K. SAITO'S CONJECTURE FOR NONNEGATIVE ETA PRODUCTS AND ANALOGOUS RESULTS FOR OTHER INFINITE PRODUCTS

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Abstract. We prove that the Fourier coefficients of a certain general eta product considered by K. Saito are nonnegative. The proof is elementary and depends on a multidimensional theta function identity. The $z = 1$ case is an identity for the generating function for p-cores due to Klyachko [17] and Garvan, Kim and Stanton [10]. A number of other infinite products are shown to have nonnegative coefficients. In the process a new generalization of the quintuple product identity is derived.

1. Introduction

Throughout this paper $q = \exp(2\pi i\tau)$ with $\Im\tau > 0$ so that $|q| < 1$. As usual the Dedekind eta function is defined as

$$\eta(\tau) := \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi in\tau)) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{1.1}$$

An eta product is a finite product of the form

$$\prod_k \eta(k\tau)^{e(k)}, \tag{1.2}$$

where the $e(k)$ are integers. K. Saito [21] considered eta products that are connected with elliptic root systems and considered the problem of determining when all the Fourier coefficients of such eta products are nonnegative. Subsequent work contains the following

Conjecture 1.1. (K. Saito [23]) *Let N be a positive integer. The eta product*

$$S_N(\tau) := \frac{\eta(N\tau)^{\phi(N)}}{\prod_{d|N}\eta(d\tau)^{\mu(d)}} \tag{1.3}$$

has nonnegative Fourier coefficients.

The conjecture has been proved for $N = 2, 3, 4, 5, 6, 7, 10$ by K. Saito [21], [22], [23], [24], [25], for prime powers $N = p^\alpha$ by T. Ibukiyama [15], and for $\gcd(N, 6) > 1$ by K. Saito and S. Yasuda [26], who also showed that for general N, the coefficient of q^n in $S_N(\tau)$ is nonnegative for sufficiently large n. We prove the conjecture for general N.

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The case $N = p$ (prime) occurs in the study of p-cores. A partition is a p-core if it has no hooks of length p [10], [16]. p-cores are important in the study of p-modular representations of the symmetric group S_n. Define

$$E(q) := \prod_{n=1}^{\infty}(1-q^n), \tag{1.4}$$

and let $a_t(n)$ denote the number of partitions of n that are t-cores. It is well known that for any positive integer t

$$\sum_{n\geq 0} a_t(n)q^n = \frac{E(q^t)^t}{E(q)}. \tag{1.5}$$

This result is originally due to Littlewood [18]. See [10] for a combinatorial proof. Thus (1.5) implies that Conjecture 1.1 holds for $N = p$ prime since

$$S_p(\tau) = \frac{\eta(p\tau)^p}{\eta(\tau)} = q^{(p^2-1)/24}\frac{E(q^p)^p}{E(q)}. \tag{1.6}$$

Granville and Ono [13] have proved that $a_t(n) > 0$ for all $t \geq 4$ and all n. We also need the following identity due to Klyachko [17]

$$\sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\cdot\vec{1}_t=0}} q^{\frac{t}{2}\vec{n}\cdot\vec{n}+\vec{b}_t\cdot\vec{n}} = \frac{E(q^t)^t}{E(q)}, \tag{1.7}$$

where $\vec{1}_t = (1,1,\ldots,1) \in \mathbb{Z}^t$, $\vec{b}_t = (0,1,2,\ldots,t-1)$, and t is any positive integer. See [10] for a combinatorial proof. See also [11, Prop.1.29] and [6, §2]. Our proof of K. Saito's Conjecture depends on the following z-analogue of (1.7).

Theorem 1.2. *Let $a \geq 2$ be an integer. Then for $z \neq 0$ and $|q| < 1$ we have*

$$\begin{aligned} C_a(z;q) &:= \sum_{\substack{\vec{n}=(n_0,n_1,\ldots,n_{a-1})\in\mathbb{Z}^a\\ \vec{n}\cdot\vec{1}_a=0}} q^{\frac{a}{2}\vec{n}\cdot\vec{n}+\vec{b}_a\cdot\vec{n}}\sum_{j=0}^{a-1} z^{an_j+j} \\ &= E(q)E(q^a)^{a-2}\prod_{n=1}^{\infty}\frac{(1-z^aq^{a(n-1)})(1-z^{-a}q^{an})}{(1-zq^{n-1})(1-z^{-1}q^n)} \end{aligned} \tag{1.8}$$

We note that (1.7) follows from (1.8) by letting $a = t$ and $z \to 1$. The case $a = 3$ is equivalent to [14, (1.23)]. See [5, §3.3]. The case $a = 2$ can be written as

$$\sum_{n\in\mathbb{Z}} q^{2n^2+n}\left(z^{2n}+z^{2n+1}\right) = \prod_{n=1}^{\infty}(1+zq^{(n-1)})(1+z^{-1}q^n)(1-q^n), \tag{1.9}$$

which follows easily from Jacobi's triple product identity [1, (2.2.10)]. Klyachko [17] proved (1.7) by showing that it was a special case of the Macdonald identity [19] for the affine root system A_{t-1}. Hjalmar Rosengren [20] has observed that Theorem 1.2 (i.e. (1.8)) is also a special case of the Macdonald identity for A_{a-1}.

