GENERALIZATIONS OF DYSON'S RANK AND NON-ROGERS-RAMANUJAN PARTITIONS

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Abstract. For any fixed integer $k \geq 2$, we define a statistic on partitions called the k-rank. The definition involves the decomposition into successive Durfee squares. Dyson's rank corresponds to the 2-rank. Generating function identities are given. The sign of the k-rank is reversed by an involution which we call k-conjugation. We prove the following partition theorem: the number of self-$2k$-conjugate partitions of n is equal to the number of partitions of n with no parts divisible by $2k$ and the parts congruent to $k \pmod{2k}$ are distinct. This generalizes the well-known result: the number of self-conjugate partitions of n is equal to the number of partitions into distinct odd parts.

1. Introduction

Let $p(n)$ denote the number of unrestricted partitions of n [A2]. Ramanujan discovered and later proved

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Dyson [D1], [D3] discovered remarkable combinatorial interpretations of (1.1) and (1.2). He defined an integral statistic on partitions, called the *rank*, whose value modulo 5 split

The author was supported in part by NSF Grant DMS-9208813.

1980 *Mathematics Subject Classification* (1991 *Revision*) Primary 11P83,

Key words and phrases. Partitions, congruences, q-series, combinatorial idenities, Rogers-Ramanujan identities, Durfee squares, basic hypergeometric functions, Bailey chains, involutions, self-conjugate partitions.

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the set of partitions of $5n+4$ into 5 equal classes thus giving a combinatorial refinement of (1.1). He further conjectured that the rank modulo 7 gave an analogous combinatorial refinement of (1.2) and hypothesized a statistic, called the *crank*, which would likewise give a combinatorial refinement of (1.3). Atkin and Swinnerton-Dyer [A-SD] proved Dyson's conjecture for 5 and 7. In [Ga1], [Ga2] a crank for 11 as well as new cranks for 5 and 7 were found relative to vector partitions. In [A-G1], Andrews and Garvan completed the solution of Dyson's crank conjecture. Garvan, Kim and Stanton [G-K-S] found different cranks in terms of t-cores and found a crank which gave a combinatorial refinement of Ramanujan's partition congruence modulo 25:

$$p(25n+24) \equiv 0 \pmod{25}. \tag{1.4}$$

In [Ga3], some relations between the rank and the crank modulo 5, 7 were proved. Later, relations for the crank modulo 8, 9, 10 were found [Ga4]. In [L1], Lewis conjectured 88 linear relations involving the rank and the crank. The new relations involved the moduli 4, 8, 9, 12. These were subsequently proved in a series of papers by Lewis and Santa-Gadea [L2], [L3], [SG1], [SG2], [L-SG].

The mock-theta conjectures [A-G2] are combinatorial identities that relate the rank modulo 5 and Ramanujan's mock-theta functions of order 5. These were proved by Hickerson [H1] who also found connections between the rank modulo 7 and the mock-theta functions of order 7 [H2]. Lewis and Santa-Gadea found connections between their relations and mock-theta functions of order 3 [W2].

In this paper we generalize Dyson's rank. Given a partition Dyson's *rank* is defined to be the largest part minus the number of parts. If we let $N(m,n)$ denote the number of partitions of n with rank m we have the following partition identities:

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} N(m,n) z^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} N(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+|m|n}(1-q^n). \tag{1.6}$$

Here we have employed the usual q-notation $(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ and $(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n$ for $|q|<1$. Equation (1.5) is given in [Ga2, (7.4)]. Equation (1.6) is due to Dyson and was proved by Atkin and Swinnerton-Dyer [A-SD]. In [Ga2] we showed how (1.6) follows from Watson's [W1] q-analog of Whipple's theorem. We note that in (1.5) we have taken the rank of the empty partition to be 0, but in (1.6) we have defined

$N(m,0) = 0$. Equation (1.6) provides an effective means of computing the coefficients $N(m,n)$ and is the starting point of Atkin and Swinnerton-Dyer's proof of the Dyson conjectures.

The Andrews-Garvan *crank* is defined as follows. For a partition we define the crank to be the largest part if the partition contains no ones, otherwise it is the difference between the number of parts larger than the number of ones and the number of ones. We let $M(m,n)$ denote the number of partitions of n with crank m. Our derivation of the crank depended on some previous work on vector partitions [Ga1], [Ga2]. We define $N_V(m,n)$ by

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}. \tag{1.7}$$

The $N_V(m,n)$ may be interpreted as counting certain weighted triples of partitions (vector partitions). We have the following identities:

$$M(m,n) = N_V(m,n) \qquad \text{for } n > 1, \tag{1.8}$$

$$\sum_{n=0}^{\infty} N_V(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+|m|n}(1-q^n). \tag{1.9}$$

Equation (1.8) is the main result of [A-G1] and equation (1.9) is [Ga2, Theorem (7.19)].

