SHIFTED AND SHIFTLESS PARTITION IDENTITIES II

FRANK G. GARVAN AND HAMZA YESILYURT

Abstract. Let S and T be sets of positive integers and let a be a fixed positive integer. An a-shifted partition identity has the form

$$p(S,n) = p(T,n-a), \quad \text{for all } n \geq a.$$

Here $p(S,n)$ is the number partitions of n whose parts are elements of S. For all known nontrivial shifted partition identities, the sets S and T are unions of arithmetic progressions modulo M for some M. In 1987, Andrews found two 1-shifted examples ($M = 32$, 40) and asked whether there were any more. In 1989, Kalvade responded with a further six. In 2000, the first author found 59 new 1-shifted identities using a computer search and showed how these could be proved using the theory of modular functions.

Modular transformation of certain shifted identities leads to shiftless partition identities. Again let a be a fixed positive integer, and S, T be distinct sets of positive integers. A shiftless partition identity has the form

$$p(S,n) = p(T,n), \quad \text{for all } n \neq a.$$

In this paper, we show, except in one case, how all known 1-shifted and shiftless identities follow from a four parameter theta function identity due to Jacobi. New shifted and shiftless partition identities are proved.

1. Introduction

Let S and T be sets of positive integers. Let a be a fixed positive integer. An *a-shifted partition identity* has the form

$$p(S,n) = p(T,n-a), \qquad \text{for all } n \geq a.$$

Andrews [2] found the first nontrivial 1-shifted identities:

$$S = \{n : n \text{ odd or } n \equiv \pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}\},$$
$$T = \{n : n \text{ odd or } n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}\};$$

and

$$S = \{n : n \equiv \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \pm 13, \pm 15, \pm 16, \pm 19 \pmod{40}\},$$
$$T = \{n : n \equiv \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \pm 14, \pm 15, \pm 16, \pm 17, \pm 19 \pmod{40}\},$$

In these examples, each S and T is the union of arithmetic progressions modulo M for some M, namely $M = 32$ and $M = 40$. In fact, each is a union of 24 such arithmetic progressions. In 1989, Kalvade [6] found six more identities with $M = 42$, 48 and 60, each also involving

2000 *Mathematics Subject Classification.* Primary: 11P83; Secondary: 05A17.

Key words and phrases. partitions, theta functions, shifted partition identities.

the union of 24 arithmetic progressions. In 1996, Alladi [1] considered more general shifted partition identities in which parts in some residue classes are distinct. In 2000, the first author [5] found 59 new 1-shifted identities using a computer search and showed how such identities could be proved using the theory of modular functions. One case [5, (4.1)] was proved in detail.

The generating function for $p(S,n)$ is an infinite product:

$$\sum_{n\geq 0} p(S,n)q^m = \prod_{n\in S} \frac{1}{(1-q^n)},$$

where $|q|<1$. We can write an a-shifted partition identity as an equivalent q-series identity:

$$\prod_{n\in S} \frac{1}{(1-q^n)} - q^a \prod_{n\in T} \frac{1}{(1-q^n)} = 1. \tag{1.1}$$

The following two identities of Ramanujan which were later proved by Rogers [7] are well-known examples of shifted partition identities

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1 \tag{1.2}$$

and

$$H(q^2)G(q^7) - qG(q^2)H(q^7) = \prod_{n=1}^{\infty} \frac{(1-q^{2n-1})}{(1-q^{14n-7})}, \tag{1.3}$$

where

$$G(q) = \prod_{n\equiv\pm 1 \pmod 5} \frac{1}{(1-q^n)} \quad \text{and} \quad H(q) = \prod_{n\equiv\pm 2 \pmod 5} \frac{1}{(1-q^n)}$$

are the Rogers-Ramanujan functions. Equation (1.2) is equivalent to a 2-shifted identity observed by Bressoud [4]. See also Andrews [2, (3)].

In [5], the effect of modular transformations on shifted identities was studied. This led to certain shiftless partition identities. As before, let a be a fixed positive integer, and let S, T be distinct sets of positive integers. A *shiftless partition identity* has the form:

$$p(S,n) = p(T,n), \qquad \text{for all } n \neq a.$$

The simplest shiftless identity has $M=40$ and $a=2$. See [5, (1.5)] or Theorem 4.1 below. Each shiftless partition identity, that we find, can be written as an equivalent q-series identity

$$\frac{1}{q^a}\prod_{n\in S} \frac{1}{(1-q^n)} - \frac{1}{q^a}\prod_{n\in T} \frac{1}{(1-q^n)} = 1. \tag{1.4}$$

In this paper we show, except for one case, that all known 1-shifted and shiftless partition identities follow from a certain four parameter theta function identity (equation (2.1)) due to Jacobi. The exceptional case which is given in Theorem 18.2 is similar to the identities (1.2) and (1.3) above.

We note that in all identities considered the two products that occur on the left sides of each of (1.1) and (1.4) can be written as a quotient of an eta-product and a product of theta-functions. When we apply a modular transformation $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(M)$, we obtain a shifted or shiftless identity. Here $\gcd(\alpha, M) = 1$, and the effect on S and T is to basically multiply residue classes by α and reduce the products modulo M. Thus we say two identities are equivalent under multiplication by the group $U(M) = (\mathbb{Z}/M\mathbb{Z})^{\times}$ if one can be obtained from the other by applying a modular transformation in $\Gamma_0(M)$. This is explained in more detail with examples in [5]. All shifted and shiftless identities considered can be grouped into equivalence classes. One needs only prove one identity in each class. Many of these identities are just special cases of the four parameter Jacobi identity (2.1). Others follow from several theta function identities each of which is a special case of (2.1).

We find new shifted and shiftless partition identities which are not equivalent to 1-shifted identities under modular transformation. In any shifted or shiftless identity of given modulus M we may replace q by q^k and get another identity of modulus kM. Hence, we may assume throughout that $\gcd(S \cup T) = 1$.