If we rewrite the right side of (1.7) in terms of eta products, and apply Jacobi's transformation $\tau \mapsto -1/\tau$ then we are led to the identity

$$\sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\cdot\vec{1}_t=0}} \omega_t^{\vec{b}_t\cdot\vec{n}} q^{\frac{1}{2}\vec{n}\cdot\vec{n}} = \frac{E(q)^t}{E(q^t)}, \tag{1.10}$$

where $\omega_t := \exp(2\pi i/t)$. The proof uses well known transformation formulas for the eta function [3, Thm.3.1] and multidimensional theta functions [27, (5),p.205]. See [11, Prop.2.29] for an elementary proof. Equation (1.10) has the following z-analogue.

Theorem 1.3. *Let a and j be integers where $a \ge 2$ and $0 \le j \le a-1$. Then for $z \ne 0$ and $|q| < 1$ we have*

$$B_{j,a}(z;q) := \sum_{\substack{\vec{n}=(n_0,n_1,\ldots,n_{a-1})\in\mathbb{Z}^a \\ \vec{n}\cdot\vec{1}_a=0}} z^{n_j}\omega_a^{\vec{b}_a\cdot\vec{n}}q^{\frac{1}{2}\vec{n}\cdot\vec{n}} \tag{1.11}$$

$$= E(q)^{a-2}E(q^a)\prod_{n=1}^{\infty}\frac{(1-zq^{n-1})(1-z^{-1}q^n)}{(1-zq^{a(n-1)})(1-z^{-1}q^{an})}$$

Clearly, (1.10) follows from (1.11) by letting $a = t$ and $z \to 1$. Again the $a = 2$ case follows easily from Jacobi's triple product identity.

Notation. We use the following notation for finite products

$$(z;q)_n = (z)_n = \begin{cases} \prod_{j=0}^{n-1}(1-zq^j), & n>0 \\ 1, & n=0. \end{cases}$$

For infinite products we use

$$(z;q)_\infty = (z)_\infty = \lim_{n\to\infty}(z;q)_n = \prod_{n=1}^{\infty}(1-zq^{(n-1)}),$$

$$(z_1,z_2,\ldots,z_k;q)_\infty = (z_1;q)_\infty(z_2;q)_\infty\cdots(z_k;q)_\infty,$$

$$[z;q]_\infty = (z;q)_\infty(z^{-1}q;q)_\infty = \prod_{n=1}^{\infty}(1-zq^{(n-1)})(1-z^{-1}q^n),$$

$$[z_1,z_2,\ldots,z_k;q]_\infty = [z_1;q]_\infty[z_2;q]_\infty\cdots[z_k;q]_\infty,$$

for $|q| < 1$ and z, z_1, z_2,..., $z_k \ne 0$.

2. Proof of Theorems 1.2 and 1.3

Suppose $a \ge 2$. The idea is to show both sides of (1.8) satisfy the same functional equation as $z \to zq$ and agree for enough values of z. Define

$$R_a(z;q) = E(q)E(q^a)^{a-2}\frac{[z^a;q^a]_\infty}{[z;q]_\infty}, \tag{2.1}$$

which is the right side of (1.8). An easy calculation gives

$$R_a(zq;q) = z^{-(a-1)}R_a(z;q). \tag{2.2}$$

We show that basically the a terms in the definition of $C_a(z;q)$ are permuted cyclically as $z \to zq$. To this end we define

$$Q_a(\vec{n}) = \frac{a}{2}\vec{n}\cdot\vec{n} + \vec{b}_a\cdot\vec{n}, \tag{2.3}$$

$$F_j(z;q) := \sum_{\substack{\vec{n}=(n_0,n_1,\ldots,n_{a-1})\in\mathbb{Z}^a \\ \vec{n}\cdot\vec{1}_a=0}} z^{an_j+j}q^{Q_a(\vec{n})} \qquad (0 \le j \le a-1). \tag{2.4}$$

Now suppose $1 \leq j \leq a-1$. Let $\vec{e}_0 = (1, 0, \ldots, 0)$, $\vec{e}_1 = (0, 1, \ldots, 0)$, $\ldots$, $\vec{e}_{a-1} = (0, 0, \ldots, 0, 1)$ be the standard unit vectors, $\vec{n} = (n_0, n_1, \ldots, n_{a-1}) \in \mathbb{Z}^a$, and $\vec{n}\,' = (n_1, n_2, \ldots, n_{a-1}, n_0) + \vec{e}_{j-1} - \vec{e}_{a-1}$. An easy calculation gives

$$Q_a(\vec{n}\,') - Q_a(\vec{n}) = an_j + j - \vec{n} \cdot \vec{1}_a. \tag{2.5}$$

Hence

$$\begin{aligned} F_{j-1}(z;q) &= \sum_{\substack{\vec{n} \in \mathbb{Z}^a \\ \vec{n} \cdot \vec{1}_a = 0}} z^{an_{j-1} + (j-1)} q^{Q_a(\vec{n})} \\ &= \sum_{\substack{\vec{n}\,' \in \mathbb{Z}^a \\ \vec{n}\,' \cdot \vec{1}_a = 0}} z^{a(n_j+1) + (j-1)} q^{Q_a(\vec{n}\,')} \\ &= \sum_{\substack{\vec{n} \in \mathbb{Z}^a \\ \vec{n} \cdot \vec{1}_a = 0}} z^{an_j + j + (a-1)} q^{Q_a(\vec{n}) + an_j + j} \\ &= z^{(a-1)} F_j(zq;q), \end{aligned} \tag{2.6}$$

and

$$F_j(zq;q) = z^{-(a-1)} F_{j-1}(z;q). \tag{2.7}$$

Similarly we find that

$$F_0(zq;q) = z^{-(a-1)} F_{a-1}(z;q), \tag{2.8}$$

by using the result that

$$Q_a(\vec{n}\,') - Q_a(\vec{n}) = an_0 - \vec{n} \cdot \vec{1}_a, \tag{2.9}$$

where $\vec{n}\,' = (n_1, n_2, \ldots, n_{a-1}, n_0)$.