We define $N_k(m,n)$ by

$$\sum_{n=0}^{\infty} N_k(m,n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n((2k-1)n-1)/2+|m|n}(1-q^n), \tag{1.10}$$

for any positive integer k. The problem we consider in this paper is to find a combinatorial interpretation of $N_k(m,n)$. The answer is given below in Theorem (1.12). We observe that the $k=2$ case corresponds to Dyson's rank and the $k=1$ case corresponds to the Andrews-Garvan crank. For $k \ge 2$, the interpretation of $N_k(m,n)$ is in terms of the Ferrers graph [A2] of a partition and its decomposition into successive Durfee squares [A4], [A5],[A7]. For fixed $k \ge 2$ we define the *k-rank* of a partition. To do this we need to define some statistics on partitions. For a partition, π, we define $n_1(\pi)$, $n_2(\pi)$, $\ldots$ to be the sizes of the successive Durfee squares of π. We note the $n_\ell(\pi) = 0$ when the number of successive

Durfee squares of π is less then ℓ. The k-rank, $r_k(\pi)$, is given by

(1.11)
$$\begin{aligned} r_k(\pi) \quad = \quad & \text{the number of columns in the Ferrers graph of} \\ & \pi \text{ which lie to the right of the first Durfee square} \\ & \text{and whose length} \le n_{k-1}(\pi) \\ & \text{minus} \\ & \text{the number of parts of } \pi \text{ that lie below the} \\ & (k-1)\text{-th Durfee square.} \end{aligned}$$

We note that $r_k(\pi) = 0$ if $n_{k-1}(\pi) = 0$. The main result is

Theorem (1.12). *Let $k \ge 2$ be fixed and let $N_k(m,n)$ be defined by (1.10). Then $N_k(m,n)$ is the number of partitions of n into at* <u>*least*</u> *$k-1$ successive Durfee squares with k-rank equal to m.*

An easy consequence of (1.10) is

$$N_k(m,n) = N_k(-m,n). \tag{1.13}$$

In §5 we define an involution which the reverses the sign of the k-rank thus explaining (1.13) combinatorially. We call this involution *k-conjugation* since it depends on k and when $k = 2$ it corresponds to ordinary conjugation. We call a partition *k-self-conjugate* if it is fixed by k-conjugation. We prove the following partition theorem:

Theorem (1.14). *Let $k \ge 1$ be fixed. Then the number of self-$2k$-conjugate partitions of n is equal to the number of partitions of n with no parts divisible by $2k$ and all parts which are congruent to k modulo $2k$ are distinct.*

This theorem generalizes the well-known result that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts [S-W, Theorem 3.3, p.72], [A7, p.24].

2. Multiple basic hypergeometric series

In this section we set up the needed q-series identities needed for the proofs of our combinatorial results. The basic hypergeometric series ${}_m\phi_n$ is defined by

$${}_m\phi_n \left[\begin{matrix} a_1, & a_2, & \ldots, & a_m \\ b_1, & b_2, & \ldots, & b_n \end{matrix} ; q, z \right] := \sum_{j \ge 0} \frac{(a_1)_j \cdots (a_m)_j z^j}{(b_1)_j \cdots (b_n)_j (q)_j}, \tag{2.1}$$

where $|z| < 1$, $|q| < 1$, $b_i \ne q^{-k}$. We will need Andrews' [A1] multiple series generalization of the q-analog of Whipple's theorem. This result is best explained in terms of Bailey chains [A6, Lecture 3]. Two sequences α_n, β_n form a *Bailey pair* (α_n, β_n) if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}, \tag{2.2}$$

for $n \geq 0$. Bailey's Lemma [A6, Theorem 3.3] gives rise to another Bailey pair (α'_n, β'_n) defined in terms of the original Bailey pair via

$$\alpha'_n = \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n}, \tag{2.3}$$

$$\beta'_n = \sum_{j=0}^{n} \frac{(\rho_1)_j(\rho_2)_j(aq/\rho_1\rho_2)_{n-j}(aq/\rho_1\rho_2)^j\beta_j}{(q)_{n-j}(aq/\rho_1)_n(aq/\rho_2)_n}. \tag{2.4}$$

Successive iterations of Bailey's Lemma produce a so-called *Bailey Chain*:

$$(\alpha_n, \beta_n) \to (\alpha'_n, \beta'_n) \to (\alpha''_n, \beta''_n) \to \cdots . \tag{2.5}$$

This is made explicit in the following result which is [A6, Theorem 3.4]. If (α_n, β_n) is a Bailey pair (ie. related by (2.2)) then

$$\begin{aligned}
&\sum_{n\geq 0} \frac{(b_1)_n(c_1)_n\cdots(b_k)_n(c_k)_n}{(aq/b_1)_n(aq/c_1)_n\cdots(aq/b_k)_n(aq/c_k)_n} \\
&\qquad \times \frac{(q^{-N})_n}{(aq^{N+1})_n}\left(\frac{a^kq^{k+N}}{b_1c_1\cdots b_kc_k}\right)^n q^{-\binom{n}{2}}(-1)^n\,\alpha_n \\
&= \frac{(aq)_N(aq/b_kc_k)_N}{(aq/b_k)_N(aq/c_k)_N}\sum_{n_k\geq\cdots\geq n_1\geq 0}\frac{(b_k)_{n_k}(c_k)_{n_k}\cdots(b_1)_{n_1}(c_1)_{n_1}}{(q)_{n_k-n_{k-1}}\cdots(q)_{n_2-n_1}} \\
&\qquad \times \frac{(q^{-N})_{n_k}(aq/b_{k-1}c_{k-1})_{n_k-n_{k-1}}\cdots(aq/b_1c_1)_{n_2-n_1}}{(b_kc_kq^{-N}/a)_{n_k}(aq/b_{k-1})_{n_k}(aq/c_{k-1})_{n_k}\cdots(aq/b_1)_{n_2}(aq/c_1)_{n_2}} \\
&\qquad \times q^{n_1+\cdots+n_k}a^{n_1+\cdots+n_{k-1}}(b_{k-1}c_{k-1})^{-n_{k-1}}\cdots(b_1c_1)^{-n_1}\;\beta_{n_1}.
\end{aligned} \tag{2.6}$$

