# SHIFTED AND SHIFTLESS PARTITION IDENTITIES

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Abstract. In 1987, George Andrews considered the following question: For which sets of positive integers $S$ and $T$ is $p(S,n) = p(T,n-1)$ for all $n \geq 1$?, where $p(S,n)$ denotes the number of partitions of $n$ into elements of $S$. Andrews found two non-trivial examples and in 1989, Kalvade found a further six. We prove a new shifted partition identity using the theory of modular functions. We consider other shifted-type identities and shiftless identities. Let $a$ be a fixed positive integer, and let $S$, $T$ be distinct sets of positive integers. A shiftless identity has the form: $p(S,T) = p(T,n)$ for all $n \neq a$. These other identities arise through certain modular transformations.

## 1. Introduction

Let $S$ and $T$ be sets of positive integers. Let $a$ be a fixed positive integer. A *shifted partition identity* has the form

$$p(S,n) = p(T,n-a), \qquad \text{for all } n \geq a.$$

We will mainly consider the case $a = 1$. Assume $a = 1$. If $S$ or $T$ is finite then it is not hard to show that $S = T = \{1\}$. Andrews [4] found the following two non-trivial examples:

$$S = \{n : n \text{ odd or } n \equiv \pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}\},$$
$$T = \{n : n \text{ odd or } n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}\};$$

and

$$\begin{aligned} S = \{n : n \equiv{} & \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \pm 13, \pm 15, \\ & \pm 16, \pm 19 \pmod{40}\}, \\ T = \{n : n \equiv{} & \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \pm 14, \pm 15, \pm 16, \\ & \pm 17, \pm 19 \pmod{40}\}, \end{aligned} \tag{1.1}$$

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In these examples, each $S$ and $T$ is the union of arithmetic progressions modulo $M$ for some $M$; namely $M = 32$ and $M = 40$. In fact, each is a union of 24 such arithmetic progressions. Later, Kalvade [10] found five more identities with $M = 42$, 48 and 60, each also involving the union of 24 arithmetic progressions. Through a computer search we have found a further 48 identities with $M = 40$, 42, 46, 48, 54, 56, 60, 62, 66, 70 and 72. All but two of these identities involve unions of 24 arithmetic progressions. The remaining two involve unions of 48 arithmetic progressions. In the present paper, we show how the theory of modular functions may be used to prove certain shifted partition identities.

The generating function for $p(S, n)$ is an infinite product:

$$\sum_{n\ge 0} p(S,n)q^m = \prod_{n\in S} \frac{1}{(1-q^n)},$$

where $|q| < 1$. For the case $a = 1$, we can write a shifted partition identity as an equivalent $q$-series identity:

$$\prod_{n\in S} \frac{1}{(1-q^n)} - q \prod_{n\in T} \frac{1}{(1-q^n)} = 1. \tag{1.2}$$

There is a simple proof of Andrews's mod 32 and 40 shifted partition identities using Jacobi's triple product identity [2, p. 21]:

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2} = \prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - q^n). \tag{1.3}$$

Let $T(z, q)$ denote the left side. By splitting $T(z, q)$ into its even and odd parts (as a function of $z$) we easily find that

$$T(z,q) = T(-z^2q, q^4) - zT(-z^{-2}q, q^4),$$

or

$$\frac{T(-z^2q, q^4)}{T(z,q)} - z\,\frac{T(-z^{-2}q, q^4)}{T(z,q)} = 1. \tag{1.4}$$

The idea is to replace $q$ by $q^m$, and let $z = q$ in (1.4) so that the numerator in each term simplifies to 1 after cancellation in the infinite products. This only occurs in two cases: $m = 4$ giving Andrews's mod 32 shifted identity, and $m = 5$ giving Andrews's mod 40 shifted identity. Other choices for $m$ do not give shifted partition identities but variants in which the parts from some residue classes must be distinct. These types of identities have been considered by Alladi [1].

When comparing shifted identities, quite often it is helpful to write (1.2) in the form:

$$\prod_{n\in T\setminus S}(1-q^n) - q\prod_{n\in S\setminus T}(1-q^n) = \prod_{n\in S\cup T}(1-q^n).$$

For example, Andrews's modulus $M = 40$ result involves

$$T\setminus S = \{n : n \equiv \pm 3, \pm 14, \pm 17 \pmod{40}\},$$
$$S\setminus T = \{n : n \equiv \pm 6, \pm 7, \pm 13 \pmod{40}\}.$$

There is another shifted result with $M = 40$ that Andrews missed. It involves

$$T\setminus S = \{n : n \equiv \pm 2, \pm 9, \pm 11, \pm 12 \pmod{40}\},$$
$$S\setminus T = \{n : n \equiv \pm 4, \pm 6, \pm 7, \pm 13 \pmod{40}\}.$$

We prove this identity using the theory of modular functions in section 4.