2. Theta function identities

For $|q| < 1$, let us define

$$(a;q)_\infty = (a)_\infty := \prod_{n=0}^{\infty}(1 - aq^n),$$

$$[a_1, \cdots a_k : q] := (a_1)_\infty (q/a_1)_\infty \cdots (a_k)_\infty (q/a_k)_\infty \text{ , and}$$
$$(a_1, \cdots a_k : q) := (-a_1)_\infty (-q/a_1)_\infty \cdots (-a_k)_\infty (-q/a_k)_\infty .$$

The main tool in our proofs is a four parameter $\theta-$function identity due to Jacobi, namely

$$[z, t, xty, zy/x : q] - [xt, zy, ty, z/x : q] = \frac{z}{x}[y, x, xt/z, zty : q]. \tag{2.1}$$

The following equivalent form of (2.1) in terms of sigma function is given in [8, p. 451]

$$\sum_{u,v,w} \sigma(z+u)\sigma(z-u)\sigma(v+w)\sigma(v-w) = 0.$$

For convenience, we use the following reformulation of (2.1) which in most general form can also be found as an exercise in [8].

$$-bc^2[b/c, a/x, a/y, xy/bc : q] + ac^2[a/c, b/x, b/y, xy/ac : q] = ab^2[a/b, c/x, c/y, xy/ab : q]. \tag{2.2}$$

In (2.2), by replacing a by $-a$ and b by $-b$, we obtain

$$-bc^2(b/c, a/x, a/y, xy/bc : q) + ac^2(a/c, b/x, b/y, xy/ac : q) = ab^2[a/b, c/x, c/y, xy/ab : q]. \tag{2.3}$$

We divide both sides of (2.3) by the right hand side of (2.3) and use the trivial identity

$$(a : q) := \frac{[a^2 : q^2]}{[a : q]} = \frac{[a^2 : q^2]}{[a : q^2][aq : q^2]}, \tag{2.4}$$

to write each infinite product with base q^2, we arrive at

$$\frac{-c^2[b^2/c^2, a^2/x^2, a^2/y^2, x^2y^2/b^2c^2 : q^2]}{ab[b/c, bq/c, a/x, aq/x, a/y, aq/y, xy/bc, xyq/bc, a/b, aq/b, c/x, cq/x, c/y, cq/y, xy/ab, xyq/ab : q^2]}$$
$$+ \frac{c^2[a^2/c^2, b^2/x^2, b^2/y^2, x^2y^2/a^2c^2 : q^2]}{b^2[a/c, aq/c, b/x, bq/x, b/y, bq/y, xy/ac, xyq/ac, a/b, aq/b, c/x, cq/x, c/y, cq/y, xy/ab, xyq/ab : q^2]}$$
$$= 1. \tag{2.5}$$

In applications of (2.5), a, b, c, x, y and q will be replaced by q^a,q^b,q^c,q^x,q^y and q^n for some positive integer n but for convenience we will refer to these parameters simply as $[a, b, c, x, y]$. Here we may assume that $\gcd(a, b, c, x, y) = 1$. Shifted partition identities are obtained when each product in the numerator cancels with a product in the denominator and the remaining products in the denominator of both terms on the left side of (2.5) are distinct. However, such cancellation imposes a bound on n and we found that (2.5) does not directly produce shifted partition identities for $n > 41$. Starting in section 3, for each modulus, we first list those shifted partition identities that are obtained directly from (2.5). For most of the moduli, we have identities that do not follow directly from (2.5) but these identities can still be proved through "iteration" by using certain special cases of (2.5) together with (2.2). In most cases, the auxiliary identities that we use for iteration are themselves shifted partition identities. For convenience, in our proofs, we will use the shorthand notation $[a : n]$ for $[q^a : q^n]$ and $(a : n)$ for $(q^a : q^n)$.

We should remark that all shifted partition identities proved by Andrews, Kalvade and Alladi are special cases of (2.5). In fact, we were able to simplify the Macdonald identity that Kalvade employed in her proofs and reduce it to the following special case of (2.1)

$$(x, y : q)\big\{(x^2/y, xy^2 : q) - \frac{x}{y}(y^2/x, x^2y : q)\big\} = [x^2, y^2, xy, x/y : q]. \tag{2.6}$$

The identity (2.6) however has many applications to shifted partition identities other than those considered by Kalvade. Replacing q by q^3 and y by xq, we obtain the famous Quintuple Product Identity [3, p. 80, Entry 28(iv)]

$$(x, xq, xq^2 : q^3)\Big\{(x^3q : q^3) - x(x^{-3}q : q^3)\Big\} = [x^2, qx^2, q^2x^2, q : q^3]. \tag{2.7}$$

Some of our proofs use (2.7) alone in the form

$$\begin{aligned}&[x^{-3}q, x^{-3}q^4, x^6q^2 : q^6] - x[x^3q, x^3q^4, x^{-6}q^2 : q^6]\\ &= [q, q^2, x, xq, xq^2, xq^3, xq^4, xq^5, x^2q, x^2q^3, x^2q^5, x^3q, x^3q^4, x^{-3}q, x^{-3}q^4 : q^6],\end{aligned} \tag{2.8}$$

which we obtain by expressing each infinite product in (2.7) with base q^6 via (2.4).

3. Modulus $M = 32$

There is only one distinct class of identities under multiplication by the group $U(32)$. The identity in Theorem 3.1 below was first given by Andrews [2].

Theorem 3.1.

Let $S \equiv \pm\{1,3,4,5,6,7,8,9,10,11,13,15\} \pmod{32}$, *and*
$T \equiv \pm\{1,2,3,5,7,8,9,11,12,13,14,15\} \pmod{32}$. *Then*
$p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

Proof. This identity follows from (2.5) with the choice of parameters $[1,2,4,12,13]$; i.e., $a = q$, $b = q^2$, $c = q^4$, $x = q^{12}$, $y = q^{13}$, and with q replaced by q^{16}. □

4. Modulus $M = 40$

There are two distinct classes of identities under multiplication by the group $U(40)$. Part (i) of Theorem 4.1 below was also given by Andrews [2].

Theorem 4.1.

(i) *Let* $S \equiv \pm\{1,4,5,6,7,9,10,11,13,15,16,19\} \pmod{40}$, *and*
$T \equiv \pm\{1,3,4,5,9,10,11,14,15,16,17,19\} \pmod{40}$. *Then*
$p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,5,7,8,9,10,11,12,13,15,17\} \pmod{40}$, *and*
$T \equiv \pm\{1,3,5,7,8,10,12,13,15,17,18,19\} \pmod{40}$. *Then*
$p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i) and (ii) follow from (2.5) with the choice of parameters $[1,2,5,15,16]$, $[1,3,4,14,16]$ and with q replaced by q^{20} in both instances. □

Theorem 4.2.

