MORE CRANKS AND t-CORES

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Abstract. In 1990, new statistics on partitions (called *cranks*) were found which combinatorially prove Ramanujan's congruences for the partition function modulo 5, 7, 11 and 25. The methods are extended to find cranks for Ramanujan's partition congruence modulo 49. A more explicit form of the crank is given for the modulo 25 congruence.

Dedicated to George Szekeres on the occasion of his 90th Birthday

1. Introduction

Let $p(n)$ be the number of partitions of n [1]. If $\alpha \geq 1$, and δ_α, λ_α, μ_α are the reciprocals of 24 modulo 5^α, 7^α, 11^α respectively, then

$$p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{1.1}$$

$$p(7^{2\alpha-1} n + \lambda_{2\alpha-1}) \equiv 0 \pmod{7^\alpha}, \tag{1.2}$$

$$p(7^{2\alpha} n + \lambda_{2\alpha}) \equiv 0 \pmod{7^{\alpha+1}}, \tag{1.3}$$

$$p(11^\alpha n + \mu_\alpha) \equiv 0 \pmod{11^\alpha}. \tag{1.4}$$

These are Ramanujan's partition congruences. Watson [9] proved (1.1), (1.2), (1.3) and Atkin [3] proved (1.4). Dyson [5] was the first to consider explaining these congruences combinatorially. Dyson defined an integral statistic on partitions, called the rank, whose value mod 5 he conjectured split the partitions of $5n + 4$ into 5 equal classes, thus giving a combinatorial refinement for the $\alpha = 1$ case of (1.1). He further conjectured that the analogous result for the rank mod 7 gave the $\alpha = 1$ case of (1.2), and that there was a statistic, called the crank, which would similarly give the $\alpha = 1$ case of (1.4). Atkin and Swinnerton-Dyer [4] proved Dyson's conjecture for 5 and 7. Andrews and Garvan [2] were able to find a crank which not only solved Dyson's crank conjecture for 11 but gave new interpretations for 5 and 7. Later, Garvan, Kim and Stanton [6] found new cranks which gave new interpretations of Ramanujan's congruences mod 5, 7, 11, and 25. Their approach was combinatorial and in terms of the t-core of a partition. They gave explicit bijections between the equinumerous classes. In the present paper we extend the methods of [6] and give a crank which is a combinatorial refinement of the $\alpha = 1$ case of (1.3), namely

$$p(49n + 47) \equiv 0 \pmod{49}. \tag{1.5}$$

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In §2 we reexamine two bijections from [6]. A crank for the partitions of $25n+24$ mod 25 was given in [6]. A more explicit form of this crank is given in Theorem 3.4. A new and explicit crank for the 7-cores of $49n+47$ is given in Theorem 3.5. This leads to a crank for the partitions of $49n+47$ (Corollary 3.1).

2. Two bijections for t-cores

We need to examine in detail the two bijections relating partitions and t-cores which were given in [6]. Following [6] we let P be the set of all partitions. For any $\lambda \in P$, let $|\lambda|$ denote the number that λ partitions. Fix a positive integer t. Let $P_{t\text{-core}}$ be the set of partitions which are t-cores. Recall, that a partition is a t-core if it has no hook numbers that are multiples of t or equivalently no rim hooks that are multiples of t. See [7] for background on t-cores, hook numbers and rim hooks. We let $a_t(n)$ denote the number of partitions of n which are t-cores.

Bijection 1. ([7, 2.7.17], [6, p.2]) There is a bijection $\phi_1 : P \to P_{t\text{-core}} \times P \times \cdots \times P$,

$$\phi_1(\lambda) = (\widetilde{\lambda}, \widehat{\lambda}_0, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_{t-1}),$$

such that

$$|\lambda| = |\widetilde{\lambda}| + t \sum_{i=0}^{t-1} |\widehat{\lambda}_i|.$$

Corollary 2.1.