By considering the effect of modular transformations on shifted identities we were led to consider shiftless partition identities. Let $a$ be a fixed positive integer, and let $S$, $T$ be distinct sets of positive integers. A *shiftless partition identity* has the form:

$$p(S,T) = p(T,n), \qquad \text{for all } n \neq a.$$

Our simplest example is for modulus $M = 40$. Let

$$\begin{aligned} S = \{n \,:\, n \equiv &\pm 1, \pm 2, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 11, \\ &\pm 12, \pm 13, \pm 15, \pm 19 \pmod{40}\}, \\ T = \{n \,:\, n \equiv &\pm 1, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 13, \\ &\pm 14, \pm 15, \pm 17, \pm 19 \pmod{40}\}. \end{aligned}$$

Then

$$p(S,n) = p(T,n), \qquad \text{for all } n \neq 2. \tag{1.5}$$

This identity follows from our shifted result with $M = 40$ by applying a modular transformation. The details are given in section 4. Other shiftless identities exist for the moduli $M = 42$, 46, 48, 54, 56, 60, 62, 66 and 72. In section 5 we present some examples of shifted and shiftless identities. Details for these other identities will be left to a later paper.

## 2. Computer search

Our new shifted partition identities were found via a computer search. The idea is two consider a finite analogue of (1.2). Let $N$ be an integer

$N > 2$. A pair of sets $[S, T]$ is called a *truncated $ST$-pair $O(q^N)$* if

$$\prod_{n\in S} \frac{1}{(1-q^n)} - q \prod_{n\in T} \frac{1}{(1-q^n)} = 1 + \mathrm{O}(q^N), \tag{2.1}$$

$$S \subset \{1, 2, 3, \dots, N-1\},$$

and

$$T \subset \{1, 2, 3, \dots, N-2\}.$$

It is clear that if $S$, $T$ give a shifted partition theorem then they also give rise to a truncated $ST$-pair $O(q^N)$, for all $N$. For $N \geq 3$, let $\mathcal{T}_N$ be the set of truncated $ST$-pairs $O(q^N)$. For a given $N$, we may find all truncated $ST$-pairs by a boot-strapping method:

0. *Initialize*: $\mathcal{T}_3 = \{[\{1\}, \{1\}]\}$
1. Suppose we are given $\mathcal{T}_N$. For each $[S, T] \in \mathcal{T}_N$, we consider four possible $[S', T']$:

   a. $S' = S$, $T' = T$
   b. $S' = S \cup \{N\}$, $T' = T$
   c. $S' = S$, $T' = T \cup \{N-1\}$
   d. $S' = S \cup \{N\}$, $T' = T \cup \{N-1\}$

   For each of these $[S', T']$, if

$$\prod_{n\in S'} \frac{1}{(1-q^n)} - q \prod_{n\in T'} \frac{1}{(1-q^n)} = 1 + \mathrm{O}(q^{N+1}), \tag{2.2}$$

   then we include $[S', T']$ in $\mathcal{T}_{N+1}$.

In this way we may construct the sets $\mathcal{T}_N$. The computation in (2.2) would be done using a computer algebra package like Maple. For small $N$, we may calculate by hand.

We give some examples.

$$\mathcal{T}_3 = \{[\{1\}, \{1\}]\}$$
$$\mathcal{T}_4 = \{[\{1,3\}, \{1,2\}],\ [\{1\}, \{1\}]\}$$
$$\mathcal{T}_5 = \{[\{1,3,4\}, \{1,2,3\}],\ [\{1,3\}, \{1,2\}],\ [\{1,4\}, \{1,3\}],\ [\{1\}, \{1\}]\}$$
$$\mathcal{T}_6 = \{[\{1,3,4,5\}, \{1,2,3\}],\ [\{1,3,5\}, \{1,2\}],\ [\{1,4,5\}, \{1,3,4\}],$$
$$[\{1,4\}, \{1,3\}],\ [\{1,5\}, \{1,4\}],\ [\{1\}, \{1\}]\}$$

We have calculated the $\mathcal{T}_N$ for $N \leq 37$. Let $t(N) = |\mathcal{T}_N|$, ie. the number of truncated $ST$-pairs $O(q^N)$. We have the following table:

| $n$ | $t(n)$ | $n$ | $t(n)$ |
|---|---|---|---|
| 3 | 1 | 21 | 2447 |
| 4 | 2 | 22 | 3425 |
| 5 | 4 | 23 | 4962 |
| 6 | 6 | 24 | 6839 |
| 7 | 11 | 25 | 10000 |
| 8 | 15 | 26 | 13989 |
| 9 | 26 | 27 | 21383 |
| 10 | 41 | 28 | 30781 |
| 11 | 67 | 29 | 48292 |
| 12 | 96 | 30 | 70456 |
| 13 | 138 | 31 | 110214 |
| 14 | 197 | 32 | 159686 |
| 15 | 300 | 33 | 253265 |
| 16 | 431 | 34 | 374385 |
| 17 | 636 | 35 | 591648 |
| 18 | 893 | 36 | 876405 |
| 19 | 1258 | 37 | 1354888 |
| 20 | 1723 | | |

It appears that $t(n)$ grows exponentially. We do not have enough data to make a real conjecture.

All the known shifted partition identities involve sets of integers $S$ and $T$ that remain unchanged on multiplication by $-1 \pmod{M}$ for a certain modulus $M$. Armed with $\mathcal{T}_{37}$, we searched for such shifted partition identities with even modulus up to $M = 74$.

## 3. Modular Functions

In this section we set up the necessary theory of modular functions to prove our shifted partition identities.

