SOME MACDONALD-MEHTA INTEGRALS BY BRUTE FORCE

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Abstract. Bombieri and Selberg showed how Mehta's [6; p. 42] integral could be evaluated using Selberg's [7] integral. Macdonald [5; §§5,6] conjectured two different generalizations of Mehta's integral formula. The first generalization is in terms of finite Coxeter groups and depends on one parameter. The second generalization is in terms of root systems and the number of parameters in equal to the number of different root lengths. In the case of Weyl groups Macdonald showed how the first generalization follows from the second. We give a proof of the $\mathcal{I}_3$ case of the first generalization and the F_4 case of the second generalization. As well we give a two parameter generalization for the dihedral group $\mathcal{H}_2^{2n}$. The parameters are constant on each of the two orbits. We note that the G_2 case of the second generalization follows from our two-parameter version for $\mathcal{H}_2^6$. Our proofs draw on ideas from Aomoto's [1] proof of Selberg's integral and Zeilberger's [10] proof of the $G_2^\vee$ case of the Macdonald Morris [5; Conj. 3.3] constant term root system conjecture. The problem is reduced to solving a system of linear equations. These equations were generated and solved by the computer algebra package MAPLE.

1. Introduction. In 1967 Mehta [6; p. 42] conjectured that

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2}|D(x)|^{2k}dx = (2\pi)^{n/2}\prod_{j=1}^{n}\frac{\Gamma(jk+1)}{\Gamma(j+1)}. \tag{1.1}$$

Here, k is any complex number with $Re(k) > 0$, $dx = dx_1 \dots dx_n$ is Lebesgue measure, $\|x\|^2 = x_1^2 + \cdots + x_n^2$, and $D(x) = \prod_{i<j}(x_i - x_j)$. E. Bombieri and Selberg showed how (1.1) follows from Selberg's integral

$$\int_{[0,1]^n}\prod_{i=1}^{n} x_i^{a-1}(1-x_i)^{b-1}|D(x)|^{2c}dx = \prod_{i=1}^{n}\frac{\Gamma(a+(n-i)c)\Gamma(b+(n-i)c)\Gamma(ic+1)}{\Gamma(a+b+(2n-i-1)c)\Gamma(c+1)}. \tag{1.2}$$

See [5; p. 1000]. Macdonald [5; §§5,6] conjectured two different generalizations of (1.1).

The first generalization is in terms of Coxeter groups. Let $\mathcal{G}$ be a finite Coxeter group, i.e. a finite group of isometries of $\mathbb{R}^n$ generated by reflections S in hyperplanes through the origin. The equations of these hyperplanes are of the

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form $h_S(x) = \sum_{i=1}^n a_i x_i = 0$. Normalize each h_S (up to sign) by requiring that $\sum a_i^2 = 2$, and let $P(x) = \prod_S h_S(x)$ be the product of these normalized linear forms, the product being over all reflections S in $\mathcal{G}$. Let d_i be the degrees of the fundamental polynomial invariants of $\mathcal{G}$. Macdonald [5; Conj. 5.1] conjectured

Macdonald-Mehta Conjecture I. *If k is any complex number with $Re(k) > 0$, then*

$$\text{(Mac-Meh I)} \qquad \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\|x\|^2/2} |P(x)|^k dx = \prod_{j=1}^n \frac{\Gamma(\frac{k}{2} d_j + 1)}{\Gamma(\frac{k}{2} + 1)}.$$

When $\mathcal{G}$ is the symmetric group S_n, acting on $\mathbb{R}^n$ by permuting the coordinates (Mac-Meh I) reduces to (1.1). A. Regev observed that when $\mathcal{G}$ is $\mathcal{B}_n$ or $\mathcal{D}_n$ then (Mac-Meh I) is true for all k, again by Selberg's integral. Macdonald showed that (Mac-Meh I) is true for k=1 and $\mathcal{G}$ a Weyl group, and for arbitrary k when $\mathcal{G}$ is dihedral. As noted by Macdonald the dihedral case can be computed by transforming to polar coordinates.

We note that (Mac-Meh I) may be generalized as follows: for a reflection $S \in \mathcal{G}$ we let k_S be any complex number with $Re(k_S) > 0$ such that $k_{S_1} = k_{S_2}$ whenever h_{S_1}, h_{S_2} belong to the same orbit when $\mathcal{G}$ acts on the set of hyperplanes. In this case if $|P(x)|^k$ in the integrand of the left side of (Mac-Meh I) is replaced by $\prod_S |h_S(x)|^{k_S}$ then the resulting integral can be evaluated as a nice product of gamma functions. If $\mathcal{G}$ is a Weyl group then this integral reduces to the one given below in (Mac-Meh II). The only other non-transitive non-Weyl irreducible Coxeter groups are the dihedral groups $\mathcal{H}_2^{2m}$. In this case there are two orbits. The integral is given below in (1.4). As in the equal parameter case the evaluation follows easily by transforming to polar coordinates. The details are given in §2.

Theorem (1.3). *If $a, b \in \mathbb{C}$ with $Re(a), Re(b) > 0$ and $m \in \mathbb{N}$ then*

(1.4)
$$\frac{2}{2\pi}^{m(a+b)/2} \int_{\mathbb{R}^2} \prod_{k=0}^{m-1} |\cos\tfrac{k\pi}{m}\, x_1 + \sin\tfrac{k\pi}{m}\, x_2|^a |\cos\tfrac{(2k+1)\pi}{2m}\, x_1 + \sin\tfrac{(2k+1)\pi}{2m} x_2|^b$$
$$\cdot\, e^{-(x_1^2+x_2^2)/2} dx_1 dx_2$$
$$= \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(\frac{m(a+b)}{2}+1)}{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)\Gamma(\frac{a+b}{2}+1)}.$$

Macdonald's second generalization is in terms of root systems. Let S be a (not necessarily reduced) root system consisting of linear forms on a real Euclidean space $\mathfrak{A}$. We normalize the linear forms $a \in S$ so that they have norm $\sqrt{2}$. Let k_α be complex-numbers with real part > 0 such that $k_\alpha = k_\beta$ if $\|\alpha\| = \|\beta\|$, and let $P(x) = \prod_{\alpha \in S^+} |\alpha(x)|^{k_\alpha}$ be the product of these normalized linear forms, weighted according to the multiplicity k_α, over the set of positive roots. Macdonald conjectured

Macdonald-Mehta Conjecture II.

$$\text{(Mac-Meh II)} \quad \int_{\mathfrak{A}} e^{-\|x\|^2/2} P(x) d\gamma(x) = \prod_{\alpha \in S^+} \frac{\Gamma(\frac{1}{2}k_\alpha + \frac{1}{4}k_{\alpha/2} + \frac{1}{2}(\rho_k, \alpha^\vee) + 1)}{\Gamma(\frac{1}{4}k_{\alpha/2} + \frac{1}{2}(\rho_k, \alpha^\vee) + 1)},$$

where $\alpha^\vee$ is the coroot $2\alpha/\|\alpha\|^2$, $k_{\alpha/2} = 0$ if $\frac{1}{2}\alpha \notin S$, $\rho_k = \frac{1}{2}\sum_{\alpha\in S^+} k_\alpha \alpha$, and γ is the Gaussian measure on $\mathfrak{A}$.

Macdonald has shown that (Mac-Meh II) is true in the following three cases:

(a) S is of classical type $(A_n, B_n, C_n, D_n, BC_n)$ (by Selberg's integral).
(b) S is the restricted root system of a symmetric space G/K and the k_α are the multiplicities m_α of the root.
(c) $S = G_2$ and the k_α are all equal.

Case (c) follows from the fact that when S is reduced and the k_α are all equal (Mac-Meh II) reduces to (Mac-Meh I) and from the fact that the Weyl group of G_2 is $\mathcal{H}_2^6$. We note that the general G_2 case follows from (1.4) with $m = 3$.