(i) *Let* $S \equiv \pm\{1,3,4,5,6,7,13,15,16,17,18,19\} \pmod{40}$, *and*
$T \equiv \pm\{1,2,3,5,9,11,12,15,16,17,18,19\} \pmod{40}$. *Then*
$p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{3,4,5,6,7,8,9,11,13,14,15,17\} \pmod{40}$, *and*
$T \equiv \pm\{1,3,5,8,9,11,12,14,15,17,18,19\} \pmod{40}$. *Then*
$p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(iii) *Let* $S \equiv \pm\{1,3,4,5,6,7,8,13,14,15,17,19\} \pmod{40}$, *and*
$T \equiv \pm\{1,2,5,6,7,8,9,11,12,13,15,19\} \pmod{40}$. *Then*
$p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iv) *Let* $S \equiv \pm\{2,3,4,5,7,9,11,13,14,15,16,17\} \pmod{40}$, *and*
$T \equiv \pm\{1,2,5,7,9,11,12,13,15,16,18,19\} \pmod{40}$. *Then*
$p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i)–(iv) follow from (2.5) with the following sets of parameters

$$[1,3,2,5,7],\ [1,5,2,7,8],\ [1,2,4,8,9],\ [1,3,4,12,17]$$

and with q replaced by q^{20} in each instance. □

5. Modulus $M = 42$

There are three distinct classes of identities under multiplication by the group $U(42)$. The first identity in Theorem 5.1 below was first given by Kalvade [6]. Kalvade also conjectured the two identities, numbered (iv) and (vi), in Theorem 5.2.

Theorem 5.1.

(i) *Let* $S \equiv \pm\{1,5,6,7,8,9,10,11,13,14,15,19\} \pmod{42}$, *and* $T \equiv \pm\{1,4,5,6,7,9,13,14,15,17,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,5,7,8,9,11,12,13,14,17,19\} \pmod{42}$, *and* $T \equiv \pm\{1,3,5,7,9,11,12,14,16,17,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iii) *Let* $S \equiv \pm\{1,3,4,5,7,11,13,14,15,16,17,18\} \pmod{42}$, *and* $T \equiv \pm\{1,2,3,7,10,11,13,14,15,17,18,19\} \pmod{42}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

Proof. As we mentioned above the identities (i)–(iii) follow from (2.7) or equivalently from (2.5) with the following sets of parameters

$$[1,3,2,7,9],\ [1,4,2,6,9],\ [1,2,4,12,18]$$

and with q replaced by q^{21} in each instance. □

Theorem 5.2.

(i) *Let* $S \equiv \pm\{1,4,5,7,8,9,11,12,13,15,16,19\} \pmod{42}$, *and* $T \equiv \pm\{1,3,4,7,8,11,12,13,15,17,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,4,5,7,9,11,13,17,18,19,20\} \pmod{42}$, *and* $T \equiv \pm\{1,2,5,7,9,11,13,15,16,18,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iii) *Let* $S \equiv \pm\{1,2,3,4,5,6,7,9,10,11,17,19\} \pmod{42}$, *and* $T \equiv \pm\{1,2,3,4,5,6,7,8,11,13,15,17\} \pmod{42}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 8$.

(iv) *Let* $S \equiv \pm\{1,3,5,7,8,10,11,12,13,15,17,20\} \pmod{42}$, *and* $T \equiv \pm\{1,2,5,7,9,10,12,13,15,17,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(v) *Let* $S \equiv \pm\{3,4,5,6,7,9,10,11,13,16,17,19\} \pmod{42}$, *and* $T \equiv \pm\{1,3,6,7,10,11,13,15,16,17,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(vi) *Let* $S \equiv \pm\{1,3,5,7,8,9,10,11,16,17,18,19\} \pmod{42}$, *and* $T \equiv \pm\{1,2,5,7,8,9,13,15,16,17,18,19\} \pmod{42}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

Proof. We prove (i) which we can express as

$$\begin{aligned}&[3,17,20:42] - q[5,9,16:42]\\&= [1,3,4,5,7,8,9,11,12,13,15,16,17,19,20:42].\end{aligned} \tag{5.1}$$

Similarly the first part of Theorem 5.1 is equivalent to

$$\begin{aligned}&[4,17,20:42] - q[8,10,11:42]\\&= [1,4,5,6,7,8,9,10,11,13,14,15,17,19,20:42].\end{aligned} \tag{5.2}$$

Comparing these two identities, we find that (5.1) is equivalent to

$$\begin{aligned}&[3,12,16:42]\big\{[4,17,20:42] - q[8,10,11:42]\big\}\\&= [6,10,14:42]\big\{[3,17,20:42] - q[5,9,16:42]\big\}.\end{aligned} \tag{5.3}$$

Rearranging the terms of (5.3), we arrive at

$$\begin{aligned}&q[10,16:42]\big\{[5,6,9,14:42] - [3,8,11,12:42]\big\}\\&= [3,17:42]\big\{[6,10,14,20:42] - [4,12,16,20:42]\big\}.\end{aligned} \tag{5.4}$$

Employing (2.2) twice with the choice of parameters $[1,3,6,9,12]$, $[1,3,7,13,17]$ and with q replaced by q^{42} in both instances, we deduce

$$[5,6,9,14:42] - [3,8,11,12:42] = q^3[2,3,6,17:42] \text{ and}$$

$$[6,10,14,20:42] - [4,12,16,20:42] = q^4[2,6,10,16:42].$$

These last two equations readily imply (5.4) and so the proof of (i) is complete. $\square$

Theorem 5.3.

(i) *Let* $S \equiv \pm\{1,3,4,5,7,9,11,16,17,18,19,20\} \pmod{42}$, *and* $T \equiv \pm\{1,2,3,7,8,11,13,15,17,18,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,2,5,6,7,9,10,11,13,15,16,19\} \pmod{42}$, *and* $T \equiv \pm\{1,3,4,5,6,7,11,13,15,16,17,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iii) *Let* $S \equiv \pm\{1,3,4,5,7,9,10,11,12,17,19,20\} \pmod{42}$, *and* $T \equiv \pm\{1,2,5,7,8,9,10,11,12,13,15,19\} \pmod{42}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iv) *Let* $S \equiv \pm\{2,3,5,7,8,9,10,11,13,17,18,19\} \pmod{42}$, *and* $T \equiv \pm\{1,3,5,7,8,11,13,15,16,17,18,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(v) *Let* $S \equiv \pm\{3,4,5,7,8,9,10,11,12,13,17,19\} \pmod{42}$, *and* $T \equiv \pm\{1,4,5,7,9,12,13,15,16,17,19,20\} \pmod{42}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(vi) *Let* $S \equiv \pm\{1,2,4,5,6,7,9,13,15,16,17,19\} \pmod{42}$, *and* $T \equiv \pm\{1,2,3,6,7,8,10,11,13,15,17,19\} \pmod{42}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 3$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[2,8,13,15:42] - q[4,5,9,16:42]\\&= [1,2,3,4,5,7,8,9,11,13,15,16,17,18,19,20:42].\end{aligned} \tag{5.5}$$

From Theorem 5.1(iii), we also have

$$\begin{aligned}&[2,10,19:42] - q[4,5,16:42]\\&= [1,2,3,4,5,7,10,11,13,14,15,16,17,18,19:42].\end{aligned} \tag{5.6}$$