3.1. **Background theory.** The necessary background theory of modular functions and modular forms may be found in [16], [17], [11] and [15]. Many of the results that we require are contained in [5], [6] and [7]. Let $\Gamma(1)$ denote the full modular group and as usual let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,:\, c \equiv 0 \pmod{N} \right\}.$$

Let $\Gamma$ be a subgroup of $\Gamma(1)$ with finite index. For a modular function $f$ on $\Gamma$ and a cusp $\zeta$ the order of $f \pmod{\Gamma}$ at $\zeta$ is denoted by $\mathrm{Ord}\,(f;\zeta;\Gamma)$ and the *invariant order* of $f$ at $\zeta$ is denoted by $\mathrm{ord}\,(f;\zeta)$. We have

$$\mathrm{Ord}\,(f;\zeta;\Gamma) = \kappa(\Gamma;\zeta)\,\mathrm{ord}\,(f;\zeta) \tag{3.1}$$

where $\kappa(\Gamma;\zeta)$ denotes the fan width of the cusp $\zeta \pmod{\Gamma}$, and

$$\operatorname{ord}(f \mid A;\zeta) = \operatorname{ord}(f;A\zeta), \tag{3.2}$$

for $A \in \Gamma(1)$. Here we use the usual stroke operator notation

$$(f \mid A)(\tau) := f(A\tau).$$

Any non-zero modular function $f$ must satisfy the valence formula:

$$\sum_{s\in\mathcal{F}} \operatorname{Ord}(f;s;\Gamma) = 0, \tag{3.3}$$

where $\mathcal{F}$ is a fundamental set of $\Gamma$ and the sum is taken over $s$ with non-zero order so that this is a finite sum. For $s \in \mathcal{H}$ the order is interpreted in the usual sense. See [16] for more details. We shall use the valence formula to prove certain modular function identities. This is a standard technique.

3.2. **Theta products and eta products.** We need results from [7] on transformation formulae and multiplier systems for the Dedekind eta function

$$\eta(\tau) = \exp(\pi i/12) \prod_{n=1}^{\infty}(1-\exp(2\pi i\tau)), \tag{3.4}$$

and the theta function

$$\vartheta_1(v\,|\,\tau) = -i \sum_{m=-\infty}^{\infty} (-1)^m \exp(\pi i\tau(m+\tfrac{1}{2})^2)\,\exp(2\pi i v(m+\tfrac{1}{2})),$$

where $\tau\in\mathcal{H}$ and $v\in\mathbb{C}$.

For $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(1)$, $\eta(\tau)$ and $\vartheta(\tau)$ have well-known transformation formulae

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta(A)\,\sqrt{c\tau+d}\,\eta(\tau), \tag{3.5}$$

and

$$\vartheta_1\left(\frac{v}{c\tau+d}\,\Big|\,\frac{a\tau+b}{c\tau+d}\right) = \nu_{\vartheta_1}(A)\,\sqrt{c\tau+d}\,\exp(\pi i c v^2/(c\tau+d))\,\vartheta_1(v\,|\,\tau).$$

Here $\nu_\eta(A)$, $\nu_{\vartheta_1}(A)$ are explicit 24-th and 8-th roots of unity, respectively. Formulae for these multipliers are given in [7, §2]. We have

$$\nu_{\vartheta_1}(A) = \nu_\eta^3(A). \tag{3.6}$$

Let $q=\exp(2\pi i\tau)$. Let $N$, $\rho$ be integers, $N\ge 1$, $\rho\nmid N$. We define the theta function $\theta_{\rho;N}(\tau)$ by

$$\theta_{\rho;N}(\tau) := \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{8N}(2Nm+2\rho-N)^2} \tag{3.7}$$

$$= q^{\frac{1}{8N}(N-2\rho)^2} \prod_{m=1}^{\infty} (1 - q^{Nm-\rho})(1 - q^{Nm-(N-\rho)})(1 - q^{Nm}),$$

by Jacobi's triple product identity [3, (7.1)]. Our function $\theta_{\rho;N}$ corresponds to Biagioli's [7, (2.8)] $f_{N,\rho}$. From the definition of $\theta_{\rho;N}$ we have

$$\theta_{\rho+N;N} = \theta_{-\rho;N} = \theta_{\rho;N}. \tag{3.8}$$

Biagioli [7, Lemma 2.1] gives the following transformation formula:

(3.9)

$$\theta_{\rho;N}(A\tau) = (-1)^{b\rho+\lfloor\rho a/N\rfloor+\lfloor\rho/N\rfloor} \exp(\rho^2\pi iab/N)\, \nu_{\vartheta_1}\left({}^N A\right) \sqrt{c\tau+d}\, \theta_{a\rho;N}(\tau),$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here

$${}^N A = \begin{pmatrix} a & Nb \\ c/N & d \end{pmatrix}.$$

Similarly, from (3.5) we find that

$$\eta(N\tau) \mid A = \nu_\eta\left({}^N A\right) \sqrt{c\tau+d}\, \eta(N\tau). \tag{3.10}$$

For a set of integers $\boldsymbol{\rho} = \{\rho_1, \rho_2, \ldots, \rho_k\}$, where each $\rho_j \nmid N$, we define the following theta product

$$\theta_{\boldsymbol{\rho};N}(\tau) := \prod_{j=1}^{k} \theta_{\rho_j;N}(\tau).$$