$$\sum_{n \geq 0} a_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}.$$

Given a partition λ we label a cell in the i-th row and j-th column by $j-i \pmod{t}$. The resulting diagram is called a t-residue diagram [7, p.84]. We form the extended t-residue diagram by adding an infinite column 0 labelled in the same way. A region r of the extended diagram is the set of cells (i,j) with $t(r-1) \leq j-i < tr$. A cell is *exposed* if it is at the end of a row. The partition λ is a t-core if and only if for each exposed cell labeled i in region r there is an exposed cell labeled i in each region $< r$. Now we construct t bi-infinite words W_0, W_1, $\ldots$, W_{t-1} of words of two letters N (not exposed) and E (exposed):

$$\text{The } j\text{-th element of } W_i = \begin{cases} N & \text{if } i \text{ is not exposed in region } j, \\ E & \text{if } i \text{ is exposed in region } j. \end{cases}$$

We now give the bijection. For each i we do the following steps:

Step 1. Find the right most E.
Step 2. Find the right most N to the left of this E. If no such N exists then END.
Step 3. Remove the rim hook whose head is at E and whose tail is one cell to the right of the N. Place a part of size (rim hook removed)$/t$ in λ_i.
Step 4. Go to Step 1.

The operation in Step 3 above changes a substring of W_i of the form $NEE\ldots EEN$ to $EEE\ldots ENN$, i.e. the N is pushed right.The other words W_j are left unchanged by removing this rim hook, and we can process the i's in any order. Steps 1–4 create a partition λ_i starting from the smallest part to the largest part and the process is easily reversible. At the end when all the W_i have been processed we are left with the required t-core $\widetilde{\lambda}$.

Bijection 2. ([6, p.3]) There is a bijection $\phi_2 : P_{t\text{-core}} \to \{\vec{n} = (n_0, n_1, \ldots, n_{t-1}) : n_i \in \mathbb{Z}, n_0 + \cdots + n_{t-1}\}$, where

$$|\widetilde{\lambda}| = t||\vec{n}||^2/2 + \vec{b} \cdot \vec{n}, \quad \vec{b} = (0, 1, \ldots, t-1).$$

For a partition λ, we let $r_k(\lambda)$ denote the number of cells in the t-residue diagram labeled $k \pmod{t}$, and call

$$\vec{r} = (r_0, r_1, \ldots, r_{t-1})$$

the r-vector of λ. Bijection 2 is given by

$$\phi_2(\widetilde{\lambda}) = \vec{n} = (r_0 - r_1, r_1 - r_2, \ldots, r_{t-1} - r_0). \tag{2.1}$$

Let $[x]$ denote the greatest integer $\leq x$. We will need the following

Lemma 2.1. *Let $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ be a partition and suppose*

$$\phi_1(\lambda) = (\widetilde{\lambda}, \widehat{\lambda}_0, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_{t-1}).$$

Then

$$\sum_{i=0}^{t-1} |\widehat{\lambda}_i| = r_0 - \left(\sum_{i=0}^{t-1} r_i^2 - r_i r_{i+1} \right), \tag{2.2}$$

and

(2.3)

$$\sum_{i=0}^{t-1} i|\widehat{\lambda}_i| \equiv \sum_{j=1}^{m} (\lambda_j - j) \left[\frac{\lambda_j - j}{t} \right] - \sum_{i=1}^{t-1} i d_i \left(\tfrac{1}{2}(d_i + 1) + \left[\frac{m - i - 1}{t} \right] \right) \pmod{t},$$

where d_i is the number of elements of the sequence

$$\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_m - m,$$

which are congruent to $i \pmod{t}$.

Proof. For t-cores, we have

$$r_0 = \sum_{i=0}^{t-1} (r_i^2 - r_i r_{i+1}).$$

See [6, p.6]. Now suppose $\vec{r}$ is the r-vector of λ and $\vec{r'}$ is the r-vector of its t-core $\widetilde{\lambda}$. The partition $\widetilde{\lambda}$ is obtained from λ by the removal of rim hooks whose lengths are multiples of t. Each rim hook of length t contains cells with distinct t-residues. It follows that

$$r_i' + s = r_i$$

where

$$s = \sum_{j=0}^{t-1} |\widehat{\lambda}_j|.$$

Since $\vec{r'}$ is the r-vector of a t-core we have

$$r'_0 = \sum_{i=0}^{t-1} ({r'}_i^2 - {r'}_i {r'}_{i+1}),$$

$$r_0 - s = \sum_{i=0}^{t-1} ((r_i - s)^2 - (r_i - s)(r_{i+1} - s)),$$

$$= \sum_{i=0}^{t-1} (r_i^2 - r_i r_{i+1}),$$

and (2.2) follows.