In §3 we introduce some notation and prove some preliminary results. In §4 we present a computer approach for handling (Mac-Meh I) for a given Coxeter group. In §5 we describe some modifications of this approach so as to handle (Mac-Meh II). We have sucessfully implemented this approach on the computer to prove the $\mathcal{I}_3$ case of the first conjecture and the F_4 case of the second conjecture. The details of the $\mathcal{I}_3$ case are given in §6. Some details of the F_4 case are given in §7.

Our computer programs are written in FORTRAN or MAPLE and were run on an APOLLO DN-5800 at the I.M.A., University of Minnesota, Minneapolis. All computer programs or files used in this paper are available from the author on request.

2. An extra parameter for the dihedral case.

In this section we prove (1.4) which is a generalization of (Mac-Meh I) for the dihedral group $\mathcal{H}_2^{2m}$. Our generalization has two parameters one for each orbit.

As noted in §1 the dihedral group $\mathcal{H}_2^n$ is non-transitive if and only if n is even. From [4; p. 76] a set of hyperplanes for $\mathcal{H}_2^n$ is

$$h_j(x) := \cos\tfrac{j\pi}{n}\, x_1 + \sin\tfrac{j\pi}{n}\, x_2 = 0, \qquad (0 \le j \le n-1).$$

If $n = 2m$ and $\mathcal{H}_2^n$ acts on the set of hyperplanes there are two orbits:

$$ORB1 = \{h_j : j \text{ even}\}, \quad ORB2 = \{h_j : j \text{ odd}\}.$$

In this way (1.4) may be written as

$$\begin{aligned}(2.1)\qquad \frac{1}{2\pi}\int_{\mathbb{R}^2} \prod_{h\in ORB1} |\sqrt{2}h(x)|^a \prod_{h\in ORB2} |\sqrt{2}h(x)|^b e^{-(x_1^2+x_2^2)/2} dx_1 dx_2 \\ = \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(\frac{m(a+b)}{2}+1)}{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)\Gamma(\frac{a+b}{2}+1)}.\end{aligned}$$

As with the $a = b$ case the general case of (2.1) follows easily by transforming to polar coordinates. If we let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $z = re^{i\theta}$ then

(2.2)
$$\begin{aligned}\prod_{h\in ORB1}|h(x)| &= \prod_{k=0}^{m-1}|r\cos(\frac{k\pi}{m}-\theta)| \\ &= \frac{r^m}{2^m}\prod_{k=0}^{m-1}|e^{-i\theta}(1-Xe^{-2\pi ik/m})| \qquad (\text{where } X = e^{2i\theta}) \\ &= \frac{r^m}{2^m}|e^{-im\theta}(1-X^m)| \\ &= \frac{r^m}{2^m}|e^{im\theta}-e^{-im\theta}| \\ &= \frac{1}{2^m}|z^m-\overline{z}^m|.\end{aligned}$$

Similarly,

$$\prod_{h\in ORB2}|h(x)| = \frac{1}{2^m}|z^m+\overline{z}^m|. \tag{2.3}$$

Hence on conversion to polar coordinates we find that the left side of (2.1) becomes

$$\begin{aligned}&\frac{1}{2\pi}2^{(a+b)(1-\frac{m}{2})}\int_0^\infty r^{ma+mb+1}e^{-r^2/2}dr\int_0^{2\pi}|\sin^b m\theta||\cos^a m\theta|d\theta \\ &= \frac{2^{a+b}\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})\Gamma(\frac{ma+mb}{2}+1)}{\pi\Gamma(\frac{a+b}{2}+1)} \\ &= \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(\frac{m(a+b)}{2}+1)}{\Gamma(\frac{a}{2}+1)\Gamma(\frac{b}{2}+1)\Gamma(\frac{a+b}{2}+1)},\end{aligned}$$

as required. In the last step we have used the duplication formula [10; p. 240]:

$$2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\Gamma(2z). \tag{2.5}$$

3. Preliminaries. Before describing our computer approach we need to introduce some notation and develop some preliminary results.

First we need to write (Mac-Meh I) and (Mac-Meh II) in the same form. To do this we define a *root system* of a finite Coxeter group $\mathcal{G}$. Our definition follows Benson and Grove [4; §4.1]. Let V be a finite dimensional real Euclidean space, with inner product (,). For a hyperplane $\mathcal{P}_r = \{x : (x,r) = 0\}$ $(r \neq 0)$ the reflection S_r is the unique linear transformation fixing $\mathcal{P}_r$ and sending r to $-r$. S_r is given by

$$S_r(x) = x - \frac{2(x,r)}{(r,r)}r. \tag{3.1}$$

Suppose $\mathcal{G}$ is generated by $\mathcal{S} \subset \mathcal{G}$ a finite set of reflections. For each $S \in \mathcal{S}$ choose $r \neq 0$ such that $S = S_r$, (usually $\|r\| = 1$ but not necessarily). The vectors $\pm r$ are the *roots* corresponding to S. The *root system* $\Delta = \Delta(\mathcal{G})$ is the set of all roots corresponding to the generating reflections, together with all images of these roots under all transformations in $\mathcal{G}$. In Appendix A we give a table of root systems for each finite Coxeter group. We note that each of these root systems is embedded in $\mathbb{R}^n$ for some n.

We may partition Δ into two subsets as follows. Choose a $t \in V$ such that $(t, r) \neq 0$ for every $r \in \Delta$. Let

$$\Delta^+ = \Delta_t^+ = \{r \in \Delta : (t, r) > 0\} \tag{3.2a}$$

and

$$\Delta^- = \Delta_t^- = \{r \in \Delta : (t, r) < 0\}. \tag{3.2b}$$

Then

$$\Delta^- = -\Delta^+ \text{ and } \Delta = \Delta^+ \cup \Delta^-.$$

For $r \in \Delta^+$ the equation of the hyperplane orthogonal to r is given by

$$r(x) := (x, r) = 0. \tag{3.3}$$

For example if $\Delta = \Delta(\mathcal{A}_{n-1})$, $r = e_i - e_j$ then

$$r(x) = r(x_1, x_2, \ldots, x_n) = x_i - x_j.$$

In this way we may write each of the integrals in (Mac-Meh I) and (Mac-Meh II) in the form:

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \prod_{r \in \Delta^+} \mid \frac{\sqrt{2}}{\|r\|} r(x) \mid^{k_r} e^{-\|x\|^2/2} dx. \tag{3.4}$$

We introduce some notation to describe the conjectures for the equal parameter case (i.e. $k_r = k_s$ for all $r, s \in \Delta^+$). In §5 we describe the general case. We assume $k_r = 2a$ for all $r \in \Delta^+$ and let

$$F(a) = F_{\mathcal{G}}(a) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \prod_{r \in \Delta^+} \mid \frac{\sqrt{2}}{\|r\|} r(x) \mid^{2a} e^{-\|x\|^2/2} dx, \tag{3.5}$$

and

$$f(a) = f_{\mathcal{G}}(a) := \prod_{j=1}^{n} \frac{\Gamma(a d_j + 1)}{\Gamma(a + 1)}, \tag{3.6}$$

which is the right side of either (Mac-Meh I) (with $k = 2a$) or (Mac-Meh II) (with $k_\alpha = 2a$).