Since

$$C_a(z;q) = \sum_{j=0}^{a-1} F_j(z;q), \tag{2.10}$$

we have

$$C_a(zq;q) = z^{-(a-1)} C_a(z;q). \tag{2.11}$$

In view of [4, Lemma 2] or [14, Lemma 1] it suffices to show that (1.8) holds for a distinct values of z with $|q| < |z| \leq 1$. It is clear that

$$C_a(z;q) = R_a(z;q) = 0, \tag{2.12}$$

for $z = \exp(2\pi i k/a)$ for $1 \leq k \leq a-1$. Finally, (1.8) holds for $z = 1$ since

$$C_a(1;q) = a \sum_{\substack{\vec{n} \in \mathbb{Z}^a \\ \vec{n} \cdot \vec{1}_a = 0}} q^{\frac{a}{2} \vec{n} \cdot \vec{n} + \vec{b}_a \cdot \vec{n}} = a \frac{E(q^a)^a}{E(q)} = R_a(1;q), \tag{2.13}$$

by (1.7) with $t = a$. This completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is similar. First by considering a cyclic permutation one can show that the definition of $B_{j,a}(z;q)$ is independent of j. Next one show that both sides of (1.11) satisfy the same functional equation

$$\Phi_a(zq^a;q) = q^{-\binom{a}{2}} (-z)^{-(a-1)} \Phi_a(z;q). \tag{2.14}$$

For the right side this is easy. For left side one uses the transformation $\vec{n} \mapsto \vec{n}\,' = \vec{n} + \vec{1}_a - a\vec{e}_j$. Since both sides satisfy the same functional equation (2.14) and are analytic for $z \neq 0$, one needs only to verify that the identity holds for a distinct values of z on the region $|q|^a < |z| \leq 1$. For $1 \leq k \leq a-1$, one uses the transformation $\vec{n} \mapsto \vec{n}\,' = \vec{n} + k(\vec{e}_{j+1} - \vec{e}_j)$ to find that

$$B_{j,a}(q^k;q) = \omega_a^k B_{j+1,a}(q^k;q) = \omega_a^k B_{j,a}(q^k;q), \tag{2.15}$$

so that

$$B_{j,a}(q^k;q) = 0. \tag{2.16}$$

Hence, both sides of (1.11) are zero for $z = q^k$ $(1 \leq k \leq a)$ and agree at $z = 1$ by (1.10) with $t = a$. The theorem follows.

3. Proof of K. Saito's Conjecture

First we show that

$$\prod_{d|M} E(q^d)^{\mu(d)} = \prod_{\substack{n \geq 1 \\ (n,M)=1}} (1 - q^n), \tag{3.1}$$

for any positive integer M. Now

$$\prod_{d|M} E(q^d)^{\mu(d)} = \prod_{d|M} \prod_{m=1}^{\infty} (1 - q^{dm})^{\mu(d)} = \prod_{n=1}^{\infty} (1 - q^n)^{\varepsilon(n)}, \tag{3.2}$$

where

$$\varepsilon(n) = \sum_{d|M \,\&\, d|n} \mu(d) = \sum_{d|(M,n)} \mu(d) = \begin{cases} 1 & \text{if } (M,n) = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

by a well known property of the Möbius function, and we have (3.1).

For any positive integer N we define

$$\widetilde{S}_N(q) := \frac{E(q^N)^{\phi(N)}}{\prod_{d|N} E(q^d)^{\mu(d)}}. \tag{3.4}$$

We wish to show that all coefficients in the q-expansion of $\widetilde{S}_N(q)$ are nonnegative. We consider three cases.

Case 1. $N = p^\alpha$ where p is prime. This case was proved by Ibukiyama [15]. Alternatively, the case $\alpha = 1$ follows from (1.5) and then use an easy induction on α.

Case 2. $N = p\,M$, where p is prime, M is odd and $p \nmid M$. We have

$$\prod_{d|N} E(q^d)^{\mu(d)} = \prod_{d|M} E(q^d)^{\mu(d)} E(q^{pd})^{\mu(pd)} = \prod_{d|M} \left(\frac{E(q^d)}{E(q^{pd})} \right)^{\mu(d)}. \tag{3.5}$$

By (3.1) we have

$$\prod_{d|M} E(q^d)^{\mu(d)} = \prod_{\substack{n \geq 1 \\ (n,M)=1}} (1 - q^n) = \prod_{n \geq 0} \prod_{\substack{(r,M)=1 \\ 1 \leq r \leq M-1}} (1 - q^{Mn+r}) = \prod_{\substack{(r,M)=1 \\ 1 \leq r \leq \frac{M-1}{2}}} [q^r; q^M]_\infty. \tag{3.6}$$