If we let N, b_1, ... , b_k, c_1, ... , c_k all tend to infinity then we obtain the following result which is [A6, Theorem 3.5]. If (α_n, β_n) is a Bailey pair then

$$\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty} q^{kn^2}a^{kn}\alpha_n = \sum_{n_k\geq\cdots\geq n_1\geq 0}\frac{a^{n_1+\cdots+n_k}q^{n_1^2+\cdots+n_k^2}\beta_{n_1}}{(q)_{n_k-n_{k-1}}\cdots(q)_{n_2-n_1}}. \tag{2.7}$$

It is well-known that the following form a Bailey pair

$$\beta_n = \begin{cases} 1, & n = 0, \\ 0, & n > 0, \end{cases} \tag{2.8}$$

$$\alpha_n = \frac{(-1)^n q^{\binom{n}{2}} (1 - aq^{2n})(a)_n}{(1-a)(q)_n}. \tag{2.9}$$

As noted in [A6, Theorem 3.6] taking $k = 2$ in (2.6) and using the Bailey pair in (2.8)–(2.9) gives Watson's q-analog of Whipple's theorem. For general k we obtain Andrews' generalization [A1]:

(2.10)

$$\begin{aligned} &{}_{2k+4}\phi_{2k+3} \left[\begin{matrix} a, & q\sqrt{a}, -q\sqrt{a}, & b_1, c_1, & \dots, & b_k, c_k, q^{-N} \\ & \sqrt{a}, -\sqrt{a}, & aq/b_1, aq/c_1, & \dots & aq/b_k, aq/c_k, aq^{N+1} \end{matrix} ; q, \frac{a^k q^{k+N}}{b_1 c_1 \cdots b_k c_k} \right] \\ &= \frac{(aq)_N (aq/b_k c_k)_N}{(aq/b_k)_N (aq/c_k)_N} \sum_{n_{k-1} \geq \dots \geq n_1 \geq 0} \frac{(b_k)_{n_{k-1}} (c_k)_{n_{k-1}} \cdots (b_2)_{n_1} (c_2)_{n_1}}{(q)_{n_{k-1} - n_{k-2}} \cdots (q)_{n_2 - n_1} (q)_{n_1}} \\ &\quad \times \frac{(q^{-N})_{n_{k-1}} (aq/b_{k-1} c_{k-1})_{n_{k-1} - n_{k-2}} \cdots (aq/b_2 c_2)_{n_2 - n_1} (aq/b-1c_1)_{n_1}}{(b_k c_k q^{-N}/a)_{n_{k-1}} (aq/b_{k-1})_{n_{k-1}} (aq/c_{k-1})_{n_{k-1}} \cdots (aq/b_1)_{n_1} (aq/c_1)_{n_1}} \\ &\quad \times q^{n_1 + \dots + n_{k-1}} a^{n_1 + \dots + n_{k-2}} (b_{k-1} c_{k-1})^{-n_{k-2}} \cdots (b_2 c_2)^{-n_1}. \end{aligned}$$

3. The 3-rank and non-Rogers-Ramanujan Partitions

In this section we show how we found the definition of the k-rank for $k = 3$. The Rogers-Ramanujan identities are

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-1})(1 - q^{5n-4})}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-2})(1 - q^{5n-3})}. \tag{3.2}$$

See [A2, Chapter 7]. Watson [W1] showed how (3.1), (3.2) follow from his q-analog of Whipple's theorem (ie. (2.10) with $k = 2$). Andrews [A4] has generalized (3.1), (3.2).

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_a + \dots + N_{k-1}}}{(q)_{n_1} (q)_{n_2} \dots (q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm a \pmod{2k+1}}}^{\infty} \frac{1}{1 - q^n}, \tag{3.3}$$

where $N_j = n_j + n_{j+1} + \dots + n_{k-1}$ with $1 \leq a \leq k$. The identities (3.1)–(3.3) have a number of combinatorial interpretations.

Theorem (3.4). *(B. Gordon* [Go]*) Let $B_{k,a}(n)$ denote the number of partitions of n of the form $b_1 + b_2 + \cdots + b_s$ where $b_j \geq b_{j+1}$, $b_j - b_{j+k-1} \geq 2$ and at most $a-1$ of the b_j equal 1. Let $A_{k,a}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm a \pmod{2k+1}$. Then*

$$A_{k,a}(n) = B_{k,a}(n), \qquad \textit{for all } n.$$

It is clear that $A_{k,a}(n)$ enumerates the right side of (3.3). Bressoud has shown how the left side enumerates $B_{k,a}(n)$. Andrews [A4] has found another interpretation of the left side of (3.3). We relate the $a = k$ case.

We define Andrews' concept of successive Durfee squares. For each partition π we find the largest square (starting from the upper left-hand corner) of dots contained in its graphical representation. This square is called the *Durfee square* (after W.P. Durfee). For example, if π is the partition $9+6+4+3+3+2+1$, then its graphical representation is given below in FIGURE I.

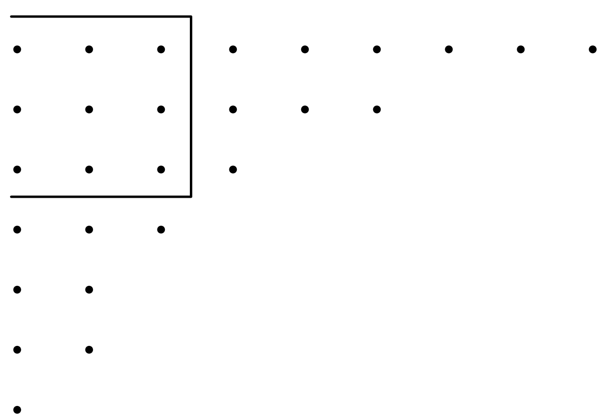


FIGURE I

and the 3×3 "Durfee" square is indicated. Once the Durfee square is determined, it splits the given partition into 3 parts: (1) the square itself, (2) a smaller partition to the right of the square, and (3) a smaller partition below the square. If the smaller partition below the Durfee square is non-empty then one can determine a "second Durfee square". Clearly third, fourth, etc. Durfee squares can be determined as long as the lower portion of the partition is not exhausted. Thus we see in FIGURE II that the partition $9+6+4+3+3+2+1$ has four successive Durfee squares.