From (3.9) we have the following transformation formula. For $A \in \Gamma_0(N)$,

(3.11)

$$\theta_{\boldsymbol{\rho};N}(A\tau) = (-1)^{bL(\boldsymbol{\rho})+M(\boldsymbol{\rho},a)} \exp(Q(\boldsymbol{\rho})\pi iab/N)\, \nu_{\vartheta_1}^k({}^N A)\, (c\tau+d)^{k/2}\, \theta_{a\boldsymbol{\rho};N}(\tau),$$

where

$$L(\boldsymbol{\rho}) := \sum_j \rho_j,$$

$$M(\boldsymbol{\rho}, x) := \sum_j \lfloor \rho_j x/N\rfloor + \lfloor\rho_j/N\rfloor,$$

$$Q(\boldsymbol{\rho}) := \sum_j \rho_j^2,$$

and where $a\boldsymbol{\rho} = \{a\rho_1, \ldots, a\rho_k\}$.

In view of (3.8) we define a relation on $\mathbb{Z}$. Let $N$ be a fixed positive integer. For $x$, $y \in \mathbb{Z}$ we define $x \sim_N y$ if either $x \equiv y \pmod N$ or $x \equiv -y \pmod N$. This is an equivalence relation. If $x \sim_N y$ then

$$\theta_{x;N} = \theta_{y;N}, \tag{3.12}$$

by (3.8). This equivalence relation extends naturally to certain sets of integers. We consider finite sets of integers whose elements are inequivalent mod $\sim_N$. For any two such sets of integers $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ we define $\boldsymbol{\rho} \sim_N \boldsymbol{\sigma}$ if each element $\rho$ of $\boldsymbol{\rho}$ is equivalent to some element $\sigma$ of $\boldsymbol{\sigma}$ modulo $\sim_N$ and vice versa. For example,

$$\{3, 9, 12, 15, 18, 21, 39, 45, 48, 51\} \sim_{40} \{1, 3, 5, 8, 9, 11, 12, 15, 18, 19\}$$

The analogue of (3.12) holds for theta products. Thus, if $\boldsymbol{\rho} \sim_N \boldsymbol{\sigma}$ then

$$\theta_{\boldsymbol{\rho};N} = \theta_{\boldsymbol{\sigma};N}, \tag{3.13}$$

There is a group action of $\mathbb{Z}_N^{\times}$ on equivalence classes $\boldsymbol{\rho}$. For $(a, N) = 1$, we define $a\,\boldsymbol{\rho}$ in the natural way by

$$a\,\boldsymbol{\rho} = \{a\rho \,:\, \rho \in \boldsymbol{\rho}\}. \tag{3.14}$$

Since $(a, N) = 1$, the $a\rho$ are distinct modulo $\sim_N$. In later sections we will need to compute the orbit of certain sets $\boldsymbol{\rho}$.

3.3. **Orders at cusps.** We will need the following lemmas

LEMMA 3.1, [7]. *If $(r, s) = 1$, then the fan width of $\Gamma_0(N)$ at $\frac{r}{s}$ is*

$$\kappa\left(\Gamma_0(N); \frac{r}{s}\right) = \frac{N}{(N, s^2)}. \tag{3.15}$$

LEMMA 3.2, [9]. *Let $S_N$ be the set of integer pairs $(c, a)$ satisfying*

(0) $(1, 0) \in S_N$*;*
(1) $c > 1$, $c \mid N$, $1 \le a < c$, $\gcd(c, a) = 1$, *and;*
(2) *If* $(c, a)$, $(c, a') \in S_N$ *and* $a' \equiv a \pmod{\gcd(c, N/c)}$, *then* $a = a'$.

*Then the set*

$$\{\frac{a}{c} \,:\, (c, a) \in S_N\}$$

*is a complete set of inequivalent cusps for $\Gamma_0(N)$.*

We will need results on the orders at cusps of eta-products and theta-functions. Newman [14] has found necessary and sufficient conditions under which an eta-product is a modular function on $\Gamma_0(N)$. Ligozat [12] has computed the order of a general eta-product at the cusps of $\Gamma_0(N)$. We need the behavior of the modular form $\eta(N\tau)$ near each cusp of $\Gamma_0(N)$. The following result follows from [9, Lemma 3.5].

LEMMA 3.4, [12]. *The order at the cusp $s = \frac{b}{c}$ (assuming $(b, c) = 1$) of the eta function $\eta(N\tau)$ is*

$$\operatorname{ord}\left(\eta(N\tau); s\right) = \frac{c^2}{24N}. \tag{3.16}$$

Biagoli [7] has computed the order of the theta function $\theta_{\rho;N}$ at any cusp.