Now for a a positive integer, $|q| < 1$ and $z \neq 0$ we let

$$D_a(z;q) := \frac{E(q^a)^a}{E(q)} C_a(z;q) = E(q^a)^{2a-2} \frac{[z^a;q^a]_\infty}{[z;q]_\infty}, \tag{3.7}$$

so that

$$\begin{aligned}
\prod_{\substack{(r,M)=1 \\ 1 \le r \le \frac{M-1}{2}}} D_p(q^r;q^M) &= \left(E(q^{pM})^{2p-2}\right)^{\phi(M)/2} \prod_{\substack{(r,M)=1 \\ 1 \le r \le \frac{M-1}{2}}} \frac{[q^{pr};q^{pM}]_\infty}{[q^r;q^M]_\infty} \\
&= E(q^N)^{\phi(N)} \prod_{d|M} \frac{E(q^{pd})^{\mu(d)}}{E(q^d)^{\mu(d)}} \qquad \text{(by (3.6))} \\
&= \frac{E(q^N)^{\phi(N)}}{\prod_{d|N} E(q^d)^{\mu(d)}} \qquad \text{(by (3.5))} \\
&= \widetilde{S}_N(q).
\end{aligned} \tag{3.8}$$

K. Saito's Conjecture holds in this case since each $C_p(q^r;q^M)$ has nonnegative coefficients by Theorem 1.2, and $E(q^{pM})^p/E(q^M)$ has nonnegative coefficients by (1.5) so that each $D_p(q^r;q^M)$ has nonnegative coefficients.

Case 3. $N = p^\alpha M$, where p is prime, M is odd, $p \nmid M$, and $\alpha \ge 2$. We let $N' = pM$. It is clear that

$$\prod_{d|N} E(q^d)^{\mu(d)} = \prod_{d|N'} E(q^d)^{\mu(d)}. \tag{3.9}$$

Hence

$$\widetilde{S}_N(q) = \frac{E(q^N)^{\phi(N)}}{E(q^{N'})^{\phi(N')}} \widetilde{S}_{N'}(q) = \left(\frac{E(q^{p^{\alpha-1}N'})^{p^{\alpha-1}}}{E(q^{N'})} \right)^{(p-1)\phi(M)} \widetilde{S}_{N'}(q). \tag{3.10}$$

Here $\widetilde{S}_N(q)$ is the product of two terms. The second term $\widetilde{S}_{N'}(q)$ has nonnegative coefficients from Case 2. The first term has nonnegative coefficients using (1.5) with q replaced with $q^{N'}$ and $t = p^{\alpha-1}$. Thus K. Saito's Conjecture holds in this case.

4. Other Products with Nonnegative Coefficients

In this section we prove a number of results for coefficients of other infinite products. For a formal power series

$$F(q) := \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Z}[[q]]$$

we write

$$F(q) \succeq 0,$$

if $a_n \ge 0$ for all $n \ge 0$. For a formal power series $F(z_1, z_2, \ldots, z_n; q)$ in more than one variable we interpret $F(z_1, z_2, \ldots, z_n; q) \succeq 0$ in the natural way. The following result follows from the q-binomial theorem [1, Thm2.1].

Proposition 4.1. *If* $|q|$, $|t| < 1$ *then*

$$\frac{(at;q)_\infty}{(a;q)_\infty (t;q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(aq^n;q)_\infty (q)_n} \succeq 0. \tag{4.1}$$

Corollary 4.2. *If a, b, M are positive integers then*

$$\prod_{n=0}^{\infty} \frac{(1-q^{Mn+a+b})}{(1-q^{Mn+a})(1-q^{Mn+b})} \succeq 0. \tag{4.2}$$

Proposition 4.1 has a finite analogue. For $0 \le m \le n$ the Gaussian polynomial [1, p.33] is defined by

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = \frac{(q)_{m+n}}{(q)_n (q)_m} = \frac{(1-q^{n+1})\cdots(1-q^{n+m})}{(q)_m} \tag{4.3}$$

Since it is the generating function for partitions with at most m parts each $\le n$ it is a polynomial (in q) with positive integer coefficients. We have

Proposition 4.3. *If $L \ge 0$ then*

$$\frac{(z_1 z_2; q)_L}{(z_1; q)_L (z_2; q)_L} = \sum_{j=0}^{L} \begin{bmatrix} L \\ j \end{bmatrix}_q \frac{z_1^j}{(z_1 q^{L-j}; q)_j (z_2 q^j; q)_{L-j}} \succeq 0. \tag{4.4}$$

This proposition follows from [12, Ex1.3(i), p.20].