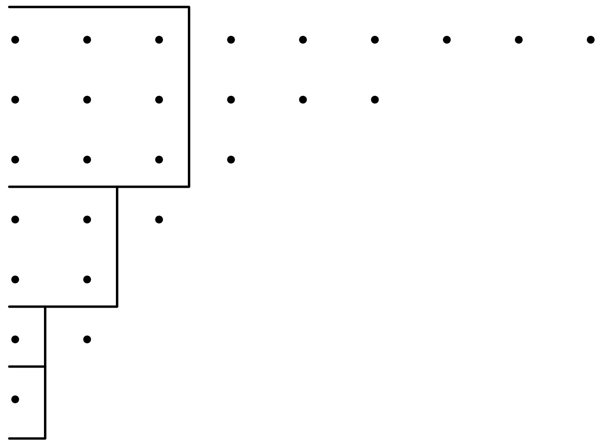


FIGURE II

Thus for $a = k$ we find the following interpretation of (3.3).

Theorem (3.5). [A4, Theorem 1] *The number of partitions of n with at most $k-1$ successive Durfee squares equals the number of partitions of n into parts $\not\equiv 0, \pm k \pmod{2k+1}$.*

Andrews [A4, Theorem 2] also found interpretations of (3.3) for general a.

As mentioned before, in [Ga2], we showed how (1.6) follows from Watson's q-analog of Whipple's theorem. To interpret the right side of (1.10) with $k = 3$ we play the same game except we utilize (2.10) with $k = 3$. We need the following lemma.

Lemma (3.6). *For $n \geq 1$, and $|q| < |z| < |q|^{-1}$ we have*

$$\frac{q^n(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)} = 1 - \frac{(1-q^n)}{(1+q^n)} \sum_{m=0}^{\infty} z^m q^{mn} - \frac{(1-q^n)}{(1+q^n)} \sum_{m=1}^{\infty} z^{-m} q^{mn}. \tag{3.7}$$

Letting $k=3$, $b_1=z$, $c_1=z^{-1}$, b_2, c_2, b_3, c_3, $N\to\infty$, and $a\to 1$ in (2.10) we obtain

(3.8)
$$\begin{aligned}
&\sum_{n_2\geq n_1\geq 0}\frac{q^{n_1^2+n_2^2}}{(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}}\\
&=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n(5n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right)\\
&=\frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^nq^{n(5n-1)/2}(1+q^n)\left\{1-\frac{(1-q^n)}{(1+q^n)}\left(\sum_{m=0}^{\infty}z^mq^{mn}+\sum_{m=1}^{\infty}z^{-m}q^{mn}\right)\right\}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma (3.6))}\\
&=\frac{\sum_{n=-\infty}^{\infty}(-1)^nq^{n(5n-1)/2}}{(q)_\infty}\\
&\qquad+\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(5n-1)/2}(1-q^n)\left(\sum_{m=0}^{\infty}z^mq^{mn}+\sum_{m=1}^{\infty}z^{-m}q^{mn}\right)\\
&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}+\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(5n-1)/2}(1-q^n)\left(\sum_{m=0}^{\infty}z^mq^{mn}+\sum_{m=1}^{\infty}z^{-m}q^{mn}\right).
\end{aligned}$$

In the last step we used the following.

(3.9)
$$\begin{aligned}
\frac{\sum_{n=-\infty}^{\infty}(-1)^nq^{n(5n-1)/2}}{(q)_\infty}&=\prod_{n=1}^{\infty}\frac{(1-q^{5n})(1-q^{5n-2})(1-q^{5n-3})}{(1-q^n)}\\
&\qquad\text{(by Jacobi's triple product identity [A2, p.21])}\\
&=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-1})(1-q^{5n-4})}\\
&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}\qquad\text{(by the first Rogers-Ramanujan identity (3.1)).}
\end{aligned}$$

We observe that the term corresponding to $n_1=0$ in the summation on the left side of

(3.8) also occurs on the right side. If we subtract this term from both sides we have

(3.10)
$$\sum_{n_2\geq n_1\geq 1}\frac{q^{n_1^2+n_2^2}}{(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}}$$
$$=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(5n-1)/2}(1-q^n)\left(\sum_{m=0}^{\infty}z^mq^{mn}+\sum_{m=1}^{\infty}z^{-m}q^{mn}\right).$$

In the analysis below, we will see that the left side of (3.10) (with $z=1$) enumerates partitions with at least two successive Durfee squares. We note, by Theorem (3.5), that the left side of the first Rogers-Ramanujan identity (3.1) enumerates partitions with at most one Durfee square. Thus we call partitions with at least two successive Durfee squares *non-Rogers-Ramanujan partitions.* We need to determine how the parameter z is counting these non-Rogers-Ramanujan partitions.

