CONGRUENCES FOR GENERALIZED FROBENIUS PARTITIONS WITH AN ARBITRARILY LARGE NUMBER OF COLORS

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Abstract

In his 1984 AMS Memoir, George Andrews defined the family of k–colored generalized Frobenius partition functions. These are enumerated by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. In that Memoir, Andrews proved (among many other things) that, for all $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod 5$. Soon after, many authors proved congruence properties for various k–colored generalized Frobenius partition functions, typically with a small number of colors.

Work on Ramanujan–like congruence properties satisfied by the functions $c\phi_k(n)$ continues, with recent works completed by Baruah and Sarmah as well as the author. Unfortunately, in all cases, the authors restrict their attention to small values of k. This is often due to the difficulty in finding a "nice" representation of the generating function for $c\phi_k(n)$ for large k. Because of this, no Ramanujan–like congruences are known where k is large. In this note, we rectify this situation by proving several infinite families of congruences for $c\phi_k(n)$ where k is allowed to grow arbitrarily large. The proof is truly elementary, relying on a generating function representation which appears in Andrews' Memoir but has gone relatively unnoticed.

1. Introduction

In his 1984 AMS Memoir, George Andrews [2] defined the family of k–colored generalized Frobenius partition functions which are enumerated by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. These combinatorial objects serve as a natural generalization of ordinary integer partitions. We provide a brief explanation here.

The Ferrers graph associated with a partition

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r$$

of n with $\lambda_i \geq \lambda_{i+1}$ is generally represented as a set of left–justified rows of dots where the i^{th} row contains λ_i dots. For example, the Ferrers graph of the partition $7 + 7 + 5 + 4 + 2 + 2$ is given by the following:

$$\begin{array}{lllllll}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & & & \\
\bullet & \bullet & & & & &
\end{array}$$

From here we consider the Frobenius symbol associated with an integer partition. Given the Ferrers graph of a partition, note that the rows of dots strictly to the right of the r diagonal elements can be enumerated to provide one strictly decreasing sequence of r nonnegative integers (the r^{th} row might be empty, producing a value of 0). The remaining dots strictly below the main diagonal can be enumerated by columns to provide a second strictly decreasing sequence of r nonnegative integers. The resulting two sequences are then written in the form of a two–rowed array. For example, the partition $7 + 7 + 5 + 4 + 2 + 2$ of 27 mentioned above is represented by the Frobenius symbol

$$\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}.$$

From here we can describe the generalized Frobenius partitions of n using k colors. Consider k copies of the nonnegative integers written j_i where $j \geq 0$ and $1 \leq i \leq k$. We then say that $j_i < l_m$ precisely when $j < l$ or $j = l$ and $i < m$. Moreover, j_i is equal to l_m if and only if $j = l$ and $i = m$.

Then $c\phi_k(n)$ counts the number of generalized Frobenius partitions of n under the conditions that the parts are "decreasing" (using the ordering above). Thus, for example, $c\phi_2(2) = 9$:

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix} \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix} \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix} \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix} \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}$$

$$\begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix} \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix} \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix} \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}$$

Among many things, Andrews [2, Corollary 10.1] proved that, for all $n \geq 0$, $c\phi_2(5n + 3) \equiv 0 \pmod 5$. Soon after, many authors proved similar congruence properties for various k–colored generalized Frobenius partition functions, typically for a small number of colors k. See, for example, [5, 6, 7, 9, 10, 11, 12, 13, 15].

In recent years, this work has continued. Baruah and Sarmah [3] proved a number of congruence properties for $c\phi_4$, all with moduli which are powers of 4. Motivated by this work of Baruah and Sarmah, the author [14] further studied 4–colored generalized Frobenius partitions and proved that for all $n \geq 0$, $c\phi_4(10n+6) \equiv 0 \pmod 5$.

Unfortunately, in all the works mentioned above, the authors restrict their attention to small values of k. This is often due to the difficulty in finding a "nice" representation of the generating function for $c\phi_k(n)$ for large k. Because of this, no Ramanujan–like congruences are known where k is large. The goal of this brief note is to rectify this situation by proving several infinite families of congruences for $c\phi_k(n)$ where k is allowed to grow arbitrarily large. The proof is truly elementary, relying on a generating function representation which appears in Andrews' Memoir but has gone relatively unnoticed.