Therefore, it suffices to show that

$$\begin{aligned}&[8,9,20:42]\big\{[2,10,19:42] - q[4,5,16:42]\big\}\\&= [10,14:42]\big\{[2,8,13,15:42] - q[4,5,9,16:42]\big\}.\end{aligned} \tag{5.7}$$

After rearrangement of the terms of (5.3), we arrive at

$$\begin{aligned}&[2,8,10:42]\big\{[9,19,20:42] - [13,14,15:42]\big\}\\&= q[4,5,16,9:42]\big\{[8,20:42] - [10,14:42]\big\}.\end{aligned} \tag{5.8}$$

Employing (2.2) twice with the choice of parameters $[1,5,14,19,20]$, $[1,3,11,17,19]$ and with q replaced by q^{42} in both instances, we find that

$$[9,18,19,20:42] - [13,14,15,18:42] = -q^9[4,5,6,9:42] \text{ and}$$
$$[8,16,18,20:42] - [10,14,16,18:42] = -q^8[2,6,8,10:42].$$

Now, (5.8) follows easily from these last two equations and so the proof of (i) is complete. $\square$

6. Modulus $M = 46$

There is only one class of identities under multiplication by the group $U(46)$.

Theorem 6.1.

(i) *Let* $S \equiv \pm\{1,3,4,5,7,13,15,16,17,18,19,22\} \pmod{46}$, *and* $T \equiv \pm\{1,2,3,7,11,12,15,16,17,19,20,21\} \pmod{46}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,2,3,5,7,8,9,11,12,15,20,21\} \pmod{46}$, *and* $T \equiv \pm\{1,2,3,5,6,9,10,11,13,14,17,21\} \pmod{46}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 6$.

(iii) *Let* $S \equiv \pm\{3,5,7,8,9,10,11,12,13,14,15,17\} \pmod{46}$, *and* $T \equiv \pm\{2,3,5,7,11,12,15,17,18,19,20,21\} \pmod{46}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(iv) *Let* $S \equiv \pm\{2,3,5,7,8,9,13,14,15,19,20,21\} \pmod{46}$, *and* $T \equiv \pm\{1,3,5,7,11,12,13,16,18,19,20,21\} \pmod{46}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(v) *Let* $S \equiv \pm\{3,4,5,6,7,9,13,15,16,17,18,19\} \pmod{46}$, *and* $T \equiv \pm\{1,3,6,9,10,13,14,15,17,19,21,22\} \pmod{46}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(vi) *Let* $S \equiv \pm\{2,3,5,8,9,11,12,13,14,15,19,21\} \pmod{46}$, *and* $T \equiv \pm\{1,3,6,8,10,11,13,15,17,19,21,22\} \pmod{46}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(vii) *Let* $S \equiv \pm\{1,4,6,7,9,10,11,13,15,17,19,22\} \pmod{46}$, *and* $T \equiv \pm\{1,3,5,7,9,11,13,16,17,18,20,22\} \pmod{46}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(viii) *Let* $S \equiv \pm\{1,5,6,8,9,10,11,13,14,15,17,21\} \pmod{46}$, *and* $T \equiv \pm\{1,4,5,7,9,10,13,15,16,19,21,22\} \pmod{46}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ix) *Let* $S \equiv \pm\{1,4,5,6,7,9,13,16,17,19,21,22\} \pmod{46}$, *and* $T \equiv \pm\{1,3,4,5,11,12,13,17,18,19,20,21\} \pmod{46}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(x) *Let* $S \equiv \pm\{1,3,4,5,7,9,11,16,17,18,19,20\} \pmod{46}$, *and* $T \equiv \pm\{1,2,5,7,8,9,11,12,15,18,19,21\} \pmod{46}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(xi) *Let* $S \equiv \pm\{2,3,7,8,9,10,11,13,14,15,17,21\} \pmod{46}$, *and* $T \equiv \pm\{1,4,6,7,9,11,14,15,17,19,21,22\} \pmod{46}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i)–(xi) follow from (2.5) with the following choice of parameters

$$[1,2,4,12,20],\ [1,2,8,10,11],\ [1,6,3,9,10],\ [1,4,2,6,9],$$
$$[1,4,5,14,20],\ [1,4,2,6,10],\ [1,2,5,17,18],\ [1,3,2,7,10],$$
$$[1,3,2,6,8], [1,2,4,8,9], [1,3,4,16,18],$$

and with q replaced by q^{23} in each instance. □

7. Modulus $M = 48$

There are seven distinct classes of identities under multiplication by the group $U(48)$. The first identity in Theorem 7.1 below was first given by Kalvade [6] and the second identity in Theorem 7.3 was first given by Alladi [1].

Theorem 7.1.

(i) *Let* $S \equiv \pm\{1,3,4,5,7,14,16,17,18,19,21,23\} \pmod{48}$, *and* $T \equiv \pm\{1,2,3,7,11,13,16,17,18,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{4,5,6,7,9,10,11,13,15,16,17,19\} \pmod{48}$, *and* $T \equiv \pm\{1,5,6,9,11,13,15,16,19,20,22,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

Proof. The identities (i) and (ii) follow from (2.5) with the choice of parameters $[1,2,4,12,21]$, $[1,6,2,8,10]$ and with q replaced by q^{24} in both instances. □

Theorem 7.2.

(i) *Let* $S \equiv \pm\{1,4,5,6,7,9,15,17,19,20,22,23\} \pmod{48}$, *and* $T \equiv \pm\{1,3,4,5,11,13,14,18,19,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,4,5,7,11,13,17,18,19,20,21\} \pmod{48}$, *and* $T \equiv \pm\{1,4,6,7,9,10,11,13,15,17,20,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 1$.

Proof. The identities (i) and (ii) follow from (2.5) with the choice of parameters $[1,3,2,6,8]$, $[1,3,4,8,10]$ and with q replaced by q^{24} in both instances. □

Theorem 7.3.

(i) *Let* $S \equiv \pm\{1,6,7,8,9,10,11,13,15,16,17,23\} \pmod{48}$, *and* $T \equiv \pm\{1,5,6,7,8,9,15,16,17,19,22,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,5,7,8,11,13,16,17,18,19,21\} \pmod{48}$, *and* $T \equiv \pm\{1,3,5,8,11,13,14,16,18,19,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i) and (ii) follow from (2.5) with the choice of parameters $[1,3,2,8,10]$, $[1,3,4,14,20]$ and with q replaced by q^{24} in both instances. □

Theorem 7.4.