Let us recall that the Gaussian polynomial

$$\begin{bmatrix} n+m \\ m \end{bmatrix}=\frac{(q)_{m+n}}{(q)_n(q)_m}=\frac{(1-q^{n+1})\cdots(1-q^{n+m})}{(q)_m} \tag{3.11}$$

is the generating function for partitions with at most m parts each $\leq n$ [A2, p.33]. We rewrite the left side of (3.10) as

(3.12)
$$\sum_{n_2\geq n_1\geq 1} q^{n_2^2}\times\frac{1}{(1-q^{n_1+1})\cdots(1-q^{n_2})}\times\frac{1}{(1-zq)(1-zq^2)\cdots(1-zq^{n_1})}$$
$$\times\, q^{n_1^2}\times\frac{(1-q^{n_1+1})\cdots(1-q^{n_2})}{(1-q)\cdots(1-q^{n_2-n_1})}\times\frac{1}{(1-z^{-1}q)(1-z^{-1}q^2)\cdots(1-z^{-1}q^{n_1})}.$$

Each of the six terms in the summand above corresponds to one of six regions in the graphical representation of a partition with at least 2 successive Durfee squares. In FIGURE III we consider a generic non-Rogers-Ramanujan partition with successive Durfee squares of sizes $n_2\geq n_1\geq 1$.

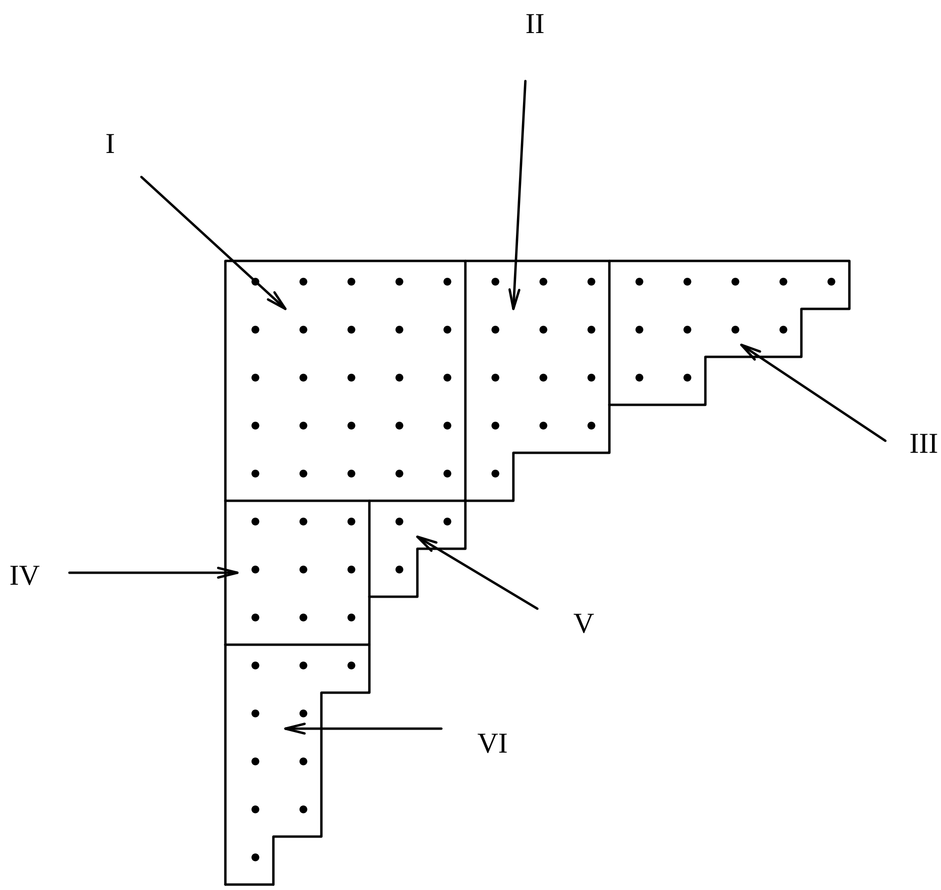


FIGURE III

We examine each term in (3.12). The term $q^{n_2^2}$ corresponds to region I which is the first Durfee square. The second term $\frac{1}{(1-q^{n_1+1})\cdots(1-q^{n_2})}$ is the generating function with with parts greater than n_1 and less than or equal to n_2 and it enumerates the columns in region II. The term $\frac{1}{(1-zq)(1-zq^2)\cdots(1-zq^{n_1})}$ is the generating function for partitions with parts $\leq n_1$ in which the power of z keeps track of the number of parts. Thus the third term enumerates the columns in region III with the power of z keeping track of the number of columns. The term $q^{n_1^2}$ corresponds to region IV which is the second Durfee square. The term

$$\frac{(1-q^{n_1+1})\cdots(1-q^{n_2})}{(1-q)\cdots(1-q^{n_2-n_1})} = \begin{bmatrix} n_2 \\ n_2-n_1 \end{bmatrix} = \begin{bmatrix} n_1+(n_2-n_1) \\ n_2-n_1 \end{bmatrix}$$

is the generating function for partitions with at most $n_2 - n_1$ parts each $\le n_1$ and enumerates the columns in region V. The term $\frac{1}{(1-z^{-1}q)(1-z^{-1}q^2)\cdots(1-z^{-1}q^{n_1})}$ enumerates the rows in region VI with the power of z^{-1} keeping track of the number of rows. Hence, considering (3.10), we are led to define the 3-rank as the number of columns in region III minus the number of rows in region VI. This coincides with the definition of $r_3(\pi)$ given in (1.11). Thus we have

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{\infty} N_3(m,n) z^m q^n = \sum_{n_2 \ge n_1 \ge 1} \frac{q^{n_1^2+n_2^2}}{(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}}. \tag{3.13}$$

The case $k = 3$ of Theorem (1.12) follows from (3.10) and (3.13). The proof for general k is completely analogous and is given in the next section.