The case $(t, a) = (z, qz^{-1})$ of Proposition 4.1 is

$$\frac{(q;q)_\infty}{(z;q)_\infty (q/z;q)_\infty} = \frac{E(q)}{[z;q]_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q)_n (z^{-1} q^{n+1}; q)_\infty} \succeq 0, \tag{4.5}$$

and is related to the crank of partitions [2]. See also [9, Eq.(5.7),p.43]. The crank of a partition is the largest part if the partition contains no ones, and is otherwise the number of parts larger than the number of ones minus the number of ones. Let $M(m,n)$ denote the number of partitions of n with crank m. Then

$$\begin{aligned} (1-z)\frac{E(q)}{[z;q]_\infty} &= \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} \\ &= 1 + (z + z^{-1} - 1)q + \sum_{n \ge 2} \left(\sum_{m=-n}^{n} M(m,n) z^m \right) q^n. \end{aligned} \tag{4.6}$$

This result follows from (1.11) and Theorem 1 in [2]. We note the coefficients on the right side of (4.6) are nonnegative except for the coefficient of $z^0 q^1$. By observing that

$$(1 + z + z^2 + \cdots + z^{m-1})(z + z^{-1} - 1) = z^{-1} + \sum_{j=1}^{m-2} z^j + z^m \qquad (m \ge 2)$$

we have

Proposition 4.4. *If $|q| < 1$, $z \ne 0$ and $m \ge 2$ then*

$$(1-z^m)\frac{E(q)}{[z;q]_\infty} = (1 + z + z^2 + \cdots + z^{m-1}) \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} \succeq 0. \tag{4.7}$$

We will also need

$$\frac{E(q^t)^t}{E(q)} \succeq 0, \quad \text{for any positive integer } t. \tag{4.8}$$

This follows from(1.5).

The quintuple product identity [12, Ex5.6, p.134] can be written as

$$\frac{[z^2;q]_\infty E(q)}{[z,z^3;q]_\infty} = \frac{E(q^3)}{[z^3,q^2z^3;q^3]_\infty} + z\,\frac{E(q^3)}{[z^3,qz^3;q^3]_\infty}. \tag{4.9}$$

Ekin [7] used this form of the quintuple product identity to prove a number of inequalities for the crank of partitions mod 7 and 11. In the following Proposition we give a generalization of the quintuple product identity. The case $a=2$ is (4.9).

Theorem 4.5. (Generalization of the Quintuple Product Identity) *Suppose a is a positive integer, $|q|<1$ and $z\neq 0$. Then*

$$\frac{[z^a;q]_\infty}{[z,z^{a+1};q]_\infty} = \frac{E(q^{a+1})^2}{E(q)^2}\sum_{j=0}^{a-1} z^j\,\frac{[q^{a-j};q^{a+1}]_\infty}{[z^{a+1},z^{a+1}q^{a-j};q^{a+1}]_\infty}. \tag{4.10}$$

Proof. We rewrite (4.10) as

$$\begin{aligned}&\frac{E(q)^2}{E(q^{a+1})^2}\frac{[z^a;q]_\infty}{[z;q]_\infty} \\ &\quad = [z^{a+1}q,z^{a+1}q^2,\ldots,z^{a+1}q^a;q^{a+1}]_\infty \sum_{j=0}^{a-1} z^j\,\frac{[q^{a-j};q^{a+1}]_\infty}{[z^{a+1}q^{a-j};q^{a+1}]_\infty}.\end{aligned} \tag{4.11}$$

Using the fact that

$$[zq^k;q]_\infty = (-1)^k z^{-k} q^{-\binom{k}{2}}[z;q]_\infty \tag{4.12}$$

it is straighforward to show that both sides of (4.11) satisfy the functional equation

$$\Phi_a(zq;q) = (-1)^{a-1} q^{-\binom{a}{2}} z^{1-a^2}\Phi_a(z;q). \tag{4.13}$$

Therefore, since both sides of (4.11) are analytic for $z\neq 0$ we need only to verify (4.11) for a^2 distinct values of z in the region $|q|<z\le 1$. For $1\le k\le a$, and $0\le n\le a$ we let

$$z = q^{k/(a+1)}e^{2\pi i n/(a+1)}, \qquad \text{so that} \quad z^{a+1}=q^k \quad \text{and} \quad z^a = z^{-1}q^k.$$

We find that each term in the sum on the right side of (4.11) is zero except the term corresponding to $j=k-1$, and that both sides simplify to

$$(-1)^{k+1}q^{-\binom{k}{2}}z^{k-1}\frac{E(q)^2}{E(q^{a+1})^2}.$$

Thus both sides of (4.11) agree for a^2+a distinct values of z and the result follows. □

Remark 4.6. This theorem can also be proved using Ramanujan's ${}_1\Psi_1$-summation.

Remark 4.7. Ekin's identity [7, (38), p.287]

$$\frac{E(q)}{[z;q]_\infty} = \frac{1}{[z^2;q^2]_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n(n-1)/2}, \tag{4.14}$$

implies

$$\frac{E(q)E(q^2)}{[z;q]_\infty} = \frac{1}{[z^4;q^4]_\infty}\sum_{n_1,n_2=-\infty}^{\infty} z^{n_1+2n_2}q^{n_1(n_1-1)/2+n_2(n_2-1)} \succeq 0. \tag{4.15}$$

We can iterate (4.14) to obtain

$$\frac{E(q)E(q^2)E(q^4)E(q^8)\cdots}{[z;q]_\infty} \succeq 0. \tag{4.16}$$

Corollary 4.8. *Let $|q|<1$ and $z\neq 0$. If a and k are integers with a and $k\geq 2$, then*

$$(1-z^{k(a+1)})E(q)E(q^{a+1})^{\lfloor (a+1)/2\rfloor}\frac{[z^a;q]_\infty}{[z,z^{a+1};q]_\infty}\succeq 0. \tag{4.17}$$