We add t dummy zeros to the parts of λ:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0 \geq \cdots \geq 0,$$

and form the sequence

$$\bar{\lambda}: \ \lambda_1 - 1 > \lambda_2 - 2 > \cdots > \lambda_m - m > -m - 1 > \cdots > -m - t.$$

Let

$$\bar{\mu_i}: \ \mu_{i,1} > \mu_{i,2} > \cdots > \mu_{i,k_i}$$

be the terms of the sequence $\bar{\lambda}$ that are congruent to $i \pmod t$. Here k_i is the terms so that $d_i = k_i - 1$. Each $\mu_{i,k}$ corresponds to an exposed cell labeled i in region $[\mu_{i,k}/t] + 1$. In Bijection 1, the numbers

$$n_{i,k} = \left[\frac{\mu_{i,k}}{t}\right] - \left[\frac{\mu_{i,k+1}}{t}\right] - 1$$

correspond to a string of $n_{i,k}$ consecutive N's in the word W_i. Since these N's are shifted as far as possible to the right we find that the sum of parts of the $(i+1)$th component

$$\begin{aligned}
|\widehat{\lambda}_i| &= n_{i,1} + 2n_{i,2} + \cdots (k_i - 1) n_{i,k-1} \\
&= \left[\frac{\mu_{i,1}}{t}\right] + \cdots + \left[\frac{\mu_{i,k_i-1}}{t}\right] \\
&\quad - \tfrac{1}{2} k_i (k_i - 1) - (k_i - 1) \left[\frac{\mu_{i,k_i}}{t}\right].
\end{aligned}$$

Since $\mu_{i,k} \equiv i \pmod t$ we find that

$$\begin{aligned}
&\sum_{i=0}^{t-1} i \left(\left[\frac{\mu_{i,1}}{t}\right] + \cdots + \left[\frac{\mu_{i,k_i-1}}{t}\right] \right) \\
&\qquad \equiv \sum_{j=1}^{m} (\lambda_j - j) \left[\frac{\lambda_j - j}{t}\right] \pmod t.
\end{aligned}$$

The desired result (2.3) follows from the fact that $d_i = k_i - 1$ and that

$$\left[\frac{\mu_{i,k_i}}{t}\right] = \left[\frac{m - i - 1}{t}\right].$$

□

3. Cranks for t-cores and partitions

We need the crank results in [6]. The following theorem follows from [6, Theorem 1].

Theorem 3.1. *([6]) If $(t, \delta) = (5, 4)$, $(7, 5)$ or $(11, 6)$, then*

$$\sum_{n \geq 0} a_t(tn + \delta) q^{n+1} = \sum_{\vec{\alpha} \in \mathbb{Z}^t, \vec{\alpha} \cdot \vec{1} = 1} q^{Q(\vec{\alpha})},$$

where

$$Q(\vec{\alpha}) = ||\vec{\alpha}||^2 - \sum_{i=0}^{t-1} \alpha_i \alpha_{i+1}.$$

The form $Q(\vec{\alpha})$ remains invariant under a cyclic permutation of the α_i. This induces a t-cycle on t-cores of $tn+\delta$, which in turn induces a t-cycle on partitions of $tn+\delta$ via Bijection 1. For the form $Q(\vec{\alpha})$ the associated crank statistic is $\sum_{i=0}^{t-1} i\alpha_i$. This leads to crank statistics for t-cores of $tn+\delta$, and for partitions of $tn+\delta$.

3.1. **Cranks for partitions of $5n+4$ and $25n+24$.**

Theorem 3.2. *([6, p.7]) Let $\vec{r} = (r_0, r_1, \ldots, r_4)$ be the r-vector of λ, a 5-core of $5n+4$. Then*

$$c_1(\lambda) := 2r_1 - r_2 + r_3 - 2r_4 \pmod 5 \in \mathbb{Z}_5 \tag{3.1}$$

is a crank for 5-cores of $5n+4$.