4. The k-rank

In this section we prove Theorem (1.12). Let $k \ge 2$ be an integer. In (2.10) we let $b_1 = z$, $c_1 = z^{-1}$, b_2, c_2, $\dots$, b_k, c_k, $N \to \infty$, and $a \to 1$ to obtain

(4.1)
$$\begin{aligned}
&\sum_{n_{k-1}\ge n_{k-2}\ge\cdots\ge n_1\ge 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}} \\
&= \frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n((2k-1)n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
&= \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{n((2k-1)n-1)/2}}{(q)_\infty} \\
&\quad + \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2}(1-q^n)\left(\sum_{m=0}^{\infty} z^m q^{mn} + \sum_{m=1}^{\infty} z^{-m}q^{mn}\right),
\end{aligned}$$

by Lemma (3.6). We have followed the recipe of (3.8) with “5” replaced by “$2k-1$”. By

Jacobi's triple product [A2, p.21]

(4.2)
$$\begin{aligned}\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{n((2k-1)n-1)/2}}{(q)_\infty} &= \prod_{n=1}^{\infty}\frac{(1-q^{(2k-1)n})(1-q^{(2k-1)n-k})(1-q^{(2k-1)n-k+1})}{(1-q^n)}\\ &= \prod_{\substack{n=1\\ n\not\equiv 0,\pm(k-1)\ (\mathrm{mod}\ 2k-1)}}^{\infty}\frac{1}{(1-q^n)}\\ &= \sum_{n_{k-1}\geq\cdots\geq n_2\geq 0}\frac{q^{n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_3-n_2}(q)_{n_2}},\end{aligned}$$

by (3.3) with k replaced by $k-1$ and $a=k-1$. We note that the last term in (4.2) corresponds to the part of the sum on the left side of (4.1) with $n_1=0$. Hence if we subtract this term from both sides of (4.1) and use (4.2) we obtain

(4.3)
$$\begin{aligned}&\sum_{n_{k-1}\geq\cdots\geq n_1\geq 1}\frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(zq)_{n_1}(z^{-1}q)_{n_1}}\\ &=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2}(1-q^n)\left(\sum_{m=0}^{\infty}z^m q^{mn}+\sum_{m=1}^{\infty}z^{-m}q^{mn}\right).\end{aligned}$$

We rewrite the left side of (4.3) as

(4.4)
$$\begin{aligned}\sum_{n_1\geq\cdots\geq n_{k-1}\geq 1} & q^{n_1^2}\times\frac{1}{(1-q^{n_{k-1}+1})\cdots(1-q^{n_1})}\times\frac{1}{(1-zq)\cdots(1-zq^{n_{k-1}})}\\ &\times q^{n_2^2}\begin{bmatrix}n_1\\ n_1-n_2\end{bmatrix}\times\cdots\times q^{n_{k-1}^2}\begin{bmatrix}n_{k-2}\\ n_{k-2}-n_{k-1}\end{bmatrix}\\ &\times\frac{1}{(1-z^{-1}q)\cdots(1-z^{-1}q^{n_{k-1}})}.\end{aligned}$$

We note that in (4.3) we have replaced $n_1, n_2, \ldots, n_{k-1}$ by $n_{k-1}, n_{k-2}, \ldots, n_1$. We show that the function in (4.4) is the generating function for the k-rank. Consider partitions into at least $k-1$ successive Durfee squares with sides $n_1\geq n_2\geq\cdots\geq n_{k-1}\geq 1$. In the graphical representation of the partition we say the smaller partition to the right of a Durfee square is *associated* with the square. The term

$$q^{n_1^2}\times\frac{1}{(1-q^{n_{k-1}+1})\cdots(1-q^{n_1})}\times\frac{1}{(1-zq)\cdots(1-zq^{n_{k-1}})}$$

is the generating function for that portion of the partition that is associated with the first Durfee square (of side n_1) and the power of z counts the number of columns of length $\leq n_{k-1}$ (the side of the $k-1$-th Durfee square). For $2 \leq j \leq k-1$

$$q^{n_j^2} \begin{bmatrix} n_{j-1} \\ n_{j-1} - n_j \end{bmatrix}$$

generates those parts of the partition associated with the j-th Durfee square by considering columns of dots. Finally, the term

$$\frac{1}{(1-z^{-1}q)\cdots(1-z^{-1}q^{n_{k-1}})}$$

is the generating function for the portion of the partition that lies below the $k-1$-th Durfee square with the power of z^{-1} counting the number of parts. Hence the function in (4.4) is the generating function for the k-rank (see (1.11)) and we have

(4.5)
$$\sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} N_k(m,n) z^m q^n = \sum_{n_{k-1} \geq n_{k-2} \geq \cdots \geq n_1 \geq 1} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2-n_1} (zq)_{n_1} (z^{-1}q)_{n_1}}.$$

Theorem (1.12) follows from (4.3) and (4.5).

5. A family of involutions

From (1.10) we have

$$N_k(m,n) = N_k(-m,n). \tag{5.1}$$

We define an involution that explains (5.1) combinatorially. In this section we also prove Theorem (1.14). For fixed $k \geq 2$ we define an involution which we call *k-conjugation* and which acts on all partitions. If the number of successive Durfee squares is less than $k-1$ then k-conjugation is the identity. If the partition has at least $k-1$ successive Durfee squares we consider its graphical representation. See the FIGURE IV below.