LEMMA 3.5, [7, Lemma 3.2, p.285]. *The order at the cusp $s = \frac{b}{c}$ (assuming $(b,c) = 1$) of the theta function $\theta_{\rho;N}(\tau)$ (defined above and assuming $\rho \nmid N$) is*

$$\operatorname{ord}(\theta_{\rho;N}(\tau); s) = \frac{e^2}{2N}\left(\frac{b\rho}{e} - \left\lfloor \frac{b\rho}{e} \right\rfloor - \frac{1}{2}\right)^2, \tag{3.17}$$

*where $e = (N,c)$ and $\lfloor \ \rfloor$ is the greatest integer function.*

## 4. Proof of a shifted partition identity with $M = 40$

There are two shifted partition identities with modulus $M = 40$ and $a = 1$. The first one is the identity (1.1) found by Andrews. We found the second one via a computer search. Let

$$\begin{aligned} S = \{n \,:\, &\pm 1, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 13, \pm 15, \pm 16, \pm 17, \\ &\pm 18, \pm 19 \pmod{40}\}, \\ T = \{n \,:\, &\pm 1, \pm 2, \pm 3, \pm 5, \pm 9, \pm 11, \pm 12, \pm 15, \pm 16, \pm 17, \\ &\pm 18, \pm 19 \pmod{40}\}. \end{aligned} \tag{4.1}$$

Then

$$p(S,n) = p(T, n-1),$$

for all $n \geq 1$. In this section we prove (4.1) in detail. We show how this identity is equivalent to other shifted partition identities and how it leads to a shiftless partition identity (1.5). The proofs of our other shifted partition identities are analogous. Some of the details are given in the next section.

We identify $S$ with the set of "positive" residue classes mod 40:

$$S \leftrightarrow \boldsymbol{\rho}_1 := \{1, 3, 4, 5, 6, 7, 13, 15, 16, 17, 18, 19\},$$

and $T$ with another set of "positive" residue class mod 40:

$$T \leftrightarrow \boldsymbol{\sigma}_1 := \{1, 2, 3, 5, 9, 11, 12, 15, 16, 17, 18, 19\}.$$

From (3.4), (3.7), we have

$$\sum_{n\geq 0} p(S,n) q^n = \frac{\eta^{12}(40\tau)}{\theta_{\boldsymbol{\rho}_1;40}(\tau)},$$

and

$$q \sum_{n\geq 0} p(T,n) q^n = \frac{\eta^{12}(40\tau)}{\theta_{\boldsymbol{\sigma}_1;40}(\tau)}.$$

Hence, our shifted partition identity is equivalent to showing that

$$\left(\frac{1}{\theta_{\boldsymbol{\rho}_1;40}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_1;40}(\tau)}\right)\eta^{12}(40\tau) = 1. \tag{4.2}$$

In view of (3.11) we calculate the set of theta functions $\theta_{a\boldsymbol{\rho}_1;40}$ for $(a,40)=1$. This reduces to calculating the orbit of the equivalence class $\boldsymbol{\rho}_1$ modulo $\sim_{40}$ under multiplication by the group $\mathbb{Z}_{40}^{\times}$. A calculation shows that this orbit is

$$\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_3, \boldsymbol{\rho}_7, \boldsymbol{\rho}_9\},$$

where

$$\begin{aligned}
\boldsymbol{\rho}_3 &:= \{1,3,5,8,9,11,12,14,15,17,18,19\} \sim_{40} 3\boldsymbol{\rho}_1,\\
\boldsymbol{\rho}_7 &:= \{1,2,5,6,7,8,9,11,12,13,15,19\} \sim_{40} 7\boldsymbol{\rho}_1,\\
\boldsymbol{\rho}_9 &:= \{2,3,4,5,7,9,11,13,14,15,16,17\} \sim_{40} 9\boldsymbol{\rho}_1.
\end{aligned}$$

We may extend the definition of $\boldsymbol{\rho}_j$ to all $j \in \mathbb{Z}_{40}^{\times}$, by

$$\boldsymbol{\rho}_m = \boldsymbol{\rho}_{-m}, \qquad \boldsymbol{\rho}_{m+20} = \boldsymbol{\rho}_m.$$

In this way, we find that

$$a\boldsymbol{\rho}_j \sim_{40} \boldsymbol{\rho}_{aj},$$

for all $a$, $j \in \mathbb{Z}_{40}^{\times}$. Similarly, we find that the orbit of the equivalence class $\boldsymbol{\sigma}_1$ modulo $\sim_{40}$ under multiplication by the group $\mathbb{Z}_{40}^{\times}$ is

$$\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_7, \boldsymbol{\sigma}_9\},$$

where

$$\begin{aligned}
\boldsymbol{\sigma}_3 &:= \{3,4,5,6,7,8,9,11,13,14,15,17\} \sim_{40} 3\boldsymbol{\sigma}_1,\\
\boldsymbol{\sigma}_7 &:= \{1,3,4,5,6,7,8,13,14,15,17,19\} \sim_{40} 7\boldsymbol{\sigma}_1,\\
\boldsymbol{\sigma}_9 &:= \{1,2,5,7,9,11,12,13,15,16,18,19\} \sim_{40} 9\boldsymbol{\sigma}_1,
\end{aligned}$$

and we may extend the definition of $\boldsymbol{\sigma}_j$ in analogous way so that

$$a\boldsymbol{\sigma}_j \sim_{40} \boldsymbol{\sigma}_{aj},$$

for all $a$, $j \in \mathbb{Z}_{40}^{\times}$.