We make explicit the 5-cycle σ that acts of 5-cores of $5n+4$. We let $P_{t\text{-core}}(m)$ denote the set of t-cores of m. For $0 \le j \le 4$ we let $P^j_{t\text{-core}}(m)$ denote the set of t-cores $\widetilde{\lambda}$ of m, with crank $c_1(\widetilde{\lambda}) \equiv j \pmod 5$. For a t-core $\widetilde{\lambda}$ we call $\vec{n} = \phi_2(\widetilde{\lambda})$ its n-vector. We define the 5-cycle σ in terms of n-vectors. The map

$$\sigma \,:\, P_{5\text{-core}}(5n+4) \longrightarrow P_{5\text{-core}}(5n+4)$$

is defined by

$$\vec{n} \mapsto \Big(-\frac{2\,n_0}{5} + \frac{n_1}{5} + \frac{4\,n_2}{5} + \frac{2\,n_3}{5} + \frac{3}{5}, -n_3, -\frac{3\,n_0}{5} - \frac{6\,n_1}{5} - \frac{4\,n_2}{5} - \frac{2\,n_3}{5} + \frac{2}{5},$$
$$-\frac{n_0}{5} + \frac{3\,n_1}{5} - \frac{3\,n_2}{5} + \frac{n_3}{5} - \frac{1}{5}, \frac{6\,n_0}{5} + \frac{2\,n_1}{5} + \frac{3\,n_2}{5} + \frac{4\,n_3}{5} - \frac{4}{5} \Big)$$

For each $0 \le j \le 4$, the map

$$\sigma \,:\, P^j_{5\text{-core}}(5n+4) \longrightarrow P^{j+1}_{5\text{-core}}(5n+4)$$

is a bijection.

The key to finding a crank for partitions of $25n+24$ in [6] was a bijective proof of the identity

$$a_5(5n+4) = 5a_5(n). \tag{3.2}$$

The map

$$\theta \,:\, P_{5\text{-core}}(n) \longrightarrow P^0_{5\text{-core}}(5n+4)$$

defined by

$$\vec{n} \mapsto (n_1 + 2n_2 + 2n_4 + 1, -n_1 - n_2 + n_3 + n_4 + 1, 2n_1 + n_2 + 2n_3,$$
$$-2n_2 - 2n_3 - n_4 - 1, -2n_1 - n_3 - 2n_4 - 1)$$

is a bijection. See [6, p.8]. This together with Theorem 3.2 yields a combinatorial proof of (3.2).

We now describe the crank for 5-cores of $25n+24$ found in [6]. For $\lambda \in P_{5\text{-core}}(25n+24)$ choose the unique $\lambda' \in P^0_{5\text{-core}}(25n+24)$ which is in the same orbit as λ under the 5-cycle σ. Define

$$c_2(\lambda) := c_1(\theta^{-1}(\lambda')). \tag{3.3}$$

Let $\vec{n} = \theta^{-1}(\lambda')$. By (2.1)

$$c_2(\lambda') = c_1(\vec{n}) = 2n_1 + n_2 + 2n_3.$$

Observe that this is the third component in the n-vector of $\theta(\vec{n}) = \lambda'$. It follows that

$$c_2(\lambda') = r_2 - r_3, \tag{3.4}$$

where $\vec{r}$ is the r-vector of λ'. Unfortunately, it is not true in general that $c_2(\lambda') \equiv c_2(\lambda) \pmod 5$. Nonetheless we can find a crank for 5-cores of $25n+24$ independent of the two maps σ and θ. We have the following

Theorem 3.3. *Let $\vec{r} = (r_0, r_1, \ldots, r_4)$ be the r-vector of λ, a 5-core of $25n+24$. Then*

$$\begin{aligned} c(\lambda) &:= (c_1(\lambda), c_2(\lambda)) \\ &= (2r_1 - r_2 + r_3 - 2r_4, r_2 - r_3) \pmod 5 \in \mathbb{Z}_5 \times \mathbb{Z}_5 \end{aligned} \tag{3.5}$$

is a crank for 5-cores of $25n+24$.