With Δ embedded in $\mathbb{R}^n$ the symmetric group $\mathcal{S}_n$ acts on $\mathbb{R}^n$ as follows. For $\pi \in \mathcal{S}_n$ and $r = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$

$$\pi r = (r_{\pi(1)}, r_{\pi(2)}, \ldots, r_{\pi(n)}). \tag{3.7}$$

Let,

$$SYM = \{\pi \in \mathcal{S}_n : \pi(\Delta) = \Delta\} < \mathcal{S}_n. \tag{3.8}$$

Choose a fundamental region [4; p. 27], FUN for SYM in V, so that for all $r \in V$ there is a $\pi \in SYM$ such that $\pi r \in \overline{\mathrm{FUN}}$ and let

$$w(x) = w(x,a) := \frac{1}{(2\pi)^{n/2}} \prod_{r\in\Delta^+} |\frac{\sqrt{2}}{\|r\|} r(x)|^{k_r} e^{-\|x\|^2/2} \tag{3.9}$$

so that $F(a) = \int_{\mathbb{R}^n} w(x)dx$. For $x \in \mathbb{R}^n$, the elements of $\mathcal{G}$ and SYM act on functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$Tf(x) = f(T(x)) \qquad \text{(for } T \in \mathcal{G} \text{ or } SYM). \tag{3.10}$$

For $\alpha \in \mathbb{N}^n$ say $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ we let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}. \tag{3.11}$$

The elements of SYM act on the monomials x^α in a nice way:

$$\pi x^\alpha = (\pi(x))^\alpha = x^{\pi(\alpha)} \qquad (\pi \in SYM). \tag{3.12}$$

We note that in general the elements of $\mathcal{G}$ do not act in such a nice way since they do not necessarily send monomials to monomials. Since the elements of SYM and $\mathcal{G}$ are orthogonal transformations which leave Δ invariant we have the following Lemmas:

Lemma (3.13). *Let $S \in \mathcal{G}$ and $p(x) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ then*

$$\int_{\mathbb{R}^n} p(S(x))w(x)dx = \int_{\mathbb{R}^n} p(x)w(x)dx. \tag{3.14}$$

Lemma(3.15). *For each $\alpha \in \mathbb{N}^n$ there is an $\alpha_0 \in \mathbb{N}^n \cap \overline{FUN}$ and a $\pi \in SYM$ such that $\pi\alpha = \alpha_0$ and*

$$\int_{\mathbb{R}^n} x^\alpha w(x)dx = \int_{\mathbb{R}^n} x^{\alpha_0} w(x)dx. \tag{3.16}$$

For $\alpha \in \mathbb{N}^n$ we let $\mathrm{sum}(\alpha) = \sum_{i=1}^n \alpha_i$. Since $w(-x) = w(x)$ we have

Lemma (3.17). *If sum(α) is odd then*

$$\int_{\mathbb{R}^n} x^\alpha w(x)dx = 0. \tag{3.18}$$

We note that for many root systems there are other monomials for which (3.18) is true. In particular, we have

Lemma(3.19). *If $e_i \in \Delta(\mathcal{G})$ $(1 \le i \le n)$, $\alpha \in \mathbb{N}^n$ and $\alpha \not\equiv 0$ (mod 2) then*

$$\int_{\mathbb{R}^n} x^\alpha w(x)dx = 0. \tag{3.20}$$

The proof of the lemma is analogous to that of the previous one. We note that $\Delta(\mathcal{G})$ contains e_i $(1 \le i \le n)$ for $\mathcal{G} = \mathcal{B}_n$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{F}_4$.

4. The computer approach.

In this section we give a computer approach for handling (Mac-Meh I) (or (Mac-Meh II)) for any given finite Coxeter group (or root system) in the equal parameter case excluding the dihedral and $\mathcal{G}_2$ cases. In §5 we give the modifications of our approach that will handle the two parameter case.

Our goal is to prove

$$F(a) = f(a) \qquad (\text{for } Re(a) > 0), \tag{4.1}$$

where $F(a), f(a)$ are defined in (3.5), (3.6). Using Carlson's Theorem [9; p. 186] it is enough to show that (4.1) is true for $a \in \mathbb{N} = \{0, 1, 2, \dots\}$. The details are given in Appendix B. The idea is to proceed by induction on a. That is, we want to prove

$$\frac{F(a+1)}{F(a)} = \frac{f(a+1)}{f(a)}. \tag{4.2}$$

This will be enough since the case $a = 0$ is trivial.

The flavor of our proof resembles our [2] proof of the $F_4\,(q = 1)$ case of the Macdonald-Morris constant term root system case which in turn was inspired by Zeilberger's [11] proof of the $G_2^\vee$ case. Zeilberger's proof was inspired by Stembridge's [8] proof of the A_n case.

(4.3)

$$\begin{aligned}
F(a+1) &= \int_{\mathbb{R}^n} w(x, a+1)dx \\
&= \int_{\mathbb{R}^n} \prod_{r\in\Delta^+} |\frac{\sqrt{2}}{\|r\|} r(x)|^2 w(x,a)dx \qquad (\text{by } (3.9)) \\
&= \sum_{\alpha\in L'} \int_{\mathbb{R}^n} a'_\alpha x^\alpha w(x,a)dx \qquad (\text{for some finite } L' \subset \mathbb{N}^n) \\
&= \sum_{d\in L} \int_{\mathbb{R}^n} a_\alpha x^\alpha w(x,a)dx,
\end{aligned}$$

for some $L \subset \overline{\mathrm{FUN}} \cap \{\alpha \in \mathbb{N}^n : \mathrm{sum}(\alpha) = |\Delta|\}$ by Lemma (3.15). The idea is to write a computer program to expand $\prod_{r\in\Delta^+}(r(x))^2$ and do the reduction in (4.3).

For $m = 0, 1, \dots, |\Delta^+| - 1$ let

$$L(m) = \begin{cases} \overline{\mathrm{FUN}} \cap \{\alpha \in \mathbb{N}^n : \mathrm{sum}(\alpha) = 2m\}, & \text{if } \mathcal{G} \neq \mathcal{B}_n, \mathcal{I}_3, \mathcal{I}_4, \mathcal{F}_4, \\ \overline{\mathrm{FUN}} \cap \{\alpha \in (2\mathbb{N})^n : \mathrm{sum}(\alpha) = 2m\}, & \text{otherwise,} \end{cases} \tag{4.4a}$$

and let

$$L(|\Delta^+|) = L. \tag{4.4b}$$

Suppose $|L(m)| = c_m$ and

$$L(m) = \{\alpha_{m,1}, \alpha_{m,2}, \dots, \alpha_{m,c_m}\}. \tag{4.5}$$

Let,

$$un(m,j) = un(m,j,a) := \int_{\mathbb{R}^n} x^{\alpha_{m,j}} w(x,a)dx. \tag{4.6}$$

The problem is to get each of the $un(m,j)$ in terms of $un(0,1) = \int_{\mathbb{R}^n} w(x,a)dx = F(a)$. Once we have done this (4.2) should follow from (4.3). We find that each $un(m,j)$ can be gotten in terms of the $un(m-1,k)$ $(1 \le k \le c_{m-1})$.

Theorem (4.7). *Let $\mathcal{G}$ be a finite irreducible Coxeter group, $\mathcal{G} \neq \mathcal{H}_2^n$, $\mathcal{G}_2$. For $1 \leq m \leq |\Delta^+|$, $\quad 1 \leq j \leq c_m$ there exist $d_{m,k}, e_{m,k} \in \mathbb{R}$ $\quad (1 \leq k \leq c_{m-1})$ such that*

$$un(m,j) = \sum_{k=1}^{c_{m-1}} (d_{m,k}a + e_{m,k}) un(m-1,k). \tag{4.8}$$

Proof. By definition we have

$$un(m,j) = \int_{\mathbb{R}^n} x^{\alpha_{m,j}} w(x,a) dx. \tag{4.9}$$

We describe an algorithm with 4 steps that converts equation (4.9) into the desired form of equation (4.8).