It is clear how we should define the k-conjugate. We consider two regions of the partition. The first region consists of those columns to the right of the first Durfee square whose length $\leq n_{k-1}$ (the side of the $k-1$-th Durfee square). The second region is that portion of the partition below the $k-1$-th Durfee square. We take the conjugate of each region and interchange. This is k-conjugation. This operation clearly reverses the sign of the k-rank and (5.1) follows. Also we note that 2-conjugation is ordinary conjugation. We illustrate with an example for $k=3$ in FIGURE V below.

We see that the 3-conjugate of $4+2+1+1$ is $3+2+1+1+1$. Recall that a partition is *self-conjugate* if it is fixed by conjugation. The following partition theorem is well-known.

Theorem (5.2). [S-W, Theorem 3.3, p.72] *The number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.*

We will generalize this theorem. We call a partition *self-k-conjugate* if it is fixed by k-conjugation. For example the partition $4+2+1+1+1$ is self-3-conjugate. We note that if a partition has no more than $k-2$ successive Durfee squares then it is self-k-conjugate. We let $\mathrm{sc}_k(n)$ denote the number of self-k-conjugate partitions of n.

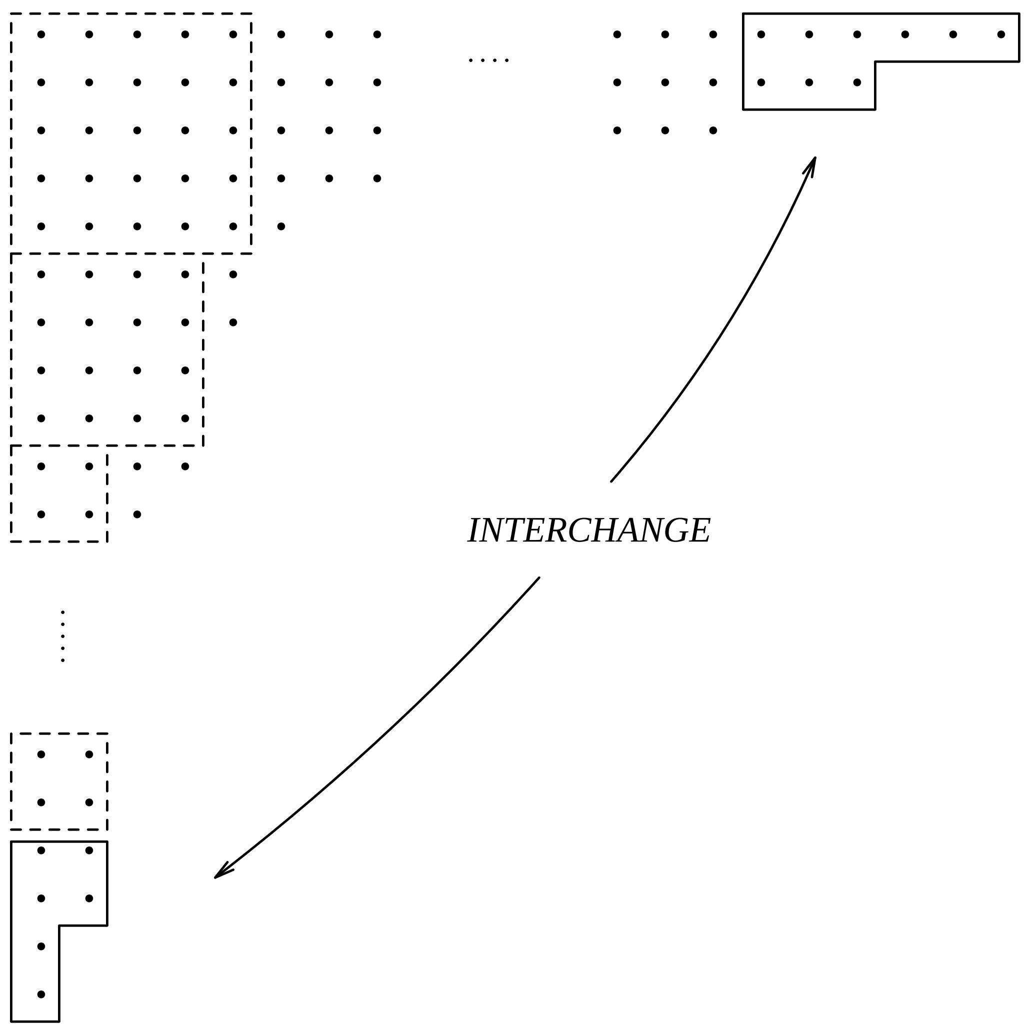


FIGURE IV

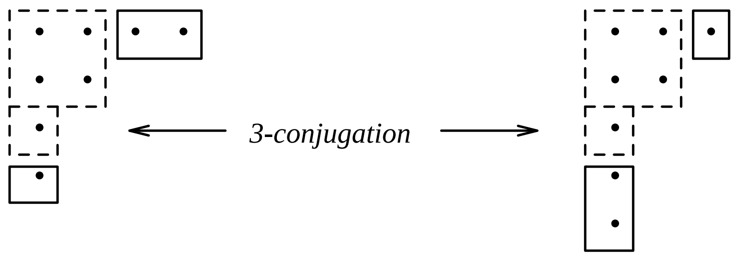


FIGURE V

The main result of this section is

Theorem (5.3). *Let $k \geq 2$ be an integer. Then*

$$\sum_{n=0}^{\infty} sc_k(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{kn})^2}{(1-q^{2kn})(1-q^n)}. \tag{5.4}$$