Proof. For each (i,j) in $\mathbb{Z}_5 \times \mathbb{Z}_5$, we let $P^{i,j}_{\text{5-core}}(25n+24)$ be the set of 5-cores λ of $25n+24$ such that $c(\lambda) \equiv (i,j) \pmod 5$. The map

$$\Psi = \theta\sigma\theta^{-1} \,:\, P^{0,j}_{\text{5-core}}(25n+24) \longrightarrow P^{0,j+1}_{\text{5-core}}(25n+24)$$

is a bijection. We have calculated the effect σ has on our crank statistics c_1, c_2. A calculation shows that the map

$$\sigma \,:\, P^{i,j}_{\text{5-core}}(25n+24) \longrightarrow P^{i+1,i^2+i+j+2}_{\text{5-core}}(25n+24)$$

is a bijection. We omit the details. We note that the indices are reduced mod 5. Using the maps Ψ and σ we find that

$$\left|P^{i,j}_{\text{5-core}}(25n+24)\right| = \left|P^{0,0}_{\text{5-core}}(25n+24)\right| = \frac{1}{25}\, a_5(25n+24),$$

for $0 \le i,j \le 4$. Hence $c = (c_1, c_2) \pmod 5$ is a crank for 5-cores of $25n+24$ $\pmod{25}$. ∎

A crank for partitions of $25n+24$ is given in [6, Theorem 6]. This crank is algorithmic in nature. It depends on Bijection 1, and the map θ. In view of Lemma 2.1 and Theorem 3.3, we may define a crank independent of these maps. For a partition $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m$, with r-vector $\vec{r} = (r_0, \ldots, r_{t-1})$, the definition of $c_1(\lambda)$ and $c_2(\lambda)$ is analogous to that given for t-cores in (3.1), (3.4) respectively. We need two more statistics. We define

$$s(\lambda) := r_0 - \left(\sum_{i=0}^{t-1} r_i^2 - r_i r_{i+1}\right), \tag{3.6}$$

and

$$c_3(\lambda) := \sum_{j=1}^{m} (\lambda_j - j)\left[\frac{\lambda_j - j}{t}\right] - \sum_{i=1}^{t-1} i d_i \left(\tfrac{1}{2}(d_i+1) + \left[\frac{m-i-1}{t}\right]\right), \tag{3.7}$$

where $d_i(\lambda)$ is the number of elements of the sequence

$$\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_m - m,$$

which are congruent to $i \pmod t$. Now let λ be any partition of $25n + 24$, and suppose

$$\phi_1(\lambda) = (\widetilde{\lambda}, \widehat{\lambda}_0, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_4).$$

Then by Lemma 2.1,

$$\sum_{i=0}^{4} |\widehat{\lambda}_i| = s(\lambda),$$

and

$$\sum_{i=0}^{4} i|\widehat{\lambda}_i| \equiv c_3(\lambda) \pmod 5.$$

If $s(\lambda) \equiv 0 \pmod 5$, then $\widetilde{\lambda}$ is a 5-core with $|\widetilde{\lambda}| \equiv 24 \pmod{25}$ and

$$c(\lambda) := (c_1(\lambda), c_2(\lambda)) \equiv (c_1(\widetilde{\lambda}), c_2(\widetilde{\lambda})) \pmod 5,$$

since the sum of the coefficients in the definitions of c_1, c_2 is zero. By rewriting [6, Theorem 6] in terms of our new statistics we obtain a bijection independent crank.

Theorem 3.4. *Let $\vec{r} = (r_0, r_1, \ldots, r_4)$ be the r-vector of a partition λ of $25n + 24$. We define a crank $c(\lambda) \in \mathbb{Z}_5 \times \mathbb{Z}_5$ as follows.*

If $s(\lambda) \equiv 0 \pmod 5$ we define

$$\begin{aligned} c(\lambda) &:= (c_1(\lambda), c_2(\lambda)) \\ &= (2r_1 - r_2 + r_3 - 2r_4, r_2 - r_3). \end{aligned} \tag{3.8}$$

If $s(\lambda) \not\equiv 0 \pmod 5$ we define

$$c(\lambda) := (c_1(\lambda), c_3(\lambda)). \tag{3.9}$$

Then $c(\lambda)$ is a crank for the partitions of $25n + 24$ mod 25.

The proof utilises Theorem 3.3 and follows from [6, Theorem 6].