STEP 1. Suppose $\alpha_{m,j} = (k_1, k_2, \ldots, k_n)$. Find the first nonzero coordinate of $\alpha_{m,j}$, say k_i and use

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} \frac{x^{\alpha_{m,j}}}{x_i} w(x,a) dx = 0. \tag{4.10}$$

This idea was used by Aomoto[1] in his proof of Selberg's [7] integral. (4.10) gives rise to the following equations:

$$\begin{aligned} un(m,j) &= \int_{\mathbb{R}^n} x^{\alpha_{m,j}} w(x,a) dx \\ &= (k_i - 1) \int_{\mathbb{R}^n} \frac{x^{\alpha_{m,j}}}{x_i^2} w(x,a) dx \\ &+ 2a \sum_{r \in \Delta^+} \int_{\mathbb{R}^n} \frac{r_i x^{\alpha_{m,j}}/x_i}{r(x)} w(x,a) dx. \end{aligned} \tag{4.11}$$

STEP 2. Use the group $\mathcal{G}$ to reduce the number of types of terms arising in STEP 1 (i.e. in the right side of (4.11)).

When $\mathcal{G}$ acts on the root system Δ there are at most 2 orbits. By examining Table I (Appendix A) we find that each orbit contains one of the vectors in the following set:

$$\text{NICEVECS} = \{e_1, e_1 - e_2\}. \tag{4.12}$$

Thus for each $r \in \Delta^+$ there is a $T_r \in \mathcal{G}$ such that $T_r(r) \in$ NICEVECS. Hence we find that each integral in the summation part of the right side of (4.11) may be written in the form

$$\begin{aligned} &\int_{\mathbb{R}^n} T_r^{-1} \left(\frac{x^{\alpha_{m,j}}/x_i}{r(x)} \right) w(x,a) dx \qquad \text{(by Lemma (3.13))} \\ &\quad = \int_{\mathbb{R}^n} \frac{T_r^{-1}(x^{\alpha_{m,j}}/x_i)}{T_r(r)(x)} w(x,a) dx, \end{aligned} \tag{4.13}$$

where $T_r(r)(x) = x_1$ or $x_1 - x_2$.

STEP 3. Get rid of the denominator appearing in the right side of (4.13) by using straight division or utilizing symmetry in x_1, x_2. This will mean each integral may be written in the form:

$$\sum_{\substack{\alpha \\ \mathrm{sum}(\alpha)=2m-2}} \int_{\mathbb{R}^n} d'_\alpha x^\alpha w(x,a)dx, \tag{4.14}$$

where the $d'_\alpha \in \mathbb{R}$. There are two cases:

CASE(I) $T_r(r)(x) = x_1$.

In this case $e_1 \in \Delta$. By considering the reflection S_{e_1} we find $w(x_1, x_2, \dots, x_n; a) = w(-x_1, x_2, \dots, x_n; a)$. It follows that $\int_{\mathbb{R}^n} x^\alpha w(x)dx = 0$ if α_1 is odd where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n$ (as long as the denominator of x^α divides the polynomial part of $w(x)$). Hence we expand $T_r^{-1}(x^{\alpha_{m,j}}/x_i)$ as a sum of monomials, carry out the division by x_1 and toss out all monomials in which the exponent of x_1 is -1. In this way the integral in (4.13) may be written in the desired form (4.14).

CASE(II) $T_r(r)(x) = x_1 - x_2$.

In this case $e_1 - e_2 \in \Delta$ and it follows that $w(x,a)$ is symmetric in x_1, x_2. Hence,

(4.15)

$$\int_{\mathbb{R}^n} \frac{T_r^{-1}(x^{\alpha_{m,j}}/x_i)}{T_r(r)(x)} w(x,a)dx$$
$$= \frac{1}{2}\int_{\mathbb{R}^n} \frac{(T_r^{-1}(x^{\alpha_{m,j}}/x_i) - (12)\{T_r^{-1}(x^{\alpha_{m,j}}/x_i\})}{(x_1 - x_2)} w(x,a)dx \qquad (\text{since } (12) \in \mathrm{SYM}).$$

Now expand $(T_r^{-1}(x^{\alpha_{m,j}}/x_i) - (12)\{T_r^{-1}(x^{\alpha_{m,j}}/x_i)\}$ and carry out the division by $(x_1 - x_2)$. In this way the integral in (4.13) may be written in the desired form (4.14).

STEP 4. After STEP 3 all integrals on the right side of (4.11) are in the form of (4.14). Use the group SYM to write each of these integrals in the form:

$$\sum_{\alpha \in L(m-1)} \int_{\mathbb{R}^n} d_\alpha x^\alpha w(x,a)dx. \tag{4.16}$$

We note that if $\mathcal{G} = \mathcal{B}_n, \mathcal{I}_3, \mathcal{I}_4$, or $\mathcal{F}_4$ we were able to omit any vector α with an odd component in view of Lemma (3.19). Thus (4.11) can be written in the form of (4.8) as required. □

Remark. Steps 1-4 in the proof of Theorem (4.7) provide an algorithm for getting $un(m,j)$ in terms of $un(m-1,k)$ $(1 \le k \le c_m)$ and hence, by iteration, of getting $\int_{\mathbb{R}^n} x^\alpha w(x,a)\,dx$ $(\alpha \in L)$ in terms of $F(a)$. This algorithm can be easily implemented on a computer using an algebra package like MAPLE. Hence with the aid of a computer one should be able to verify (4.2) for *small* root systems. We have successfully implemented this algorithm for the icosahedral group $\mathcal{I}_3$ and a modified version for the root system F_4. We give more specific details for these cases in later sections.

It is interesting to note if (Mac-Meh I) is true for a certain finite set of values k then it is true in general.

Corollary (4.17). *Let $\mathcal{G}$ be a finite irreducible Coxeter group. If the Macdonald-Mehta Integral Conjecture (Mac-Meh I) is true for $k = 0, 2, 4, \dots, |\Delta| + 2$ then it is true in general.*

Proof. With $k = 2a$ (Mac-Meh I) is equivalent to

$$F(a) = f(a). \tag{4.18}$$

As remarked earlier it is enough to prove

$$\frac{F(a+1)}{F(a)} = \frac{f(a+1)}{f(a)} \tag{4.19}$$

for $a = 0, 1, 2, \ldots$. We may assume $\mathcal{G} \neq \mathcal{H}_2^n$, $\mathcal{G}_2$ since the dihedral and the $\mathcal{G}_2$ cases are already known. From Theorem (4.7) it follows that $\frac{F(a+1)}{F(a)}$ is a polynomial in a of degree $\leq |\Delta^+|$. Since $f(a) = \prod_{j=1}^n \binom{ad_j!}{a!}$ an easy calculation shows that $\frac{f(a+1)}{f(a)}$ is a polynomial in a of degree $= |\Delta^+|$ using the fact that

(4.20)

$$\begin{aligned}\sum_{i=1}^n (d_i - 1) &= \text{the number of reflections in } \mathcal{G} \qquad \text{(by [4; Prop. 7.4.8])} \\ &= |\Delta^+|.\end{aligned}$$

If (Mac-Meh I) is true for $k = 0, 2, 4, \ldots, |\Delta| + 2$ then (4.19) is true for $a = 0, 1, \ldots, |\Delta^+|$ and hence true for all $a \in \mathbb{N}$ since both sides of (4.19) are polynomials in a of degree $\leq |\Delta^+|$. The result follows. $\square$