We calculate $\theta_{\boldsymbol{\rho};40}(A\tau)$, for $A \in \Gamma_0(40)$, and $\boldsymbol{\rho} = \boldsymbol{\rho}_j$, $\boldsymbol{\sigma}_j$. A calculation shows that

$$Q(\boldsymbol{\rho}_j) \equiv Q(\boldsymbol{\sigma}_j) \equiv 0 \pmod{80}, \tag{4.3}$$

$$L(\boldsymbol{\rho}_j) \equiv L(\boldsymbol{\sigma}_j) \equiv 0 \pmod{2}, \tag{4.4}$$

and

$$M(\boldsymbol{\rho}_j, x) \equiv M(\boldsymbol{\sigma}_j, x) \pmod{2}, \tag{4.5}$$

for $j = 1$, 3, 7 and 9 and $(x, 10) = 1$. Assuming $(x, 10) = 1$,

$$M(\boldsymbol{\rho}_1, x) \equiv 1 \pmod{2}, \tag{4.6}$$

if and only if $x \equiv \pm 3 \pmod{20}$;

$$M(\boldsymbol{\rho}_3, x) \equiv 1 \pmod{2}, \tag{4.7}$$

if and only if $x \not\equiv \pm 1 \pmod{20}$;

$$M(\boldsymbol{\rho}_7, x) \equiv 1 \pmod{2}, \tag{4.8}$$

if and only if $x \equiv \pm 9 \pmod{20}$; and

$$M(\boldsymbol{\rho}_9, x) \equiv 1 \pmod{2}, \tag{4.9}$$

if and only if $x \equiv \pm 7 \pmod{20}$.

We define four functions:

$$f_1(\tau) := \left( \frac{1}{\theta_{\boldsymbol{\rho}_1}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_1}(\tau)} \right) \eta^{12}(40\tau), \tag{4.10}$$

$$f_3(\tau) := \left( \frac{1}{\theta_{\boldsymbol{\sigma}_3}(\tau)} - \frac{1}{\theta_{\boldsymbol{\rho}_3}(\tau)} \right) \eta^{12}(40\tau), \tag{4.11}$$

$$f_7(\tau) := \left( \frac{1}{\theta_{\boldsymbol{\rho}_7}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_7}(\tau)} \right) \eta^{12}(40\tau), \tag{4.12}$$

$$f_9(\tau) := \left( \frac{1}{\theta_{\boldsymbol{\rho}_9}(\tau)} - \frac{1}{\theta_{\boldsymbol{\sigma}_9}(\tau)} \right) \eta^{12}(40\tau). \tag{4.13}$$

We note that $\{f_1, f_3, f_7, f_9\}$ is the orbit of $f_1$ by the group $\Gamma_0(40)$. We have

Theorem 4.1. *For* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(40)$,

$$f_k(A\tau) = f_{ak}(\tau), \tag{4.14}$$

*where* $\pm ak$ *is reduced* $\pmod{20}$.

*Proof.* The functions $f_3$, $f_7$, and $f_9$ are obtained by applying a modular transformation to $f_1$. In fact, from (3.6), (3.10), (3.11) and (4.3)–(4.6) we have

$$f_1 \mid A = f_a, \tag{4.15}$$

for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(40)$. In the subscript of $f_a$ we have reduced $\pm a \pmod{20}$. For $k \in \{1, 3, 7, 9\}$, choose a fixed matrix $A_k = \left(\begin{smallmatrix} k & * \\ * & * \end{smallmatrix}\right) \in \Gamma_0(40)$, so that

$$f_k = f_1 \mid A_k.$$

For any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(40)$, and $k \in \{1, 3, 7, 9\}$,

$$f_k \mid A = (f_1 \mid A_k) \mid A$$

$$
\begin{aligned}
&= f_1 \mid (A_k A) \\
&= f_{ak},
\end{aligned}
$$

since $A_k A \equiv \left(\begin{smallmatrix} ak & * \\ * & * \end{smallmatrix}\right) \pmod{40}$. $\square$

It is well-known that the theta functions $\theta_{\boldsymbol{\rho}}$, $\theta_{\boldsymbol{\sigma}}$ and the eta function $\eta^{12}(40\tau)$, are meromorphic at all cusps $A\infty$, $A \in \Gamma(1)$. This together with the above theorem gives the

Corollary 4.2. *Any symmetric polynomial in $f_1$, $f_3$, $f_7$, $f_9$ is a modular function of $\Gamma_0(40)$.*

Now we are in a position to complete the proof of our shifted partition identity (4.1). This identity is equivalent to showing that

$$f_1 \equiv 1. \tag{4.16}$$

We define $F(\tau)$ by

$$F := (f_1 - 1)(f_3 - 1)(f_7 - 1)(f_9 - 1). \tag{4.17}$$

Our result is equivalent to showing that $F \equiv 0$. Clearly, if (4.16) holds, then $F \equiv 0$. Conversely, if $F \equiv 0$, then $f_j \equiv 0$, for some $j$. But $\Gamma_0(40)$ acts transitively on the $\{f_k\}$, and so

$$f_1 = f_3 = f_7 = f_9 \equiv 1. \tag{4.18}$$

Let $\mathcal{F}$ be a fundamental set for $\Gamma_0(40)$. By the valence formula (3.3), either $F \equiv 0$ or

$$\sum_{s \in \mathcal{F}} \operatorname{Ord}(F; s; \Gamma_0(40)) = 0. \tag{4.19}$$