Proof. We use the method of [A5] where it was shown how to generalize each of Slater's [S1],[S2] Rogers-Ramanujan type identities to multiple series identities. By Slater [S1] the following form a Bailey pair (with $a=1$):

$$\alpha_n = \begin{cases} (-1)^n 2q^{n^2}, & n > 0, \\ 1, & n = 0, \end{cases} \tag{5.5}$$

$$\beta_n = \frac{1}{(q^2;q^2)_n}. \tag{5.6}$$

We apply (2.7) with k replaced by $k-1$ and $a=1$, to obtain

(5.7)
$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2} = \sum_{n_{k-1} \geq n_{k-2} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2-n_1} (q^2;q^2)_n}$$
$$\sum_{n_1 \geq \cdots \geq n_{k-1} \geq 0} q^{n_1^2} \times \frac{1}{(1-q^{n_{k-1}+1}) \cdots (1-q^{n_1})} \times \frac{1}{(q^2;q^2)_{n_{k-1}}}$$
$$\times\, q^{n_2^2} \begin{bmatrix} n_1 \\ n_1 - n_2 \end{bmatrix} \times \cdots \times q^{n_{k-1}^2} \begin{bmatrix} n_{k-2} \\ n_{k-2} - n_{k-1} \end{bmatrix},$$

which is the generating function for self-k-conjugate partitions by an analysis analogous to that of the proof of (4.5) in §4. Hence we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{sc}_k(n) q^n &= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{kn^2} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{kn})}{(1+q^{kn})(1-q^n)} \qquad \text{(by [A2, (2.2.12), p.23])} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{kn})^2}{(1-q^{2kn})(1-q^n)},
\end{aligned} \tag{5.8}$$

which is the desired result. □

As a corollary we deduce our partition theorem (1.14).

Corollary (5.9). *Let $k \geq 1$ be an integer. Then the number of self-$2k$-conjugate partitions of n is equal to the number of partitions of n with no parts divisible by $2k$ and all parts which are congruent to k modulo $2k$ are distinct.*

Proof. From (5.4) we have

$$\sum_{n=0}^{\infty} \mathrm{sc}_{2k}(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2kn})^2}{(1-q^{4kn})(1-q^n)}. \tag{5.10}$$

We need the elementary identity

$$\prod_{n=1}^{\infty} (1-q^{2n-1}) = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{2n})}. \tag{5.11}$$

So from (5.10), (5.11) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{sc}_{2k}(n) q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{4kn-2k})(1-q^{2kn})}{(1-q^n)} \\
&= \prod_{n=1}^{\infty} \frac{(1+q^{2kn-k})(1-q^{2kn-k})(1-q^{2kn})}{(1-q^n)} \\
&= \prod_{n=1}^{\infty} (1+q^{2kn-k}) \prod_{\substack{n=1 \\ n \not\equiv 0,k \pmod{2k}}}^{\infty} \frac{1}{(1-q^n)}
\end{aligned} \tag{5.12}$$

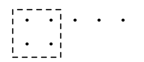
which is the generating function of partitions with no parts divisible by $2k$ and all parts which are congruent to k modulo $2k$ distinct, as required. □

In FIGURE VI below we illustrate this result with $k = 2$. There are 12 self-4-conjugate partitions of 8.

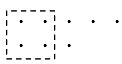
8

$7+1$

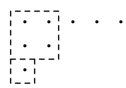
$6+2$



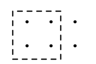
$5+3$



$5+2+1$



$4+4$



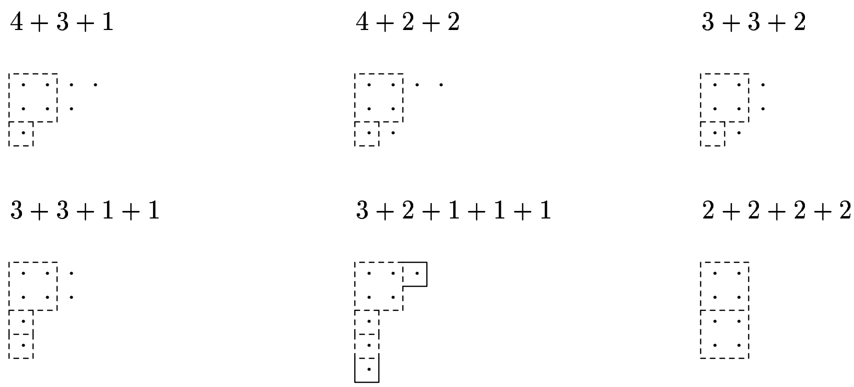


FIGURE VI

As predicted by the result, there are also 12 partitions of 8 with parts not divisible by 4 and the parts congruent to 2 modulo 8 are distinct: $7+1$, $6+2$, $6+1+1$, $5+3$, $5+2+1$, $5+1+1+1$, $3+3+2$, $3+3+1+1$, $3+2+1+1+1$, $3+1+1+1+1+1$, $2+1+1+1+1+1+1$, $1+1+1+1+1+1+1+1$.

6. Questions

Question 1. Find a combinatorial proof of Theorem (1.12).

Dyson [D2] has given a combinatorial proof of Theorem (1.12) for the case $k=2$. More recently, he has proved (1.9) combinatorially [D4].

Question 2. Find a combinatorial proof of Theorem (1.14).

The case $k=1$ has a well-known and easy combinatorial proof. See [S-W], [A7].

Lewis and Santa-Gadea have found numerous relations between $N_V(m,n)$ and $N_2(m,n)$. Unfortunately we have found no further non-trivial relations among the $N_k(m,n)$.

Acknowledgments

The author would like to thank George Andrews for suggesting the problem of generalizing Dyson's rank and for his help and encouragement.

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