5. Modifications for the two parameter case. In this section we sketch how our method can be modified to handle (Mac-Meh II) when there is more than one root length. The only non-reduced irreducible root system, BC_n, has three different root lengths. All other irreducible root systems have at most two root lengths. Our method can be adapted to handle the BC_n case for n fixed, but since (Mac-Meh II) is already known for the BC_n case we might as well assume our root system has two root lengths, short and long. Letting

$$k_r = \begin{cases} 2a, & r \text{ short}, \\ 2b, & r \text{ long}, \end{cases} \tag{5.1}$$

we define

$$\begin{aligned} F(a,b) &= F_\Delta(a,b) \\ &:= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \prod_{r \in \Delta^+_{\text{short}}} |\frac{\sqrt{2}}{\|r\|} r(x)|^{2a} \\ &\qquad\qquad \cdot \prod_{r \in \Delta^+_{\text{long}}} |\frac{\sqrt{2}}{\|r\|} r(x)|^{2b} e^{-\|x\|^2/2} dx, \end{aligned} \tag{5.2}$$

where $r(x)$ is defined in (3.3). We define $f(a,b)$ to be the right side of (Mac-Meh II) with k_r as given in (5.1). Our goal is to prove

$$F(a,b) = f(a,b) \qquad (\text{for } Re(a), Re(b) > 0). \tag{5.3}$$

Again via Carlson's Theorem it is enough to show (5.3) for $a \in \mathbb{N}$. The idea is to proceed by induction on a. First the case $a = 0$ corresponds to the sub-root system Δ_{long} (with one parameter b):

$$F_\Delta(a,b) = F_{\Delta_{\text{long}}}(b),$$

which can be handled by the method of §4. If you want to *cheat* this case can be disposed of all together since the only unknown case of (Mac-Meh II) with more then one root length is F_4 and the long roots of F_4 correspond to D_4 for which (Mac-Meh II) is already known.

What remains is to prove

$$\frac{F(a+1,b)}{F(a,b)} = \frac{f(a+1,b)}{f(a,b)}, \tag{5.4}$$

for $a \in \mathbb{N}$, $Re(b) > 0$. Letting $W(x,a,b)$ be the integrand in (5.2) we find the analog of (4.3):

$$F(a+1,b) = \sum_{\alpha \in L} \int_{\mathbb{R}^n} a_\alpha x^\alpha w(x,a,b)dx, \tag{5.5}$$

for some $L \subset \overline{\mathrm{FUN}} \cap \{\alpha \in \mathbb{N}^n : \mathrm{sum}(\alpha) = |\Delta_{\mathrm{short}}|\}$. As before we define

$$L(m) = \begin{cases} \overline{\mathrm{FUN}} \cap \{\alpha \in \mathbb{N}^n : \mathrm{sum}(\alpha) = 2m\}, & \text{if } \mathcal{G} \neq \mathcal{B}_n, \mathcal{I}_3, \mathcal{I}_4, \mathcal{F}_4, \\ \overline{\mathrm{FUN}} \cap \{\alpha \in (2\mathbb{N})^n : \mathrm{sum}(\alpha) = 2m\}, & \text{otherwise}, \end{cases} \tag{5.6a}$$

for $m = 0, 1, \ldots, |\Delta^+_{\mathrm{short}}| - 1$ and

$$L(|\Delta^+_{\mathrm{short}}|) = L. \tag{5.6b}$$

The $un(m,j) = un(m,j,a,b)$ are defined as before and everything proceeds as in §4. The $un(m,j)$ can be computed in terms of the $un(m-1,k)$ using the obvious analogs of STEPS 1-4. Hence the $\int_{\mathbb{R}^n} x^\alpha w(x,a,b)dx$ can be gotten in terms of $un(0,1) = F(a,b)$ and (5.4) should follow from (5.5), provided our computer can handle all the computations. We carry out this procedure for the F_4 case in §7.

6. The icosahedral case $\mathcal{I}_3$. In this section we prove the $\mathcal{I}_3$ case of (Mac-Meh I) by implementing our computer approach described in §4.

Theorem (6.1). *Let $Re(a) > 0$, $\alpha = \frac{1+\sqrt{5}}{4}$, $\beta = \frac{-1+\sqrt{5}}{4}$.*

$$\begin{aligned} F(a) := \frac{2^{15a}}{2\sqrt{2\pi^3}} \int_{\mathbb{R}^3} \prod_{r_2, r_3 = \pm 1} & (|\alpha x_1 + r_2 \beta x_2 + \tfrac{r_3}{2} x_3|^{2a} |\tfrac{1}{2} x_1 + r_2 \alpha x_2 + r_3 \beta x_3|^{2a} \\ & \cdot |\beta x_1 + \tfrac{r_2}{2} x_2 + r_3 \alpha x_3|^{2a}) \\ & \cdot |x_1 x_2 x_3|^{2a} e^{-(x_1^2 + x_2^2 + x_3^2)/2} dx_1 dx_2 dx_3 \end{aligned}$$

is equal to

$$f(a) := \frac{\Gamma(10a+1)\Gamma(6a+1)\Gamma(2a+1)}{\Gamma(a+1)\Gamma(a+1)\Gamma(a+1)}.$$

We must show that $F(a) = f(a)$. As noted in §4 it is enough to prove this for $a \in \mathbb{N}$. We proceed by induction on a. The case $a = 0$ is trivial. We must prove the inductive step:

$$\frac{F(a+1)}{F(a)} = \frac{f(a+1)}{f(a)}. \tag{6.2}$$

It is easily shown that

$$\begin{aligned}(6.3)\qquad \frac{f(a+1)}{f(a)} &= 2^9 3^2 5^2 (10a+9)(10a+7)(10a+3)(10a+1)\\ &\quad \cdot (5a+4)(5a+3)(5a+2)(5a+1)\\ &\quad \cdot (6a+5)(6a+1)(3a+1)(2a+1)^3.\end{aligned}$$

We observe that integrand of $F(a)$ is invariant under the cyclic permutation (123). So,

$$SYM = \{id, (123), (132)\} \tag{6.4}$$

and

$$\int_{\mathbb{R}^3} x^{\pi(\alpha)} w(x) dx = \int_{\mathbb{R}^3} x^{\alpha} w(x) dx \tag{6.5}$$

for $\pi \in SYM$. Here $x = (x_1, x_2, x_3)$ and $w(x)$ is defined in (3.9) or more simply as the integral in $F(a)$. The closure of a fundamental region for SYM is

$$\overline{\text{FUN}} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq x_2 \geq x_3 \text{ or } x_1 \geq x_3 \geq x_2\}. \tag{6.6}$$

Remark. It is possible to use a larger group than SYM namely S_3 as follows. First we observe that when $a \in \mathbb{N}, \alpha \in \mathbb{N}^3$ then $\int_{\mathbb{R}^3} x^\alpha w(x,a) dx \in \mathbb{Q}[\sqrt{5}]$. Let φ be the $\mathbb{Q}$-automorphism of $\mathbb{Q}[\sqrt{5}]$ $\sqrt{5} \mapsto -\sqrt{5}$. φ extends naturally to a ring automorphism of $\mathbb{Q}[\sqrt{5}][x_1, x_2, x_3]$. Since $w_0(x) := w(x) e^{(x_1^2+x_2^2+x_3^2)/2} \in \mathbb{Q}[\sqrt{5}][x_1, x_2, x_3]$, and a calculation shows that φ (12) $w_0 = w_0$ it follows from (6.5) that for $\pi \in S_3$, $\alpha \in \mathbb{N}^3$ we have

$$\int_{\mathbb{R}^3} x^{\pi(\alpha)} w(x) dx = \begin{cases} \varphi(\int_{\mathbb{R}^3} x^\alpha w(x) dx), & \text{if } \pi \text{ is odd},\\ \int_{\mathbb{R}^3} x^\alpha w(x) dx, & \text{if } \pi \text{ is even}.\end{cases} \tag{6.7}$$

However since the action of φ seems difficult to program in MAPLE we stick with SYM.