A set of inequivalent cusps for $\Gamma_0(40)$ is

$$\left\{ \frac{0}{1}, \frac{1}{40}, \frac{1}{20}, \frac{1}{10}, \frac{1}{8}, \frac{1}{5}, \frac{1}{4}, \frac{1}{2}. \right\} \tag{4.20}$$

From the definition of the $f_j$ we find that

$$\operatorname{ord}(f_j; s) \geq \min\{-\operatorname{ord}(\theta_{\boldsymbol{\rho}_1}; s), -\operatorname{ord}(\theta_{\boldsymbol{\sigma}_1}; s)\} + 12 \operatorname{ord}(\eta(40\tau), s).$$

Using (3.1), (3.15), (3.16) and (3.17) this gives lower bounds for $\operatorname{Ord}(f_j; s; \Gamma_0(40))$ at each cusp $s$ of $\Gamma_0(40)$, and $j = 1, 3, 7, 9$. We denote this lower bound by $\ell(j, s)$. We have

$$\operatorname{Ord}(f_j; s; \Gamma_0(40)) \geq \ell(j, s), \tag{4.21}$$

where

$$
\begin{aligned}
\ell(j, 0) &= -1, \quad \text{for all } j, \\
\ell(7, 1/40) &= -2,
\end{aligned}
$$

$$\ell(1, 1/20) = -1$$

and

$$\ell(j, s) = 0,$$

for all other $(j, s)$. Since $F$ has no poles in the complex upper half plane, we have

$$\sum_{s \neq i\infty} \operatorname{Ord}(F; s; \Gamma_0(40)) \geq -5.$$

We only need calculate the first couple of terms in the $q$-expansion of each $f_j$. The lowest power in the $q$-expansion gives the order at the cusp $i\infty$, which is equivalent to $1/40 \bmod \Gamma_0(40)$. Now,

$$\begin{aligned} f_1 &= \frac{1}{(1-q)(1-q^3)\cdots} - q\frac{1}{(1-q)(1-q^2)\cdots} \\ &= 1 + O(q^3), \end{aligned}$$

$$\begin{aligned} f_3 &= \frac{1}{(1-q^3)\cdots} - q^3\frac{1}{(1-q)\cdots} \\ &= 1 + O(q^4), \end{aligned}$$

$$\begin{aligned} f_7 &= \frac{1}{q^2(1-q)(1-q^2)\cdots} - \frac{1}{q^2(1-q)(1-q^3)\cdots} \\ &= O(1), \end{aligned}$$

and

$$\begin{aligned} f_9 &= \frac{1}{(1-q^2)(1-q^3)\cdots} - q^2\frac{1}{(1-q)(1-q^2)\cdots} \\ &= 1 + O(q^3). \end{aligned}$$

Hence,

$$\operatorname{Ord}(F, i\infty, \Gamma_0(40)) \geq 3 + 4 + 0 + 3 = 10,$$

and

$$\sum_{s \in \mathcal{F}} \operatorname{Ord}(F; s; \Gamma_0(40)) \geq 5.$$

Hence,

$$F \equiv 0,$$

and our result follows.

We see that our shifted partition identity (4.1) is equivalent to the four identities given in (4.18). We collect these together into the following

Theorem 4.2

(i) *Let* $S \equiv \pm\{1,3,4,5,6,7,13,15,16,17,18,19\} \pmod{40}$, *and* $T \equiv \pm\{1,2,3,5,9,11,12,15,16,17,18,19\} \pmod{40}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \neq 0$.

(ii) *Let* $S \equiv \pm\{3,4,5,6,7,8,9,11,13,14,15,17\} \pmod{40}$, *and* $T \equiv \pm\{1,3,5,8,9,11,12,14,15,17,18,19\} \pmod{40}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \neq 0$.

(iii) *Let* $S \equiv \pm\{1,2,5,6,7,8,9,11,12,13,15,19\} \pmod{40}$, *and* $T \equiv \pm\{1,3,4,5,6,7,8,13,14,15,17,19\} \pmod{40}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iv) *Let* $S \equiv \pm\{2,3,4,5,7,9,11,13,14,15,16,17\} \pmod{40}$, *and* $T \equiv \pm\{1,2,5,7,9,11,12,13,15,16,18,19\} \pmod{40}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \neq 0$.

Part (i) is our shifted partition identity (4.1). Part (iii) is equivalent to the identity

$$f_7 \equiv 1,$$

and is our shiftless partition identity (1.5).

## 5. Other shifted and shiftless partition identities

In this section we present some other shifted and shiftless partition identities. Details of proofs of these and other identities we have found will be presented in a later paper.

(i) Let $S \equiv \pm\{1,4,5,7,8,9,11,12,13,15,16,19\} \pmod{42}$, and $T \equiv \pm\{1,3,4,7,8,11,12,13,15,17,19,20\} \pmod{42}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(ii) Let $S \equiv \pm\{1,2,3,4,5,6,7,8,11,13,15,17\} \pmod{42}$, and $T \equiv \pm\{1,2,3,4,5,6,7,9,10,11,17,19\} \pmod{42}$. Then $p(S,n) = p(T,n)$, for all $n \neq 8$.