Proof. From (4.10) we have

$$\begin{aligned}(1-z^{k(a+1)})&E(q)E(q^{a+1})^{\lfloor (a+1)/2\rfloor}\frac{[z^a;q]_\infty}{[z,z^{a+1};q]_\infty}\\ &=\sum_{i=0}^{a-1} z^i\,\frac{E(q^{a+1})^{\lfloor (a+1)/2\rfloor}[q^{a-i};q^{a+1}]_\infty}{E(q)}\\ &\qquad\cdot(1-z^{k(a+1)})\frac{E(q^{a+1})^2}{[z^{a+1},z^{a+1}q^{a-i};q^{a+1}]_\infty}.\end{aligned} \tag{4.18}$$

Suppose $0\leq i\leq a-1$. By (4.7) each term

$$(1-z^{k(a+1)})\frac{E(q^{a+1})^2}{[z^{a+1},z^{a+1}q^{a-i};q^{a+1}]_\infty}\succeq 0, \tag{4.19}$$

since $k\geq 2$. It remains to show that each term

$$\frac{E(q^{a+1})^{\lfloor (a+1)/2\rfloor}[q^{a-i};q^{a+1}]_\infty}{E(q)}\succeq 0. \tag{4.20}$$

If $a\equiv 0 \pmod 2$ then we find that

$$\frac{E(q^{a+1})^{\lfloor (a+1)/2\rfloor}[q^{a-i};q^{a+1}]_\infty}{E(q)} = \prod_{\substack{j=1\\ j\notin\{a-i,i+1\}}}^{a/2}\frac{E(q^{a+1})}{[q^j;q^{a+1}]_\infty}\succeq 0, \tag{4.21}$$

by (4.5). If $a\equiv 1 \pmod 2$ and $i\neq (a-1)/2$ we find that

$$\frac{E(q^{a+1})^{\lfloor (a+1)/2\rfloor}[q^{a-i};q^{a+1}]_\infty}{E(q)} = \frac{E(q^{a+1})^2}{E(q^{(a+1)/2})}\prod_{\substack{j=1\\ j\notin\{a-i,i+1\}}}^{(a-1)/2}\frac{E(q^{a+1})}{[q^j;q^{a+1}]_\infty}\succeq 0, \tag{4.22}$$

by (4.5) and (4.8) with $t=2$ and $q\to q^{(a+1)/2}$. If $a\equiv 1 \pmod 2$ and $i=(a-1)/2$ we find

$$\frac{E(q^{a+1})^{\lfloor (a+1)/2\rfloor}[q^{a-i};q^{a+1}]_\infty}{E(q)} = \frac{E(q^{(a+1)/2})^{(a+1)/2}}{E(q)}\left(\frac{E(q^{a+1})}{E(q^{(a+1)/2})}\right)^{(a-3)/2}\succeq 0, \tag{4.23}$$

by (4.8) since in this case $a\geq 3$ and $E(q^2)/E(q)\succeq 0$.

□

Remark 4.9. Setting $q\to q^5$, $a=3$, $k=2$ and $z=q$ in in (4.17) we have

$$(1-q^8)E(q^5)E(q^{20})^2\frac{[q^2;q^5]_\infty}{[q;q^5]_\infty^2}\succeq 0. \tag{4.24}$$

This leads to an inequality for the crank of partitions mod 5. Using [2, Thm.1] and [8, (4.8)] we have

$$\sum_{n\ge0}(M(0,5,5n)-M(1,5,5n))q^n = E(q^5)\frac{[q^2;q^5]_\infty}{[q;q^5]^2_\infty}. \tag{4.25}$$

In view of (4.24), we have

$$M(0,5,5n) > M(1,5,5n) \quad \text{for } n\ge 0, \tag{4.26}$$

by checking the result for the first 8 coefficients. Here $M(k,t,n)$ is the number of partitions of n with crank congruent to k mod t. This proves [8, (8.47)] (conjectured in 1988). From [7, (13)] we have

$$\sum_{n\ge0}(M(2,11,11n+2)-M(1,11,11n+2))q^n = E(q^{11})\frac{[q^3;q^{11}]_\infty}{[q,q^4;q^{11}]_\infty}. \tag{4.27}$$

Setting $q\to q^{11}$, $a=3$, $k=2$ and $z=q$ in (4.17) and checking the first 11 cases we have

$$M(2,11,11n+2) > M(1,11,11n+2) \quad \text{for } n\ne 3. \tag{4.28}$$

Proposition 4.10. *If $|q|<1$ and $z\ne0$ then*

$$\frac{E(q^2)[z^4;q^2]_\infty}{[z^2;q^2]_\infty[qz^3;q^2]_\infty}\succeq 0. \tag{4.29}$$

$$\frac{E(q^3)(z^2;q^3)_\infty}{(q^3z^{-1};q^3)_\infty(z;q)_\infty}\succeq 0. \tag{4.30}$$

Proof. (i) First we find that

$$E(q^3)\,\frac{[z^2;q^3]_\infty}{[z;q]_\infty}\succeq 0, \tag{4.31}$$

since

$$\begin{aligned} &E(q^3)\,\frac{[z^2;q^3]_\infty}{[z;q]_\infty} \\ &= \left((1-z^2)\frac{E(q^3)}{[z;q^3]_\infty}\right)\left(\frac{(z^2q^3;q^3)_\infty}{(zq,zq^2;q^3)_\infty}\right)\left(\frac{(q^3/z^2;q^3)_\infty}{(q/z,q^2/z;q^3)_\infty}\right), \end{aligned} \tag{4.32}$$

and each of the three terms on the right side of (4.32) has nonnegative coefficients by (4.7) with $m=2$, (4.1) and (4.1) respectively. Next we can use the quintuple product identity to obtain

$$\frac{E(q^2)[z^4;q^2]_\infty}{[z^2;q^2]_\infty[qz^3;q^2]_\infty} = E(q^6)\frac{[q^2z^6;q^6]_\infty}{[qz^3;q^2]_\infty} + z^2E(q^6)\frac{[q^2/z^6;q^6]_\infty}{[q/z^3;q^2]_\infty}\succeq 0, \tag{4.33}$$

by (4.31).