From Table I (Appendix A) the root system of $\mathcal{I}_3$ is

$$\begin{aligned} I_3 = \Delta(\mathcal{I}_3) = \{e_i \quad (1 \leq i \leq 3);\ (\alpha, r_2\beta, \tfrac{1}{2} r_3) \quad r_2, r_3 = \pm 1 \\ \text{and all even permutations of coordinates}\}.\end{aligned}$$

Since the $e_i \in I_3$ we find that Lemma (3.19) applies. For $m = 0, 1, \ldots, 14 = |\Delta^+| - 1$ $L(m)$, defined in (4.4a), consists of compositions of $2m$ into at most 3 parts in which each part is even, and the parts are nonincreasing or if they are not nonincreasing the first part is the largest part and the second part is smaller than the first part and is the smallest part. We have written a MAPLE program to do the calculation in (4.3). We find that L, the set of monomial vectors α that occur in this calculation, has cardinality 31. We let the $un(m,j)$ be defined as in (4.6). On running our MAPLE program we have

$$F(a+1) = \frac{3}{512} un(15,1) + \frac{(-39 + 3\sqrt{5})}{256} un(15,2) + \cdots + \frac{26925}{256} un(15,31). \tag{6.8}$$

The elements of L together with the missing coefficients in (6.8) are given in Table II (Appendix A). STEPS 1–4 in the proof of Theorem (4.7) provide an algorithm for getting each $un(m,i)$ in terms of the $un(m-1,j)$'s and hence, by iteration, in terms of $un(0,1) = F(a)$. We have written a MAPLE program to carry this out. The program generates a triangular system of 260 equations in 261 unknowns. To help the reader write his/her own program we do STEPS 1–2.

Suppose $\delta = 2(n_1, n_2, n_3) \in L(m)$. In STEP 1 we use

$$\int_{\mathbb{R}^3} \frac{\partial}{\partial x_1} x_1^{2n_1-1} x_2^{2n_2} x_3^{2n_3} W(x,a)dx = 0. \tag{6.9}$$

Here we have used $\frac{\partial}{\partial x_1}$ since the first component of δ is always the largest. After simplification we have

(6.10)

$$\begin{aligned}
&\int_{\mathbb{R}^3} x^\delta w(x,a)dx \\
&\qquad = (2a+2n_1-1)\int_{\mathbb{R}^3} x_1^{2n_1-2} x_2^{2n_2} x_3^{2n_3} w(x,a)dx \\
&\qquad + 8a \int_{\mathbb{R}^3} \left\{ \frac{\alpha x_1}{x_1 - \beta x_2 + \frac{1}{2}x_3} + \frac{\frac{1}{2}x_1}{\frac{1}{2}x_1 + \alpha x_2 - \beta x_3} \right. \\
&\qquad\qquad \left. - \frac{\beta x_1}{-\beta x_1 + \frac{1}{2}x_2 + \alpha x_3} \right\} x_1^{2n_1-2} x_2^{2n_2} x_3^{2n_3} w(x,a)dx.
\end{aligned}$$

In STEP 2 we use the reflection $S_{\alpha,-\beta,-\frac{1}{2}}$ with action:

$$e_1 \mapsto (-\beta, \tfrac{1}{2}, \alpha), \quad e_2 \mapsto (\tfrac{1}{2}, \alpha, -\beta) \text{ and } e_3 \mapsto (\alpha, -\beta, \tfrac{1}{2}). \tag{6.11}$$

By applying Lemma (3.13) with $S = S_{\alpha,-\beta,-\frac{1}{2}}$ to the last integral in (6.10) we have

(6.12)

$$\begin{aligned}
&\int_{\mathbb{R}^3} x^\delta w(x,a)dx \\
&\quad = (2a+2n_1-1)\int_{\mathbb{R}^3} x_1^{2n_1-2} x_2^{2n_2} x_3^{2n_3} w(x,a)dx \\
&\quad + 8a\int_{\mathbb{R}^3} (-\beta x_1 + \frac{1}{2}x_2 + \alpha x_3)(-\frac{\beta}{x_1} + \frac{1}{2x_2} + \frac{\alpha}{x_3})(-\beta x_1 + \frac{1}{2}x_2 + \alpha x_3)^{2n_1-2} \\
&\qquad\quad \cdot (\frac{1}{2}x_1 + \alpha x_2 - \beta x_3)^{2n_2} (\alpha x_1 - \beta x_2 + \frac{1}{2}x_3)^{2n_3} w(x,a)dx.
\end{aligned}$$

Example. We apply the algorithm (by hand) to obtain $un(1,1)$ in terms of $un(0,1)$. $L(0) = \{(0,0,0)\}$ and $L(1) = \{(2,0,0)\}$ so taking $n_1 = 1$, $n_2 = n_3 = 0$ in (6.12) we find

(6.13)

$$\begin{aligned}
&\int_{\mathbb{R}^3} x_1^2 w(x,a)dx \\
&\qquad = (2a+1)\int_{\mathbb{R}^3} w(x,a)dx \\
&\qquad\quad + 8a\int_{\mathbb{R}^3} (-\beta x_1 + \frac{1}{2}x_2 + \alpha x_3)(-\frac{\beta}{x_1} + \frac{1}{2x_2} + \frac{\alpha}{x_3})\, w(x,a)dx \\
&\qquad = (10a+1)\int_{\mathbb{R}^3} w(x,a)dx
\end{aligned}$$

by Lemma (3.19) since $\alpha^2+\beta^2+\frac{1}{4}=1$. We obtain

$$un(1,1)=(10a+1)un(0,1)=(10a+1)F(a). \tag{6.14}$$

Our program was run on several machines to produce all the necessary equations in one big file. Later this file was read in MAPLE the equations being solved automatically by back substitution. After the equations were solved the values for the $un(15,i)$ were substituted into the right side of (6.8). The result was factored and we obtained

$$\frac{F(a+1)}{F(a)}=\frac{f(a+1)}{f(a)}, \tag{6.15}$$

as required. This completes the proof of Theorem (6.1).

7. The F_4 case. In this section we sketch a proof of the F_4 case of (Mac-Meh II). The approach is analogous to the $\mathcal{I}_3$ case of (Mac-Meh I) but with appropriate modifications. We give enough detail so that the reader may write his/her own computer programs following the recipe laid out in §§4–5. We have been successful in verifyng all the relevant results using the computer algebra package MAPLE. Copies of the computer programs are available from the author on request.

Theorem(7.1). *Let $Re(a), Re(b)>0$.*

$$\begin{aligned} F(a,b):=\frac{1}{2^{4b+2}\pi^2}\int_{\mathbb{R}^4}\prod_{1\le i<j\le 4}|x_i^2-x_j^2|^{2a}\prod_{i=1}^{4}|x_i|^{2b}\\ \cdot\prod_{r_2,r_3,r_4=\pm1}|x_1+r_2x_2+r_2x_3+r_4x_4|^{2b}\\ \cdot e^{-(x_1^2+x_2^2+x_3^2+x_4^2)/2}dx_1dx_2dx_3dx_4\end{aligned}$$

is equal to

$$\begin{aligned} f(a,b):=\frac{\Gamma(2a+1)\Gamma(3a+1)\Gamma(4a+2b+1)\Gamma(4a+4b+1)}{\Gamma(a+1)\Gamma(a+1)\Gamma(2a+b+1)\Gamma(a+b+1)}\\ \cdot\frac{\Gamma(6a+6b+1)\Gamma(2a+4b+1)\Gamma(3b+1)\Gamma(2b+1)}{\Gamma(3a+3b+1)\Gamma(a+2b+1)\Gamma(b+1)\Gamma(b+1)}.\end{aligned}$$