(iii) Let $S \equiv \pm\{1,3,4,5,7,13,15,16,17,18,19,22\} \pmod{46}$, and $T \equiv \pm\{1,2,3,7,11,12,15,16,17,19,20,21\} \pmod{46}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(iv) Let $S \equiv \pm\{1,2,3,5,6,9,10,11,13,14,17,21\} \pmod{46}$, and $T \equiv \pm\{1,2,3,5,7,8,9,11,12,15,20,21\} \pmod{46}$. Then $p(S,n) = p(T,n)$, for all $n \neq 6$.

(v) Let $S \equiv \pm\{1,5,6,8,9,10,11,13,15,19,20,23\} \pmod{48}$, and $T \equiv \pm\{1,4,5,7,9,10,15,17,18,19,20,23\} \pmod{48}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(vi) Let $S \equiv \pm\{1,2,3,4,5,6,11,13,19,20,21,23\} \pmod{48}$, and $T \equiv \pm\{1,2,3,4,5,7,8,17,18,19,21,23\} \pmod{48}$. Then $p(S,n) = p(T,n)$, for all $n \neq 6$.

(vii) Let $S \equiv \pm\{1,3,5,7,11,12,13,20,22,23,24,25\} \pmod{54}$, and $T \equiv \pm\{1,2,5,9,11,12,17,19,22,23,24,25\} \pmod{54}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(viii) Let $S \equiv \pm\{1,2,3,4,5,7,9,11,16,17,24,25\} \pmod{54}$, and $T \equiv \pm\{1,2,3,4,5,7,9,12,13,20,23,25\} \pmod{54}$. Then $p(S,n) = p(T,n)$, for all $n \neq 11$.

(ix) Let $S \equiv \pm\{1,3,4,5,7,18,21,22,23,24,25,27\} \pmod{56}$, and $T \equiv \pm\{1,2,3,7,11,17,20,21,24,25,26,27\} \pmod{56}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(x) Let $S \equiv \pm\{1,5,7,8,10,12,13,15,18,21,23,27\} \pmod{56}$, and $T \equiv \pm\{2,3,5,7,8,13,15,20,21,22,23,25\} \pmod{56}$. Then $p(S,n) = p(T,n)$, for all $n \neq 1$.

(xi) Let $S \equiv \pm\{1,7,8,10,11,13,14,17,19,20,23,29\} \pmod{60}$, and $T \equiv \pm\{1,6,7,9,10,13,17,20,21,23,28,29\} \pmod{60}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(xii) Let $S \equiv \pm\{1,2,7,10,11,13,16,17,19,20,23,29\} \pmod{60}$, and $T \equiv \pm\{1,3,4,10,11,13,17,18,19,20,27,29\} \pmod{60}$. Then $p(S,n) = p(T,n)$, for all $n \neq 2$.

(xiii) Let $S \equiv \pm\{1,3,5,7,8,19,21,23,25,26,28,30\} \pmod{62}$, and $T \equiv \pm\{1,2,5,7,13,18,21,23,24,27,29,30\} \pmod{62}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(xiv) Let $S \equiv \pm\{1,3,5,7,9,13,15,16,21,22,24,28\} \pmod{62}$, and $T \equiv \pm\{1,3,6,7,8,10,15,19,21,23,25,28\} \pmod{62}$. Then $p(S,n) = p(T,n)$, for all $n \neq 5$.

(xv) Let $S \equiv \pm\{1,10,11,12,13,14,15,16,19,21,23,25\} \pmod{66}$, and $T \equiv \pm\{1,9,10,11,12,13,14,15,25,29,31,32\} \pmod{66}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(xvi) Let $S \equiv \pm\{1,4,5,6,7,9,11,13,16,21,23,28\} \pmod{66}$, and $T \equiv \pm\{1,4,5,6,7,9,11,14,16,17,27,29\} \pmod{66}$. Then $p(S,n) = p(T,n)$, for all $n \neq 13$.

(xvii) Let $S \equiv \pm\{1,3,4,5,6,7,9,11,13,14,15,16,17,19,23,24,25,26,27,28,29,31,33,34\} \pmod{70}$, and $T \equiv \pm\{1,2,3,5,8,9,11,12,13,14,15,17,18,19,21,22,23,25,27,28,29,31,32,33\} \pmod{70}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(xix) Let $S \equiv \pm\{1,3,5,7,8,26,28,29,30,31,33,35\} \pmod{72}$, and $T \equiv \pm\{1,2,5,7,13,23,28,29,31,32,34,35\} \pmod{72}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

(xx) Let $S \equiv \pm\{1,4,5,6,11,14,15,21,25,31,32,35\} \pmod{72}$, and $T \equiv \pm\{1,4,5,7,10,11,16,25,26,29,31,35\} \pmod{72}$. Then $p(S,n) = p(T,n)$, for all $n \neq 6$.

(xxi) Let $S \equiv \pm\{1,3,5,7,8,9,10,11,12,13,14,16,17,18,19,23,25,27,29,31,32,33,34,35\} \pmod{72}$, and $T \equiv \pm\{1,2,5,7,8,9,11,12,13,15,16,17,18,19,21,22,23,25,26,27,29,31,32,35\} \pmod{72}$. Then $p(S,n) = p(T,n-1)$, for all $n \neq 0$.

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