(ii) We have

$$\frac{E(q^3)(z^2;q^3)_\infty}{(q^3z^{-1};q^3)_\infty(z;q)_\infty} = \left((1-z^2)\frac{E(q^3)}{[z;q^3]_\infty}\right)\left(\frac{(z^2q^3;q^3)_\infty}{(zq;q^3)_\infty(zq^2;q^3)_\infty}\right)\succeq 0, \tag{4.34}$$

by (4.7)(with $m=2$) and (4.1). ☐

Proposition 4.11. *Suppose n is a positive integer.*

(i) *If m is a positive odd integer then*

$$\frac{E(q^{mn})^{n(m-1)/2-m}E(q^m)^{(m+1)/2}E(q^n)}{E(q)} \succeq 0. \tag{4.35}$$

(ii) *If m is a positive even integer then*

$$\frac{E(q^{mn})^{(n-2)(m/2-1)}E(q^m)^{m/2-1}E(q^n)E(q^{m/2})}{E(q)E(q^{mn/2})} \succeq 0. \tag{4.36}$$

Proof. By (1.8) we have

$$C_t(z,q) = E(q)E(q^t)^{t-2}\frac{[z^t;q^t]_\infty}{[z;q]_\infty} \succeq 0, \tag{4.37}$$

for any positive integer t.

(i) Result true for $m=1$ so we suppose $m \geq 3$ is an odd integer. Then

$$\prod_{r=1}^{(m-1)/2}[q^r;q^m]_\infty = \frac{E(q)}{E(q^m)}. \tag{4.38}$$

Hence, by (4.37) we have

$$\prod_{r=1}^{(m-1)/2} C_n(q^r,q^m) = \frac{E(q^{mn})^{n(m-1)/2-m}E(q^m)^{(m+1)/2}E(q^n)}{E(q)} \succeq 0. \tag{4.39}$$

(ii) Result true for $m=2$ so we suppose $m \geq 4$ is an even integer. This time

$$\prod_{r=1}^{(m/2-1)}[q^r;q^m]_\infty = \frac{E(q)}{E(q^{m/2})}, \tag{4.40}$$

and

$$\prod_{r=1}^{(m/2-1)} C_n(q^r,q^m) = \frac{E(q^{mn})^{(n-2)(m/2-1)}E(q^m)^{m/2-1}E(q^n)E(q^{m/2})}{E(q)E(q^{mn/2})} \succeq 0. \tag{4.41}$$

□

Corollary 4.12. *If m and n are positive integers then*

$$\frac{E(q^{mn})^{mn-m-n}E(q^m)E(q^n)}{E(q)} \succeq 0. \tag{4.42}$$

Proof. We consider two cases.
Case 1. m is odd. By (4.35) and (4.8) (with $q \to q^m$ and $t=n$) we have

$$\begin{aligned}\frac{E(q^{mn})^{mn-m-n}E(q^m)E(q^n)}{E(q)} \\ = \left(\frac{E(q^{mn})^{n(m-1)/2-m}E(q^m)^{(m+1)/2}E(q^n)}{E(q)}\right)\left(\frac{E(q^{mn})^n}{E(q^m)}\right)^{(m-1)/2} \succeq 0.\end{aligned} \tag{4.43}$$

Case 2. m is even. By (4.36) (with $n=2$ and $m \to 2n$) we have

$$V_n(q) := \frac{E(q^{2n})^{n-2}E(q^2)E(q^n)}{E(q)} \succeq 0, \tag{4.44}$$

where n is any positive integer. We find that

$$\begin{aligned}
&\frac{E(q^{mn})^{mn-m-n}E(q^m)E(q^n)}{E(q)} \\
&= \left(\frac{E(q^{mn})^{(n-2)(m/2-1)}E(q^m)^{m/2-1}E(q^n)E(q^{m/2})}{E(q)E(q^{mn/2})}\right) \\
&\quad \left(\frac{E(q^{mn})^n}{E(q^m)}\right)^{m/2-1} V_n(q^{m/2}) \succeq 0,
\end{aligned} \tag{4.45}$$

by (4.36), (4.8) (with $q \to q^m$ and $t = n$) and (4.44). □

Remark 4.13. We note that in the case when m and n are distinct primes (4.42) is a special case of Saito's Conjecture ($N = mn$). Also, in the case when m is odd there is simple direct proof. In this case we find that

$$0 \preceq \prod_{r=1}^{(m-1)/2} D_n(q^r, q^m) = \frac{E(q^{mn})^{mn-m-n}E(q^m)E(q^n)}{E(q)}, \tag{4.46}$$

since each $D_n(q^r, q^m)$ (defined in (3.7)) has nonnegative coefficients.