2. Our Congruence Results

We begin by noting the following generating function result from Andrews' AMS Memoir [2, Equation (5.14)]:

Theorem 2.1. *For fixed k, the generating function for $c\phi_k(n)$ is the constant term (i.e., the z^0 term) in*

$$\prod_{n=0}^{\infty}(1+zq^{n+1})^k(1+z^{-1}q^n)^k.$$

Theorem 2.1 is the springboard that Andrews uses to find "nice" representations of the generating functions for $c\phi_k(n)$ for $k = 1, 2$, and 3. Theorem 2.1 rarely appears in the works written by the various authors referenced above; however, it is extremely useful in proving the following theorem, the main result of this note.

Theorem 2.2. *Let p be prime and let r be an integer such that $0 < r < p$. If*

$$c\phi_k(pn+r) \equiv 0 \pmod p$$

for all $n \geq 0$, then

$$c\phi_{pN+k}(pn+r) \equiv 0 \pmod p$$

for all $N \geq 0$ and $n \geq 0$.

Proof. Assume p is prime and r is an integer such that $0 < r < p$. Thanks to Theorem 2.1, we note that the generating function for $c\phi_{pN+k}(n)$ is the constant term (i.e., the z^0 term) in

$$\prod_{n=0}^{\infty}(1+zq^{n+1})^{pN+k}(1+z^{-1}q^n)^{pN+k}. \tag{1}$$

Since p is prime, we know (1) is congruent, modulo p, to

$$\prod_{n=0}^{\infty}(1+(zq^{n+1})^p)^N(1+(z^{-1}q^n)^p)^N\prod_{n=0}^{\infty}(1+zq^{n+1})^k(1+z^{-1}q^n)^k \tag{2}$$

thanks to the binomial theorem. Note that the first product in (2) is a function of q^p and the second product is the product from which we obtain the generating function for $c\phi_k(n)$ thanks to Theorem 2.1. Since the first product is indeed a function of q^p, and since we wish to find the generating function dissection for $c\phi_k(pn+r)$ where $0<r<p$, we see that if

$$c\phi_k(pn+r)\equiv 0 \pmod p$$

for all $n\geq 0$, then

$$c\phi_{pN+k}(pn+r)\equiv 0 \pmod p$$

for all $n\geq 0$. □

Of course, once one knows a single congruence of the form

$$c\phi_k(pn+r)\equiv 0 \pmod p$$

for all $n\geq 0$, where p be prime and r is an integer such that $0<r<p$, then one can write down an infinite family of congruences for an arbitrarily large number of colors with the same modulus p. We provide a number of such examples here.

Corollary 2.3. *For all $N\geq 0$ and for all $n\geq 0$,*

$$\begin{aligned} c\phi_{5N+1}(5n+4) &\equiv 0 \pmod 5,\\ c\phi_{7N+1}(7n+5) &\equiv 0 \pmod 7, \quad \textit{and}\\ c\phi_{11N+1}(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Proof. This corollary of Theorem 2.2 follows from the fact that $c\phi_1(n)=p(n)$ for all $n\geq 0$ as well as Ramanujan's well–known congruences for $p(n)$ modulo 5, 7, and 11. □

Corollary 2.4. *For all $N\geq 0$ and for all $n\geq 0$,*

$$c\phi_{5N+2}(5n+3)\equiv 0 \pmod 5.$$

Proof. This corollary of Theorem 2.2 follows from Andrews [2, Corollary 10.1] where he proved that, for all $n\geq 0$, $c\phi_2(5n+3)\equiv 0 \pmod 5$. □

Corollary 2.5. *For all $N\geq 1$ and all $n\geq 0$,*

$$c\phi_{3N}(3n+2)\equiv 0 \pmod 3.$$

Proof. This corollary of Theorem 2.2 follows from Kolitsch's work [9] where he proved that, for all $n \geq 0$, $c\phi_3(3n+2) \equiv 0 \pmod 3$. □

One last comment is in order. It is also clear that one can combine corollaries like those above in order to obtain some truly unique–looking congruences. For example, we note the following:

Corollary 2.6. *For all $N \geq 0$ and all $n \geq 0$,*

$$c\phi_{1155N+1002}(1155n+908) \equiv 0 \pmod{1155}.$$

Proof. The proof of this result follows from the Chinese Remainder Theorem and the fact that

$$1155 = 3 \times 5 \times 7 \times 11$$

along with a combination of the corollaries mentioned above. □

It is extremely gratifying to be able to explicitly identify such congruences satisfied by these generalized Frobenius partition functions.

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