\right)_\infty}{\left(q^3;q^3\right)_\infty}+4q^2\frac{\left(q^{18};q^{18}\right)_\infty^4}{\left(q^9;q^9\right)_\infty^2\left(q^6;q^6\right)_\infty}$$
$$-6q^2\frac{\left(q^{18};q^{18}\right)_\infty}{\left(q^9;q^9\right)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{1-q^{9n+3}}.$$

We note the expression on the right hand side of the identity in (5.9) is $\frac{3-\overline{R}(\zeta_3,q)}{2}$. We will show that

$$\frac{(1-z)(1-z^{-1})+(z+z^{-1})\overline{R}(z,q)}{2}=R_2(z,q).$$

We have

$$\frac{(1-z)(1-z^{-1})+(z+z^{-1})\overline{R}(z,q)}{2}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{z+z^{-1}}{2}+\frac{(1-z)(1-z^{-1})}{2}\frac{(q;q)_\infty}{(-q;q)_\infty}+(z+z^{-1})\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right)$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(1-z)(1-z^{-1})(-1)^nq^{n^2}+(z+z^{-1})\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right)$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}(1-z)(1-z^{-1})(-1)^nq^{n^2}\left(1+\frac{(z+z^{-1})q^n}{(1-zq^n)(1-z^{-1}q^n)}\right)\right)$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2}(1+q^{2n})}{(1-zq^n)(1-z^{-1}q^n)}\right)$$
$$=R_2(z,q).$$

This proves (5.9), noting $(1-\zeta_3)(1-\zeta_3^{-1})=3$ and $\zeta_3+\zeta_3^{-1}=-1$.

□

6. Conclusions

We see spt-crank-type functions lead to a large number of new spt-type functions, as well as congruences for these functions. We see some of these spt-crank-type functions have surprising product representations, while others have interesting Hecke-Rogers-type double series representations. Also in all cases these functions arise as the difference of a rank-like and crank-like function. While these results are quite different, they both come out of Bailey pairs and Bailey's Lemma.

As noted in Section 2, here we do not pursue interpreting the $M_X(m,n)$ as a statistic defined on the smallest parts that each function $\mathrm{spt}_X(n)$ counts. However there is reason to believe such an interpretation exists. By [12] we know each of $M_{A1}(m,n)$, $M_{A3}(m,n)$, $M_{A5}(m,n)$, and $M_{A7}(m,n)$ is nonnegative. Also by expanding the summands of $S_{E4}(z,q)$ by the q-binomial theorem,

$$\frac{q^{2n}\left(q^{2n+2};q^2\right)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}=\frac{q^{2n}\left(q^{2n};q\right)_\infty}{(1-q^{2n})\left(q^{2n+1};q^2\right)_\infty(zq^n,z^{-1}q^n;q)_\infty}$$
$$=\frac{q^{2n}}{(1-q^{2n})\left(q^{2n+1};q^2\right)_\infty}\sum_{k=0}^{\infty}\frac{z^{-k}q^{nk}}{(q;q)_k\left(zq^{n+k};q\right)_\infty},$$

we see that each $M_{E4}(m,n)$ is nonnegative. Numerical evidence suggests that also each $M_{C1}(m,n)$ and $M_{C5}(m,n)$ is nonnegative. We pose the problem of finding nice combinatorial interpretations of the coefficients $M_{C1}(m,n)$ and $M_{C5}(m,n)$ which prove nonnegativity.

In a coming paper, the second author continues this study with Bailey pairs from groups B, F, G, and J of [18] and [19]. This will cover the last of the Bailey pairs of Slater that appear to give congruences of this form via spt-crank-type functions. Of course that is not to say there are not many more spt-crank-type functions to uncover.

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