Let $S \equiv \pm\{3,4,5,6,7,9,15,16,17,18,19,21\} \pmod{48}$, *and* $T \equiv \pm\{1,3,6,9,11,13,15,16,18,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

Proof. This identity follows from (2.5) with the choice of parameters $[1,4,5,14,21]$ and with q replaced by q^{24}. □

Theorem 7.5.

(i) *Let* $S \equiv \pm\{1,5,6,8,9,10,11,13,15,19,20,23\} \pmod{48}$, *and* $T \equiv \pm\{1,4,5,7,9,10,15,17,18,19,20,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,2,3,4,5,6,11,13,19,20,21,23\} \pmod{48}$, *and* $T \equiv \pm\{1,2,3,4,5,7,8,17,18,19,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 6$.

(iii) *Let* $S \equiv \pm\{4,5,6,7,8,9,11,13,15,17,19,22\} \pmod{48}$, *and* $T \equiv \pm\{1,4,7,9,11,13,15,17,18,20,22,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(iv) *Let* $S \equiv \pm\{3,4,5,6,7,11,13,14,17,19,20,21\} \pmod{48}$, *and* $T \equiv \pm\{1,3,7,8,11,13,14,17,18,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[4,7,17,18:48]-q[6,8,11,13:48]\\&=[1,4,5,6,7,8,9,10,11,13,15,17,18,19,20,23:48].\end{aligned}$$

From Theorem 7.1(ii), we similarly have that

$$\begin{aligned}&[1,20,22,23:48]-q^4[4,7,10,17:48]\\&=[1,4,5,6,7,9,10,11,13,15,16,17,19,20,22,23:48].\end{aligned}$$

Therefore, it suffices to prove that

$$\begin{aligned}&[8,18:48]\big\{[1,20,22,23:48]-q^4[4,7,10,17:48]\big\}\\&=[16,22:48]\big\{[4,7,17,18:48]-q[6,8,11,13:48]\big\}.\end{aligned}\tag{7.1}$$

Rearranging the terms in (7.1), we derive the equivalent identity

$$\begin{aligned}&[7,17:48]\big\{[4,16,22,18:48]+q^4[4,8,10,18:48]\big\}\\&=[8,22:48]\big\{[1,18,20,23:48]+q[6,11,13,16:48]\big\}.\end{aligned}\tag{7.2}$$

We now employ (2.2) twice with the set of variables $[1,5,9,17,19]$, $[1,7,8,19,21]$ and with q replaced by q^{48} in each instance, we find that

$$[4,16,22,18:48]+q^4[4,8,10,18:48]=[8,12,14,22:48]\text{ and}\tag{7.3}$$

$$[1,18,20,23:48]+q[6,11,13,16:48]=[7,12,14,17:48].\tag{7.4}$$

We see that (7.3) together with (7.4) clearly implies (7.2) and so the proof of (i) is complete. $\square$

Theorem 7.6.

(i) *Let* $S\equiv\pm\{1,4,7,9,10,11,12,13,14,15,17,23\}\pmod{48}$, *and* $T\equiv\pm\{1,3,7,8,10,11,12,13,17,21,22,23\}\pmod{48}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(ii) *Let* $S\equiv\pm\{2,5,7,8,9,11,12,13,14,15,17,19\}\pmod{48}$, *and* $T\equiv\pm\{2,3,5,7,11,12,13,17,19,20,21,22\}\pmod{48}$. *Then* $p(S,n)=p(T,n-2)$, *for all* $n\geq 2$.

(iii) *Let* $S\equiv\pm\{1,3,5,7,8,10,12,17,19,21,22,23\}\pmod{48}$, *and* $T\equiv\pm\{1,2,5,7,9,12,15,17,19,20,22,23\}\pmod{48}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(iv) *Let* $S\equiv\pm\{1,2,5,8,9,11,12,13,14,15,19,23\}\pmod{48}$, *and* $T\equiv\pm\{1,3,4,5,10,11,12,13,14,19,21,23\}\pmod{48}$. *Then* $p(S,n)=p(T,n)$, *for all* $n\neq 2$.

Proof. We prove (i). First we write (i) in its equivalent form

$$\begin{aligned}&[3,8,21,22:48]-q[4,9,14,15:48]\\&=[1,3,4,7,8,9,10,11,12,13,14,15,17,21,22,23:48].\end{aligned}\tag{7.5}$$

Similarly from the first part of Theorem 7.1

$$\begin{aligned}&[2,11,13,20:48]-q[4,5,14,19:48]\\&=[1,2,3,4,5,7,11,13,14,16,17,18,19,20,21,23:48].\end{aligned}\tag{7.6}$$

Therefore, it suffices to show that

$$\begin{aligned}&[8,9,10,12,15,22:48]\big\{[2,11,13,20:48]-q[4,5,14,19:48]\big\}\\&=[2,5,16,18,19,20:48]\big\{[3,8,21,22:48]-q[4,9,14,15:48]\big\}.\end{aligned}\tag{7.7}$$

After rearrangement of the terms, we arrive at

$$\begin{aligned}&[2,22,8,20:48]\big\{[9,10,11,12,13,15:48]-[3,5,16,18,19,21:48]\big\}\\&=-q[4,5,9,14,15,19:48]\big\{[2,16,18,20:48]-[8,10,12,22:48]\big\}.\end{aligned}\tag{7.8}$$

By (2.2) with the choice of parameters $[1,7,9,17,19]$ and with q replaced by q^{48}, we deduce that

$$[2,16,18,20:48]-[8,10,12,22:48]=-q^2[6,8,10,20:48].\tag{7.9}$$

Using (7.9) in (7.8), we arrive at

$$\begin{aligned}&[2,22:48]\big\{[3,5,16,18,19,21:48]-[9,10,11,12,13,15:48]\big\}\\&=-q^3[4,5,6,9,10,14,15,19:48].\end{aligned}\tag{7.10}$$

Switching to base 24, we have from (7.10) that

$$\begin{aligned}&[2:24]\big\{[3,5,8,9:24](8,9:24)-[5,6,9,11:24](5,6:24)\big\}\\&=-q^3[2,3,5,9,10:24](2,3:24),\end{aligned}\tag{7.11}$$

which simplifies to

$$\begin{aligned}&[3,8:24](8,9:24)-[6,11:24](5,6:24)\\&=-q^3[3,10:24](2,3:24).\end{aligned}\tag{7.12}$$

By employing (2.2) with a, b, c, x, y and q replaced by $-q$, q^3, q^6, $-q^9$, q^9 and q^{24}, respectively, we establish (7.12) and complete the proof of (i). $\square$

Theorem 7.7.