We must show that $F(a,b)=f(a,b)$. As noted in §5 it is enough to prove this for $a\in\mathbb{N}$. We proceed by induction a. The case $a=0$ is already known since

$$\text{Long roots of } F_4\cong D_4. \tag{7.2}$$

Hence we must prove the inductive step:

$$\frac{F(a+1,b)}{F(a,b)}=\frac{f(a+1,b)}{f(a,b)}. \tag{7.3}$$

An easy calculation gives

(7.4)
$$\begin{aligned}\frac{f(a+1,b)}{f(a,b)} &= 2^{10}3^2(3a+2)(3a+1)(2a+1)\\ &\cdot(2a+2b+1)^2(2a+4b+1)(4a+2b+3)(4a+2b+1)\\ &\cdot(4a+4b+3)(4a+4b+1)(6a+6b+5)(6a+6b+1).\end{aligned}$$

The integrand of $F(a,b)$ is invariant under any permutation of the coordinates, so we take SYM to be the symmetric group S_4. Hence the closure of a fundamental region for SYM is

$$\overline{\mathrm{FUN}} = \{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 : x_1\geq x_2\geq x_3\geq x_4\}. \tag{7.5}$$

Since $e_i \in F_4$ $(1\leq i\leq 4)$ we have the following two parameter analog of Lemma(3.19):

$$\int_{\mathbb{R}^4} x^\alpha w(x;a,b)dx = 0 \qquad (\text{for } \alpha\in\mathbb{N}^4 \text{ with } \alpha\not\equiv(0,0,0,0) \pmod 2). \tag{7.6}$$

where $w(x;a,b)$ is the integrand of $F(a,b)$.

The root system, $F_4 = \Delta(\mathcal{F}_4)$, is given in Table I (Appendix A). For $m = 0,1,\ldots,11 = |\Delta^+|-1$ $L(m)$, defined in (5.6a), consists of partitions of $2m$ into at most 4 parts in which each part is even. We have written a MAPLE program to do the calculation in (5.5). We find that L, the set of monomial vectors α that occur in this calculation, has cardinality 16. The elements of L are listed in Table III (Appendix A). We let the $un(m,j)$ be defined as before. On running our MAPLE program we have

(7.7)
$$\begin{aligned}F(a+1,b) = 24(&un(12,1)-un(12,2)-un(12,3)+2un(12,4)-un(12,5)-un(12,6)\\ &+un(12,7)+2un(12,8)-2un(12,9)-2un(12,10)+2un(12,11)\\ &-un(12,12)+2un(12,13)+un(12,14)-3un(12,15)+un(12,16)).\end{aligned}$$

To help the reader write a program to produce the equations for the $un(m,i)$ we carry out STEP 1 of our algorithm. By letting $\delta = 2(n_1,n_2,n_3,n_4)\in L(m)$ and using $\frac{\partial}{\partial x_1}$ we find

(7.8)
$$\begin{aligned}&\int_{\mathbb{R}^4} x^\delta w(x;a,b)dx\\ &= (2b+2n_1-1)\int_{\mathbb{R}^4} x_1^{2n-2}x_2^{2n_2}x_3^{2n_3}x_4^{2n_4}w(x;a,b)dx\\ &\quad + 4a\sum_{j=2}^{4}\int_{\mathbb{R}^4}\frac{x_1}{x_1-x_j}x_1^{2n_1-2}x_2^{2n_2}x_3^{2n_3}x_4^{2n_4}w(x;a,b)dx\\ &\quad + 16b\int_{\mathbb{R}^4}\frac{x_1}{x_1+x_2+x_3+x_4}x_1^{2n_1-2}x_2^{2n_2}x_3^{2n_3}x_4^{2n_4}w(x;a,b)dx\end{aligned}$$

We have written a MAPLE program that implements our algorithm to generate and solve equations for the $un(m, i)$. This system consists of 275 equations in 276 unknowns (one unknown for each element of the $L(m)$).The solutions are polynomials in the two variables a, b. Again using MAPLE we have *plugged* the values of the $un(12, i)$ into (7.7) which then simplifies to the right side of (7.4) after dividing both sides by $F(a, b)$. This verifies (7.3) thus completing our *computer* proof of Theorem (7.1).

Acknowledgment. I would like to thank Doron Zeilberger for suggesting the problem of evaluating the Macdonald-Mehta integrals.

Appendix A: TABLES.

Group	$\lvert\Delta\rvert$	Root system Δ
$\mathcal{A}_n$	n^2+n	$\pm(e_i-e_j), 1\le j<i\le n+1.$
$\mathcal{B}_n$	$2n^2$	$\pm e_i, 1\le i\le n$; $\pm e_i\pm e_j, 1\le j<i\le n.$
$\mathcal{D}_n$	$2n(n-1)$	$\pm e_i\pm e_j, 1\le j<i\le n.$
$\mathcal{H}_2^n$	$2n$	$(\cos j\pi/n, \sin j\pi/n), 0\le j\le 2n-1.$
$\mathcal{G}_2$	12	$\pm(e_i-e_j), 1\le j<i\le 3;$ $\pm(1,-2,1), \pm(-2,1,1), \pm(1,1,-2).$
$\mathcal{I}_3$	30	$\pm e_i, 1\le i\le 3; \beta(\pm(2\alpha+1), \pm 1, \pm 2\alpha)$, and all even permutations of coordinates $(\alpha=\frac{1+\sqrt{5}}{4}, \beta=\frac{-1+\sqrt{5}}{4})$.
$\mathcal{I}_4$	120	$\pm e_i$, $1\le i\le 4; (1/2)(\pm1,\pm1,\pm1,\pm1);$ $\beta(\pm 2\alpha, 0, \pm(2\alpha+1), \pm 1)$, and all even permutations of coordinates.
$\mathcal{F}_4$	48	$\pm e_i$, $1\le i\le 4$; $\pm e_i\pm e_j$, $1\le j<i\le 4$; $(1/2)(\pm1,\pm1,\pm1,\pm1)$.
$\mathcal{E}_8$	240	$\pm e_i\pm e_j$, $1\le j<i\le 8$; $(1/2)\sum_1^8 \varepsilon_i e_i$, $\varepsilon_i=\pm1$, $\prod_1^8 \varepsilon_i=-1$.
$\mathcal{E}_7$	126	Those roots of $\mathcal{E}_8$ orthogonal to $(1/2)(1,1,1,1,1,1,1,-1)$.
$\mathcal{E}_6$	72	Those roots of $\mathcal{E}_7$ orthogonal to e_8-e_7.

Table I. Root systems of Coexeter groups. Table I was taken from [4; p. 76].

i	i-th element of L	coefficient of $un(15,i)$
1	26 2 2	$3/512$
2	24 4 2	$(-39+3\sqrt{5})/256$
3	24 2 4	$(-39-3\sqrt{5})/256$
4	22 6 2	$(861-111\sqrt{5})/512$
5	22 2 6	$(861+111\sqrt{5})/512$
6	22 4 4	$321/128$
7	20 8 2	$(-1269+192\sqrt{5})/128$
8	20 2 8	$(-1269-192\sqrt{5})/128$
9	20 6 4	$(-4341+393\sqrt{5})/256$
10	20 4 6	$(-4341-393\sqrt{5})/256$
11	18 10 2	$(33843-4671\sqrt{5})/1024$
12	18 2 10	$(33843+4671\sqrt{5})/1024$
13	18 8 4	$(15981-3615\sqrt{5})/256$
14	18 4 8	$(15981+3615\sqrt{5})/256$
15	18 6 6	$19047/256$
16	16 12 2	$(-8199+669\sqrt{5})/128$
17	16 2 12	$(-8199-669\sqrt{5})/128$
18	16 10 4	$(-8229+3648\sqrt{5})/64$
19	16 4 10	$(-8229-3648\sqrt{5})/64$
20	16 8 6	$(-12015+747\sqrt{5})/64$
21	16 6 8	$(-12015-747\sqrt{5})/64$
22	14 14 2	$10101/128$
23	14 12 4	$(9939-12639\sqrt{5})/128$
24	14 4 12	$(9939+12639\sqrt{5})/128$
25	14 10 6	$(130821-18471\sqrt{5})/256$
26	14 6 10	$(130821+18471\sqrt{5})/256$
27	14 8 8	$27/32$
28	12 12 6	$-43569/64$
29	12 10 8	$(-8829+46641\sqrt{5})/128$
30	12 8 10	$(-8829-46641\sqrt{5})/128$
31	10 10 10	$26925/256$

Table II. Coefficients of the $un(15,i)$ in (6.8).

i	i-th element of L				i	i-th element of L			
1	12	8	4	0	9	10	8	4	2
2	12	8	2	2	10	10	6	6	2
3	12	6	6	0	11	10	6	4	4
4	12	6	4	2	12	8	8	8	0
5	12	4	4	4	13	8	8	6	2
6	10	10	4	0	14	8	8	4	4
7	10	10	2	2	15	8	6	6	4
8	10	8	6	0	16	6	6	6	6

Table III. The elements of L for F_4.