3.2. **Cranks for partitions of $7n + 5$ and $49n + 47$.** For 7-cores of $7n + 5$ there is no analog of (3.2) and so there is no analog of the map θ. Nonetheless we are able to find a crank $c(\lambda) \in \mathbb{Z}_7 \times \mathbb{Z}_7$ for the partitions of $49n + 47$.

Theorem 3.5. *([6, p.7]) Let $\vec{r} = (r_0, r_1, \ldots, r_6)$ be the r-vector of λ, a 7-core of $7n + 5$. Then*

$$c_1(\lambda) := 5r_1 - r_2 - r_3 + r_4 + r_5 - 5r_6 \pmod 7 \in \mathbb{Z}_7 \tag{3.10}$$

is a crank for 7-cores of $7n + 5$.

We make explicit the 7-cycle σ that acts of 7-cores of $7n + 5$. We define the 7-cycle σ in terms of n-vectors. Since $\sum_{i=0}^{6} n_i = 0$, we omit the last component n_6, and let $\vec{n} = (n_0, n_1, \ldots, n_5)^T$.The map

$$\sigma : P_{7\text{-core}}(7n + 5) \longrightarrow P_{7\text{-core}}(7n + 5)$$

is defined by

$$\sigma(\vec{n}) = M\,\vec{n} + \vec{\tau},$$

where

$$M = \frac{1}{7}\begin{pmatrix} -8 & -2 & -3 & -4 & -5 & -6 \\ 1 & 2 & 3 & 4 & -2 & 6 \\ 3 & -1 & 2 & 5 & 1 & -3 \\ -2 & -4 & 1 & -1 & 4 & 2 \\ 0 & 0 & -7 & 0 & 0 & 0 \\ 2 & -3 & -1 & -6 & -4 & -2 \end{pmatrix}, \quad \vec{\tau} = \frac{1}{7}\begin{pmatrix} 5 \\ 2 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

We have the following

Theorem 3.6. *Let $\vec{r} = (r_0, r_1, \ldots, r_6)$ be the r-vector of λ, a 7-core of $49n + 47$. Then*

$$\begin{aligned} c(\lambda) &:= (c_1(\lambda), c_2(\lambda)) \\ &= (5r_1 - r_2 - r_3 + r_4 + r_5 - 5r_6, r_3 + 4r_4 - 4r_5 - r_6) \pmod 7 \in \mathbb{Z}_7 \times \mathbb{Z}_7 \end{aligned} \tag{3.11}$$

is a crank for 7-cores of $49n + 47$.

Proof. For each (i, j) in $\mathbb{Z}_7 \times \mathbb{Z}_7$, we let $P^{i,j}_{7\text{-core}}(49n + 47)$ be the set of 7-cores λ of $49n + 47$ such that $c(\lambda) \equiv (i, j) \pmod 7$. We construct 7 bijections

$$\Psi_j \,:\, P^{0,j}_{7\text{-core}}(49n+47) \longrightarrow P^{0,j+1}_{7\text{-core}}(49n+47), \qquad 0 \le j \le 6.$$

Each map Ψ_j has the form

$$\Psi_j(\vec{n}) = M_j\,\vec{n} + \vec{\tau}_j,$$

where M_j is a 6×6 matrix, and $\vec{\tau}_j$ is a constant vector, and which are given below.

$$M_0 = \frac{1}{49}\begin{pmatrix} -24 & -36 & -2 & -5 & 13 & -4 \\ 40 & 17 & -20 & -1 & 4 & 2 \\ -15 & -37 & -17 & -18 & 23 & -13 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ 15 & 2 & 38 & -3 & -2 & -22 \\ -40 & -38 & -36 & -41 & -60 & -23 \end{pmatrix}, \quad \vec{\tau}_0 = \frac{1}{49}\begin{pmatrix} 22 \\ -4 \\ 26 \\ 0 \\ -26 \\ 4 \end{pmatrix}$$

$$M_1 = \frac{1}{49}\begin{pmatrix} 32 & 1 & -30 & -5 & 6 & 10 \\ -9 & -46 & 8 & -15 & 4 & 2 \\ -36 & 12 & -17 & -11 & 16 & 8 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ 36 & 44 & 52 & 25 & 19 & 48 \\ 9 & 11 & 13 & -6 & 17 & -37 \end{pmatrix}, \vec{\tau}_1 = \frac{1}{49}\begin{pmatrix} 15 \\ 31 \\ 26 \\ 0 \\ -26 \\ -31 \end{pmatrix}$$