(i) *Let* $S\equiv\pm\{1,4,5,7,8,11,14,15,17,18,19,21\}\pmod{48}$, *and* $T\equiv\pm\{1,3,4,7,11,13,14,15,18,19,20,23\}\pmod{48}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(ii) *Let* $S\equiv\pm\{1,5,6,7,8,9,11,13,20,21,22,23\}\pmod{48}$, *and* $T\equiv\pm\{1,4,5,6,7,13,15,17,19,20,21,22\}\pmod{48}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(iii) *Let* $S \equiv \pm\{1,2,4,5,7,9,11,17,18,19,20,21\} \pmod{48}$, *and* $T \equiv \pm\{1,2,3,7,8,9,11,13,18,19,20,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 3$.

(iv) *Let* $S \equiv \pm\{4,5,6,7,8,9,10,11,17,19,21,23\} \pmod{48}$, *and* $T \equiv \pm\{1,4,6,10,11,13,15,17,19,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(v) *Let* $S \equiv \pm\{1,3,4,5,6,7,9,10,11,13,20,23\} \pmod{48}$, *and* $T \equiv \pm\{1,3,4,5,6,7,8,10,13,15,17,19\} \pmod{48}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 8$.

(vi) *Let* $S \equiv \pm\{2,3,4,5,7,13,15,17,18,19,20,23\} \pmod{48}$, *and* $T \equiv \pm\{1,2,5,8,11,13,15,17,18,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(vii) *Let* $S \equiv \pm\{3,4,5,6,7,9,11,17,19,20,22,23\} \pmod{48}$, *and* $T \equiv \pm\{1,3,6,8,11,13,15,17,19,20,22,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(viii) *Let* $S \equiv \pm\{3,4,5,7,8,9,13,14,17,18,19,23\} \pmod{48}$, *and* $T \equiv \pm\{1,4,5,9,11,13,14,17,18,20,21,23\} \pmod{48}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

Proof. We prove (i). Recall that by (7.6), we have

$$\begin{aligned}&[2,11,13,20:48] - q[4,5,14,19:48]\\ &= [1,2,3,4,5,7,11,13,14,16,17,18,19,20,21,23:48].\end{aligned} \tag{7.13}$$

We write (i) in its equivalent form

$$\begin{aligned}&[3,13,20,23:48] - q[5,8,17,21:48]\\ &= [1,3,4,5,7,8,11,13,14,15,17,18,19,20,21,23:48].\end{aligned} \tag{7.14}$$

Therefore, it suffices to prove that

$$\begin{aligned}&[8,15:48]\big\{[2,11,13,20:48] - q[4,5,14,19:48]\big\}\\ &= [2,16:48]\big\{[3,13,20,23:48] - q[5,8,17,21:48]\big\}.\end{aligned} \tag{7.15}$$

By rearranging the terms in (7.15), we arrive at

$$\begin{aligned}&[2,13:48]\big\{[8,11,15,20:48] - [3,16,20,23:48]\big\}\\ &= q[5,8:48]\big\{[4,14,15,19:48] - [2,16,17,21:48]\big\}.\end{aligned} \tag{7.16}$$

Next, we employ (2.2) twice with the choice of parameters $[1,6,9,17,21]$, $[1,3,5,17,18]$ and with q replaced by q^{48}, we find that

$$[8,11,15,20:48]-[3,16,20,23:48]=q^3[5,8,12,17:48] \text{ and}$$

$$[4,14,15,19:48]-[2,16,17,21:48]=q^2[2,12,13,17:48].$$

From these last two equations, (7.16) readily follows and so the proof of (i) is complete. □

8. Modulus $M=50$

There are four distinct classes of identities under multiplication by the group $U(50)$.

Theorem 8.1.

(i) *Let* $S \equiv \pm\{1,3,5,7,8,13,15,17,18,19,20,22\} \pmod{50}$, *and* $T \equiv \pm\{1,2,5,7,12,13,15,17,18,20,21,23\} \pmod{50}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,3,4,5,6,10,11,13,14,15,19,21\} \pmod{50}$, *and* $T \equiv \pm\{1,3,4,5,7,9,10,11,15,16,21,24\} \pmod{50}$. *Then* $p(S,n)=p(T,n)$, *for all* $n \neq 6$.

(iii) *Let* $S \equiv \pm\{1,3,5,7,9,10,11,14,15,16,19,24\} \pmod{50}$, *and* $T \equiv \pm\{1,4,5,6,7,9,10,15,17,19,21,24\} \pmod{50}$. *Then* $p(S,n)=p(T,n)$, *for all* $n \neq 3$.

(iv) *Let* $S \equiv \pm\{3,5,7,8,9,11,12,13,15,17,18,20\} \pmod{50}$, *and* $T \equiv \pm\{2,3,5,9,12,13,15,17,20,21,22,23\} \pmod{50}$. *Then* $p(S,n)=p(T,n-3)$, *for all* $n \geq 3$.

(v) *Let* $S \equiv \pm\{2,5,7,8,9,11,12,13,15,17,20,23\} \pmod{50}$, *and* $T \equiv \pm\{2,3,5,7,11,13,15,18,19,20,22,23\} \pmod{50}$. *Then* $p(S,n)=p(T,n-2)$, *for all* $n \geq 2$.

(vi) *Let* $S \equiv \pm\{3,4,5,9,10,11,13,14,15,16,19,21\} \pmod{50}$, *and* $T \equiv \pm\{1,5,6,9,10,13,15,16,19,21,23,24\} \pmod{50}$. *Then* $p(S,n)=p(T,n-3)$, *for all* $n \geq 3$.

(vii) *Let* $S \equiv \pm\{4,5,6,7,9,10,11,15,16,17,19,21\} \pmod{50}$, *and* $T \equiv \pm\{1,5,6,10,11,14,15,17,19,21,23,24\} \pmod{50}$. *Then* $p(S,n)=p(T,n-4)$, *for all* $n \geq 4$.

(viii) *Let* $S \equiv \pm\{2,3,5,7,8,11,15,17,18,19,20,23\} \pmod{50}$, *and* $T \equiv \pm\{1,3,5,8,12,13,15,17,19,20,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(ix) *Let* $S \equiv \pm\{2,3,5,7,8,9,15,17,20,21,22,23\} \pmod{50}$, *and* $T \equiv \pm\{1,3,5,7,12,13,15,18,20,21,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(x) *Let* $S \equiv \pm\{1,5,6,9,10,11,13,14,15,16,19,23\} \pmod{50}$, *and* $T \equiv \pm\{1,4,5,9,10,11,14,15,17,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

Proof. To prove the identities (i)–(x), we employ (2.5) with q replaced by q^{25} and with the following choice of parameters for a, b, c, x, and y in each instance $[1,2,4,14,21], [1,2,8,11,12], [1,2,5,10,11], [1,4,6,18,21], [1,3,6,19,20]$, $[1,5,2,7,11], [1,6,2,8,10], [1,3,4,14,21], [1,4,2,6,9]$, and $[1,3,2,7,11]$. □

Theorem 8.2.