APPENDIX B. An Application of Carlson's Theorem. We prove that (Mac-Meh I) holds for all complex k, $Re(k) > 0$, if it holds for all even integers $k = 2a$. Our proof is analogous to Mehta's [6; pp. 40-41] proof of this result for the $\mathcal{A}_n$ case. We need the following easy Corollary of Carlson's Theorem [9; p. 186].

Lemma (B.1). *Let $\delta > 0$ be fixed. If a function $f(\beta)$ is holomorphic and bounded on the half-plane $Re(\beta) > \delta$ and zero for $\beta = 1, 2, 3, \ldots$, then it is identically zero.*

Let $\mathcal{G}$ be a finite Coxeter group and $P(x)$ $(x \in \mathbb{R}^n)$ be defined as in §1. We define

$$W(x) := \frac{1}{2}\|x\|^2 - \ell n|P(x)|, \tag{B.2}$$

$$N := |\Delta^+| = \text{ the number of hyperplanes.} \tag{B.3}$$

We assume (Mac-Meh I) holds for all even integers $k = 2a$. To be consistent with Mehta's notation we replace k by β in both sides of (Mac-Meh I). In order to apply Carlson's Theorem we need to determine the behavior of both sides of (Mac-Meh I) as functions of β as $|\beta| \to \infty$. We define the analog of Mehta's function $\Psi(\beta)$ for our general Coxeter group $\mathcal{G}$.

$$\Psi(\beta) := \int_{\mathbb{R}^n} \exp(-\beta W(x))dx, \quad (Re(\beta) > 0). \tag{B.6}$$

$\Psi(\beta)$ is related to the left side of (Mac-Meh I) by

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/2}|P(x)|^\beta dx = \beta^{n/2+\beta N/2}\Psi(\beta). \tag{B.7}$$

It will turn out that $\Psi(\beta) = 0(|Y^\beta|)$ (as $|\beta| \to \infty$) for some constant Y. In order to show this we calculate the minimum of the function $W(x)$.

For the $\mathcal{A}_n$ case Mehta [6; Appendix A.4] relates the calculation of this minimum to the zeros of Hermite polynomials. We take a different approach by following

Macdonald's [5; p. 1002] argument:

(B.8)

$$\begin{aligned}\min_{x\in\mathbb{R}^n} W(x) &= -\frac{1}{2}\ell n(\max_{x\in\mathbb{R}^n} e^{-\|x\|^2}P(x)^2)\\ &= -\frac{1}{2}\ell n(\lim_{a\to\infty}(\int_{\mathbb{R}^n} e^{-a\|x\|^2}P(x)^{2a}dx)^{\frac{1}{a}})\\ &= -\frac{1}{2}\ell n(\lim_{a\to\infty}(2a)^{-N-n/a}(\int_{\mathbb{R}^n} e^{-\|y\|^2/2}P(y)^{2a}dy)^{\frac{1}{a}})\\ &= -\frac{1}{2}\ell n(\lim_{a\to\infty}(2a)^{-N}\prod_{j=1}^{n}(\frac{(ad_j)!}{a!})^{\frac{1}{a}}) \qquad \text{(since we have assumed (Mac-Meh I) is true for even integers)}\\ &= \frac{N}{2}(1+\ell n2) - \frac{1}{2}\sum_{j=1}^{n} d_j \ell n d_j,\end{aligned}$$

by Stirling's formula. Here we have also used (4.20). Hence,

$$W(x) \geq W_0 := \frac{N}{2}(1+\ell n2) - \frac{1}{2}\sum_{j=2}^{n} d_j \ell n d_j, \quad (x\in\mathbb{R}^n), \tag{B.9}$$

so that

$$0 \leq \exp(-W(x)) \leq Y := \exp(-W_0), \quad (x\in\mathbb{R}^n). \tag{B.10}$$

Now fix any $\delta > 0$. For $Re(\beta) > \delta$ we have

$$\begin{aligned}|\Psi(\beta)| &\leq \int_{\mathbb{R}^n} \exp(-(Re(\beta)-\delta)W(x))\exp(-\delta W(x))dx\\ &\leq |Y^{\beta-\delta}|\Psi(\delta) \quad (\text{by } (B.10))\\ &\leq C_\delta |Y^\beta|,\end{aligned} \tag{B.11}$$

where

$$C_\delta = Y^{-\delta}\Psi(\delta) < \infty. \tag{B.12}$$

Hence for $Re(\beta) > \delta$, $Y^{-\beta}\Psi(\beta)$ is a bounded holomorphic function.

Next we consider the right side of (Mac-Meh I). We define

$$\psi(\beta) := (2\pi)^{n/2}\beta^{-n/2-\beta N/2}[\Gamma(1+\tfrac{1}{2}\beta)]^{-n}\prod_{j=1}^{n}\Gamma(1+\tfrac{1}{2}\beta d_j). \tag{B.13}$$

Then (Mac-Meh I) is equivalent to $\Psi(\beta) = \psi(\beta)$, $(Re(\beta) > 0)$. We show that $\psi(\beta)$ has the same behavior as $\Psi(\beta)$ ($|\beta| \to \infty$) by using Stirling's formula. As $|\beta| \to \infty$ we have

(B.14)

$$\begin{aligned}\psi(\beta) &= (2\pi)^{n/2}\beta^{-n/2-\beta N/2}[\Gamma(1+\tfrac{1}{2}\beta)]^{-n}\prod_{j=1}^{n}\Gamma(1+\tfrac{1}{2}\beta d_j)\\ &\backsim (2\pi)^{n/2}\beta^{-n/2-\beta N/2}[(\tfrac{1}{2}\beta)^{\beta/2+1/2}e^{-\beta/2}\sqrt{2\pi}]^{-n}\\ &\quad \cdot \prod_{j=1}^{n}[(\tfrac{1}{2}\beta d_j)^{\beta d_j/2+1/2}e^{-\beta d_j/2}\sqrt{2\pi}]\\ &\backsim Y^\beta (2\pi)^{n/2}\beta^{-n/2}\sqrt{|\mathcal{G}|} \quad \text{(c.f. [6; (4.11)])}.\end{aligned}$$

Here we have used (4.20) and

$$\prod_{j=1}^{n} d_j = |\mathcal{G}|, \qquad \text{([4; Prop. 7.4.7]).} \tag{B.15}$$

It follows that on the half-plane $Re(\beta) > \delta$ the function $Y^{-\beta}\psi(\beta)$ is bounded and holomorphic. Now consider

$$\Delta(\beta) := \frac{1}{Y^{2\beta}}[\Psi(2\beta) - \psi(2\beta)]. \tag{B.16}$$

We know $\Delta(\beta)$ is bounded and holomorphic on the half-plane $Re(\beta) > \delta$ and is zero for $\beta = 1, 2, 3, \ldots$. The result follows by applying Lemma (B.1) since $\delta > 0$ was arbitrary.

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