We make the following

Conjecture 4.14. *Suppose* $|q| < 1$ *and* $z \neq 0$.

(i) *If* $p \geq 1$ *then*

$$\frac{E(q)}{(z;q)_\infty (qz^{-p};q)_\infty} \succeq 0. \tag{4.47}$$

(ii) *If* a, b, m, $n \geq 1$ *then*

$$\frac{E(q^{ma+nb})}{(q^a;q^{ma+nb})_\infty (q^b;q^{ma+nb})_\infty} \succeq 0. \tag{4.48}$$

(iii) *For* $n \geq 3$

$$\frac{(z, z^{n-1}q^n;q^n)_\infty}{(z;q)_\infty} \succeq 0. \tag{4.49}$$

(iv) *For* $n \geq 4$

$$\frac{(z^{n-1}q^n;q^n)_\infty}{(zq, zq^2, zq^3;q^n)_\infty} \succeq 0. \tag{4.50}$$

(v) *For* $n \geq 2$

$$E(q^n)\,\frac{[z^{n-1};q^n]_\infty}{[z;q]_\infty} \succeq 0. \tag{4.51}$$

(vi) *For* $n > 1$, $m > 0$, $a = 1$, 2

$$\frac{E(q^{nm})}{(q^a;q^m)_\infty} \succeq 0. \tag{4.52}$$

(vii) *For* $m > 1$

$$E(q^m)\,\frac{[z^2;q^m]_\infty}{[z;q^m]_\infty (zq, q/z;q^m)_\infty} \succeq 0. \tag{4.53}$$

(viii) *For $n \geq 2$*

$$E(q^n)\frac{[z^{n^2};q^n]_\infty}{[z^n;q^n]_\infty}\frac{[z^{n+1}q^n;q^n]_\infty}{[z^{n+1}q;q]_\infty} \succeq 0. \tag{4.54}$$

Remark 4.15. The case $p=1$ of (4.47), the case $m=n=1$ of (4.48) and the case $a=1$ of (4.47) are all special cases of Proposition 4.1.

Remark 4.16. We consider (4.49) and let

$$P_n(z,q) := \frac{(z, z^{n-1}q^n;q^n)_\infty}{(z;q)_\infty}, \tag{4.55}$$

for $n \geq 3$. We can show that (4.49) holds for $n=3$, 4. We have

$$P_3(z,q) = \frac{(z^2q^3;q^3)_\infty}{(zq,zq^2;q^3)_\infty} \succeq 0, \tag{4.56}$$

by (4.1) with $q \to q^3$, $a=zq$ and $t=zq^2$. Also,

$$P_4(z,q) = \frac{(z^3q^4;q^4)_\infty}{(zq;q^2)_\infty(zq^2;q^4)_\infty} = (-zq;q^2)_\infty\frac{(z^3q^4;q^4)_\infty}{(zq^2;q^4)_\infty(z^2q^2;q^4)_\infty} \succeq 0, \tag{4.57}$$

by (4.1) with $q \to q^4$, $a=zq^2$ and $t=z^2q^2$.

Remark 4.17. When $n \geq 4$ it is clear that (4.50) implies (4.49).

Remark 4.18. Finally, we consider (4.51). We observe that

$$E(q^n)\,\frac{[z^{n-1};q^n]_\infty}{[z;q]_\infty} = (1-z^{n-1})\frac{E(q^n)}{[z;q^n]_\infty}\cdot P_n(z,q)P_n(z^{-1},q). \tag{4.58}$$

We note that when $n=3$, (4.58) is (4.32). For $n \geq 3$, we see that (4.49) implies (4.51) by (4.7) with $m=n-1$ and $q \to q^n$. Thus (4.51) holds $n=3$, 4. It also holds for $n=2$ since

$$E(q^2)\,\frac{[z;q^2]_\infty}{[z;q]_\infty} = \frac{E(q^2)}{[zq;q^2]_\infty} \succeq 0, \tag{4.59}$$

by (4.5).

Remark 4.19. The case $m=1$ of (4.52) is trivial. When $m=2$ and $a=1$ we have

$$\frac{E(q^{2n})}{(q;q^2)_\infty} = \frac{E(q^2)^2}{E(q)}\cdot\frac{E(q^{2n})}{E(q^2)} \succeq 0, \tag{4.60}$$

by (4.8) with $t=2$.

Remark 4.20. The case $m=2$ of (4.53) is easy:

$$E(q^2)\,\frac{[z^2;q^2]_\infty}{[z;q^2]_\infty(zq,q/z;q^2)_\infty} = \frac{E(q^2)[z^2;q^2]_\infty}{[z;q]_\infty} = \frac{E(q^2)}{E(q)}\cdot(q,-z,-q/z;q)_\infty \succeq 0, \tag{4.61}$$

by (1.9).

Remark 4.21. The case $n=2$ of (4.54) is (4.29). It can be shown that (4.10) and (4.51) imply (4.54).

5. Concluding Remarks

As noted in the introduction both (1.7) and (1.8) are special cases of Macdonald's identity of type A. It is natural to consider the following questions.

(i) Is there a natural analog of t-core which extends (1.7) to other affine root systems?
(ii) Are there other special cases of Macdonald's identity for other affine root systems which give nice product identities analogous to (1.8)?

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