(i) *Let* $S \equiv \pm\{1,6,7,8,9,11,12,13,16,17,19,23\} \pmod{50}$, *and* $T \equiv \pm\{1,5,6,7,8,11,15,17,18,19,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,3,4,5,7,15,17,18,19,21,22,24\} \pmod{50}$, *and* $T \equiv \pm\{1,2,3,7,11,14,17,18,19,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(iii) *Let* $S \equiv \pm\{1,3,4,5,7,9,12,13,15,16,21,22\} \pmod{50}$, *and* $T \equiv \pm\{1,3,4,6,7,8,9,13,17,19,21,22\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 5$.

(iv) *Let* $S \equiv \pm\{2,3,5,9,11,12,13,14,15,16,21,23\} \pmod{50}$, *and* $T \equiv \pm\{1,3,7,9,11,12,13,16,18,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(v) *Let* $S \equiv \pm\{2,5,6,8,9,11,13,14,15,17,19,23\} \pmod{50}$, *and* $T \equiv \pm\{2,3,4,9,11,13,14,17,19,21,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. Identities (i)–(x) also follow from (2.5) with q replaced by q^{25} and with the following choice of parameters for a, b, c, x, and y in each instance

$$[1,3,2,8,10], [1,2,4,12,22], [1,2,7,10,11], [1,4,2,6,11], \text{ and } [1,3,6,18,20].$$ □

Theorem 8.3.

(i) *Let* $S \equiv \pm\{1,3,5,7,8,12,13,17,19,22,23,24\} \pmod{50}$, *and* $T \equiv \pm\{1,2,5,7,11,12,15,18,19,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{3,4,5,6,7,13,14,15,17,19,21,22\} \pmod{50}$, *and* $T \equiv \pm\{1,3,7,9,11,14,15,16,19,21,22,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(iii) *Let* $S \equiv \pm\{1,4,6,7,9,11,15,16,17,18,19,21\} \pmod{50}$, *and* $T \equiv \pm\{1,3,5,7,11,14,15,16,17,18,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(iv) *Let* $S \equiv \pm\{2,3,5,7,8,9,13,16,17,21,22,23\} \pmod{50}$, *and* $T \equiv \pm\{1,5,7,8,9,11,12,13,15,16,18,21\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 1$.

(v) *Let* $S \equiv \pm\{3,5,7,8,9,11,12,13,14,17,18,23\} \pmod{50}$, *and* $T \equiv \pm\{2,3,5,9,11,14,15,18,19,21,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(vi) *Let* $S \equiv \pm\{1,3,4,6,9,11,12,13,14,15,19,21\} \pmod{50}$, *and* $T \equiv \pm\{1,3,5,6,7,9,12,13,15,16,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 4$.

(vii) *Let* $S \equiv \pm\{4,5,6,7,8,9,13,15,16,17,19,23\} \pmod{50}$, *and* $T \equiv \pm\{1,4,8,9,11,14,15,17,19,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(viii) *Let* $S \equiv \pm\{2,3,5,6,7,11,13,17,18,19,22,23\} \pmod{50}$, *and* $T \equiv \pm\{1,5,6,8,9,11,12,13,15,17,19,22\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 1$.

(xi) *Let* $S \equiv \pm\{1,2,3,4,5,7,12,13,17,18,21,23\} \pmod{50}$, *and* $T \equiv \pm\{1,2,3,4,5,8,9,15,17,19,21,22\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 7$.

(x) *Let* $S \equiv \pm\{2,3,4,5,11,13,14,15,17,21,23,24\} \pmod{50}$, *and* $T \equiv \pm\{1,2,6,9,11,13,15,16,19,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. We prove (i) which is equivalent to

$$[2,11,15,18,21:50] - q[3,8,13,17,22:50]$$
$$= [1,2,3,5,7,8,11,12,13,15,17,18,19,21,22,23,24:50].$$

From Theorem 8.1(i), we similarly have that

$$[2,12,21,23:50] - q[3,8,19,22:50]$$
$$= [1,2,3,5,7,8,12,13,15,17,18,19,20,21,22,23:50].$$

Therefore, it suffices to prove that

$$[11,24:50]\big\{[2,12,21,23:50] - q[3,8,19,22:50]\big\}$$
$$= [20:50]\big\{[2,11,15,18,21:50] - q[3,8,13,17,22:50]\big\}. \tag{8.1}$$

After regrouping terms in (8.1), we are led to verify that

$$[21,2,11]\big\{[24,12,23:50] - [20,15,18:50]\big\}$$
$$= -q[3,8,22:50]\big\{[20,13,17:50] - [11,19,24:50]\big\}. \tag{8.2}$$

In (2.2), we replace q by q^{50} and take $a=1$, $b=4$, $c=16$, $x=22$, and $y=24$, we find that

$$[15,18,20,21:50] - [12,21,23,24:50] = q^{12}[3,6,8,9:50]. \tag{8.3}$$

Similarly, with the choice of parameters $a=1$, $b=3$, $c=14$, $x=20$, and $y=23$, we deduce that

$$[13,17,20,22:50] - [11,19,22,24:50] = q^{11}[2,6,9,11:50]. \tag{8.4}$$

Equation (8.3) together with (8.4) clearly implies (8.2) and so the proof of (i) is complete. □

Theorem 8.4.

(i) *Let* $S \equiv \pm\{1,3,5,7,11,12,13,16,17,18,22,23\} \pmod{50}$, *and* $T \equiv \pm\{1,2,5,9,11,12,15,16,17,21,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,2,3,5,6,13,14,15,16,17,19,23\} \pmod{50}$, *and* $T \equiv \pm\{1,2,3,4,9,11,14,15,16,17,19,21\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 4$.

(iii) *Let* $S \equiv \pm\{3,4,5,7,11,12,13,14,15,16,19,23\} \pmod{50}$, *and* $T \equiv \pm\{1,4,7,9,11,12,15,16,19,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(iv) *Let* $S \equiv \pm\{1,2,3,5,6,7,8,9,11,15,18,19\} \pmod{50}$, *and* $T \equiv \pm\{1,2,3,5,6,7,8,9,12,13,17,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 11$.