$$M_2 = \frac{1}{49}\begin{pmatrix} 36 & 58 & 24 & 18 & 12 & 27 \\ 24 & 6 & 16 & 12 & 57 & 18 \\ -23 & -18 & -48 & -36 & -24 & -54 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ 23 & -31 & -1 & -13 & -25 & 5 \\ -24 & -6 & 33 & -12 & -8 & -18 \end{pmatrix}, \vec{\tau}_2 = \frac{1}{49}\begin{pmatrix} -5 \\ 13 \\ 10 \\ 0 \\ -10 \\ -13 \end{pmatrix}$$

$$M_3 = \frac{1}{49}\begin{pmatrix} 4 & 8 & -44 & -19 & -8 & -25 \\ -9 & -4 & 22 & -15 & -10 & -40 \\ 48 & 54 & 46 & 31 & 37 & 50 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ -48 & 2 & -11 & -17 & -2 & 6 \\ 9 & -31 & -1 & -6 & 31 & 5 \end{pmatrix}, \vec{\tau}_3 = \frac{1}{49}\begin{pmatrix} 15 \\ 3 \\ -16 \\ 0 \\ 16 \\ -3 \end{pmatrix}$$

$$M_4 = \frac{1}{49}\begin{pmatrix} -12 & 39 & 6 & -6 & -18 & -2 \\ -36 & -30 & 18 & -18 & -5 & -6 \\ 31 & -15 & 9 & -9 & -27 & -3 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ -31 & -34 & -58 & -40 & -22 & -46 \\ 36 & 30 & 31 & 18 & 54 & 6 \end{pmatrix}, \vec{\tau}_4 = \frac{1}{49}\begin{pmatrix} 11 \\ 33 \\ -8 \\ 0 \\ 8 \\ -33 \end{pmatrix}$$

$$M_5 = \frac{1}{49}\begin{pmatrix} -31 & -34 & -58 & -40 & -22 & -46 \\ 12 & 10 & -6 & 6 & 18 & 51 \\ -36 & -30 & 18 & -18 & -5 & -6 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ 36 & 30 & 31 & 18 & 54 & 6 \\ -12 & 39 & 6 & -6 & -18 & -2 \end{pmatrix}, \vec{\tau}_5 = \frac{1}{49}\begin{pmatrix} 36 \\ 24 \\ 26 \\ 0 \\ -26 \\ -24 \end{pmatrix}$$

$$M_6 = \frac{1}{49}\begin{pmatrix} 36 & 44 & 52 & 25 & 19 & 48 \\ -32 & -22 & -26 & -37 & -62 & -31 \\ -9 & -46 & 8 & -15 & 4 & 2 \\ 0 & 0 & 0 & 49 & 0 & 0 \\ 9 & 11 & 13 & -6 & 17 & -37 \\ 32 & 1 & -30 & -5 & 6 & 10 \end{pmatrix}, \vec{\tau}_6 = \frac{1}{49}\begin{pmatrix} 2 \\ 20 \\ 24 \\ 0 \\ -24 \\ -20 \end{pmatrix}$$

Let

$$\begin{aligned} w(\vec{n}) &:= w(n_0, n_1, \ldots, n_5) \\ &= \tfrac{7}{2}(n_0^2 + \cdots + n_5^2 + (n_0 + \cdots + n_5)^2) \\ &\quad + n_1 + 2n_2 + \cdots + 5n_5 - 6(n_0 + \cdots + n_5). \end{aligned} \tag{3.12}$$

In terms of the n-vector c_1, c_2 are given by

$$\begin{aligned} c_1(\vec{n}) &= 5n_1 + 4n_2 + 3n_3 + 4n_4 + 5n_5, \\ c_2(\vec{n}) &= n_3 + 5n_4 + n_5. \end{aligned}$$

In order to show the Ψ_j are bijections, we have used computer algebra to show for each j, (i) Ψ_j preserves the form w, (ii) $\det(M_j) = \pm 1$, and (iii) $\vec{n} \in \mathbb{Z}^6$, $(c_1(\vec{n}), c_2(\vec{n})) = (0, j)$, $w(\vec{n}) \equiv 47 \pmod{49}$ implies $\Psi(\vec{n}) \in \mathbb{Z}^6$ and $(c_1(\vec{n}), c_2(\vec{n})) = (0, j+1)$. We have calculated the effect of the 7-cycle σ has on our crank statistics c_1, c_2. A calculation shows that the map

$$\sigma \,:\, P^{i,j}_{\text{7-core}}(49n + 47) \longrightarrow P^{i+1,4i+j}_{\text{7-core}}(49n + 47)$$

is a bijection. We omit the details. We note that the indices are reduced mod 7. Using the seven maps Ψ_j and the 7-cycle σ we find that

$$\left|P^{i,j}_{\text{7-core}}(49n + 47)\right| = \left|P^{0,0}_{\text{7-core}}(49n + 47)\right| = \frac{1}{49}\, a_7(49n + 47),$$

for $0 \le i, j \le 6$. Hence $c = (c_1, c_2) \pmod 7$ is a crank for 7-cores of $49n + 47 \pmod{49}$.

□

Corollary 3.1. *Let $\vec{r} = (r_0, r_1, \ldots, r_6)$ be the r-vector of a partition λ of $49n + 47$. We define a crank $c(\lambda) \in \mathbb{Z}_7 \times \mathbb{Z}_7$ as follows.*

If $s(\lambda) \equiv 0 \pmod 7$ we define

$$\begin{aligned} c(\lambda) &:= (c_1(\lambda), c_2(\lambda)) \\ &= (5r_1 - r_2 - r_3 + r_4 + r_5 - 5r_6, r_3 + 4r_4 - 4r_5 - r_6). \end{aligned} \tag{3.13}$$

If $s(\lambda) \not\equiv 0 \pmod 7$ we define

$$c(\lambda) := (c_1(\lambda), c_3(\lambda)), \tag{3.14}$$

where c_3 is defined in (3.7).

Then $c(\lambda)$ is a crank for the partitions of $49n + 47$ mod 49.

The proof is analogous to that of Theorem 3.4.

4. Remarks

Our cranks for the partitions of $25n+24$ and $49n+47$ depend crucially on finding the two crank functions c_1 and c_2. The first crank function c_1 arises naturally from the t-cycle one gets from Theorem 3.1. For 5-cores the second crank function c_2 arises from the map θ. We describe another way the second crank function arises. Let $w(\vec{n})$ be defined as in (3.12). Then since $w(\vec{n}) \equiv 5 \pmod 7$ and assuming $c_1(\vec{n}) \equiv 0 \pmod 7$, there are integers k, ℓ such that

$$\begin{aligned} n_0 &= 7k + 5 - 2n_1 - 3n_2 - 4n_3 - 5n_4 - 6n_5, \\ n_1 &= 7\ell - n_5 - 5n_2 - 2n_3 - 5n_4. \end{aligned}$$

Now assume the second crank function takes the form

$$c_2(\vec{n}) = ab_2 + n_3 + bn_4 + cn_5,$$

for some integers a, b, c. If we assume $c_2(\vec{n}) \equiv 0 \pmod 7$, then there is an integer m such that

$$n_3 = 7m - an_2 - bn_4 - cn_5.$$

We want $w(\vec{n})$ to be a linear form mod 49 in the remaining variables n_2, n_4, n_5. A calculation shows that this can only happen if

$$(a, b, c) \equiv (0, 5, 1) \pmod 7,$$

which nails down the second crank function c_2. We have considered the analogous problem for 11-cores of $121n + 116$, and found there is no second crank function of a similar form which makes the corresponding $w(\vec{n})$ linear mod 121. So if there is a crank for 11-cores of $121n + 116$ it must be more complicated.

It would be interesting to find other occurrences of pairs of crank functions (c_1, c_2) which give combinatorial congruences. Zoltan Reti [8] found a pair of crank functions which explains that the congruence

$$s(9n + 8) \equiv 0 \pmod 9,$$

where $s(n)$ is the number of partitions of n in which an even part may have two colors. It was Reti's result which led us to search for a function c_2 for 7-cores of $49n + 47$.

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