(v) *Let* $S \equiv \pm\{2,3,5,7,8,11,13,17,18,21,23,24\} \pmod{50}$, *and* $T \equiv \pm\{1,3,5,8,11,13,15,18,19,21,22,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(vi) *Let* $S \equiv \pm\{1,6,7,8,9,11,13,14,15,16,19,21\} \pmod{50}$, *and* $T \equiv \pm\{1,5,6,7,8,13,14,15,17,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(vii) *Let* $S \equiv \pm\{3,4,5,7,9,11,13,15,16,17,22,24\} \pmod{50}$, *and* $T \equiv \pm\{1,4,6,9,11,13,15,17,19,21,22,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(viii) *Let* $S \equiv \pm\{3,4,5,7,8,9,13,17,18,19,22,23\} \pmod{50}$, *and* $T \equiv \pm\{1,4,5,9,12,13,15,18,19,21,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(ix) *Let* $S \equiv \pm\{2,5,7,8,9,11,12,14,15,17,19,21\} \pmod{50}$, *and* $T \equiv \pm\{2,3,5,7,12,13,14,17,19,21,22,23\} \pmod{50}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(x) *Let* $S \equiv \pm\{3,4,5,6,7,9,15,17,18,21,23,24\} \pmod{50}$, *and* $T \equiv \pm\{1,3,6,9,11,14,15,18,19,21,23,24\} \pmod{50}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

Proof. We prove (i) by starting to write it in its equivalent form

$$\begin{aligned}&[2,9,15,21:50] - q[3,7,13,18:50]\\ &= [1,2,3,5,7,9,11,12,13,15,16,17,18,21,22,23:50].\end{aligned} \tag{8.5}$$

As in the previous proof, we use the first identity in Theorem 8.1 which we express in the form

$$\begin{aligned}&[2,12,21,23:50] - q[3,8,19,22:50]\\ &= [1,2,3,5,7,8,12,13,15,17,18,19,20,21,22,23:50].\end{aligned}$$

Upon comparing these last two equations, we see that it suffices to show

$$\begin{aligned}&[8,19,20:50]\big\{[2,9,15,21:50] - q[3,7,13,18:50]\big\}\\ =&[9,11,16:50]\big\{[2,12,21,23:50] - q[3,8,19,22:50]\big\},\end{aligned}$$

which after rearrangement takes the form

$$\begin{aligned}&[2,9,21:50]\big\{[11,12,16,23:50]-[8,15,19,20:50]\big\}\\&=q[3,8,19:50]\big\{[9,11,16,22:50]-[7,13,18,20:50]\big\}.\end{aligned}\tag{8.6}$$

We employ (2.2) twice with the sets of parameters $[1,4,12,16,20]$ and $[1,3,10,14,19]$ and with q replaced by q^{50} in both instances, we find that

$$[11,12,16,23:50]-[8,15,19,20:50]=q^8[3,4,8,19:50] \text{ and}$$

$$[9,11,16,22:50]-[7,13,18,20:50]=q^7[2,4,9,21:50].$$

These two equations clearly imply (8.6) and so the proof of (i) is complete. □

9. Modulus $M=52$

There is only one class of identities under multiplication by the group $U(52)$.

Theorem 9.1.

(i) *Let* $S\equiv\pm\{1,3,4,5,7,16,18,19,21,22,23,25\} \pmod{52}$, *and* $T\equiv\pm\{1,2,3,7,11,15,18,19,20,23,24,25\} \pmod{52}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(ii) *Let* $S\equiv\pm\{2,3,4,5,9,11,12,14,15,17,21,23\} \pmod{52}$, *and* $T\equiv\pm\{2,3,5,6,7,8,9,17,19,20,21,23\} \pmod{52}$. *Then* $p(S,n)=p(T,n)$, *for all* $n\neq 4$.

(iii) *Let* $S\equiv\pm\{3,4,5,9,10,11,14,15,16,17,21,23\} \pmod{52}$, *and* $T\equiv\pm\{1,5,6,9,11,14,15,17,20,21,24,25\} \pmod{52}$. *Then* $p(S,n)=p(T,n-3)$, *for all* $n\geq 3$.

(iv) *Let* $S\equiv\pm\{2,3,5,7,8,9,17,19,21,22,23,24\} \pmod{52}$, *and* $T\equiv\pm\{1,3,5,7,12,14,16,19,21,22,23,25\} \pmod{52}$. *Then* $p(S,n)=p(T,n-2)$, *for all* $n\geq 2$.

(v) *Let* $S\equiv\pm\{1,6,7,9,10,11,12,15,16,17,19,25\} \pmod{52}$, *and* $T\equiv\pm\{1,5,6,8,9,11,15,17,18,21,24,25\} \pmod{52}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(vi) *Let* $S\equiv\pm\{1,4,7,9,10,11,12,15,17,19,22,25\} \pmod{52}$, *and* $T\equiv\pm\{1,3,7,8,10,11,15,18,19,20,23,25\} \pmod{52}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

Proof. The identities (i)—(vi) follow from (2.5) with the following choice of parameters

$$[1, 2, 4, 12, 23],\ [1, 3, 7, 10, 12],\ [1, 5, 2, 7, 11],\ [1, 4, 2, 6, 9],\ [1, 3, 2, 8, 11],\ [1, 2, 5, 19, 20]$$

and with q replaced by q^{26} in each instance. □

10. Modulus $M = 54$

There are four classes of identities under multiplication by the group $U(54)$.

Theorem 10.1.

(i) *Let* $S \equiv \pm\{1, 4, 5, 7, 11, 13, 16, 17, 19, 20, 22, 23\} \pmod{54}$, *and* $T \equiv \pm\{1, 3, 4, 10, 11, 15, 16, 17, 19, 21, 23, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1, 2, 5, 7, 8, 11, 13, 19, 20, 23, 25, 26\} \pmod{54}$, *and* $T \equiv \pm\{1, 3, 4, 5, 7, 13, 15, 20, 21, 22, 23, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 2$.

(iii) *Let* $S \equiv \pm\{1, 4, 5, 7, 8, 11, 17, 19, 22, 23, 25, 26\} \pmod{54}$, *and* $T \equiv \pm\{1, 3, 4, 7, 11, 15, 16, 20, 21, 23, 25, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(iv) *Let* $S \equiv \pm\{2, 3, 7, 10, 11, 13, 14, 15, 16, 17, 21, 25\} \pmod{54}$, *and* $T \equiv \pm\{1, 4, 7, 10, 11, 13, 14, 17, 19, 23, 25, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n-2)$, *for all* $n \geq 2$.

(v) *Let* $S \equiv \pm\{2, 5, 7, 8, 10, 11, 13, 16, 17, 19, 23, 25\} \pmod{54}$, *and* $T \equiv \pm\{2, 3, 5, 8, 13, 14, 15, 19, 21, 22, 23, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n-2)$, *for all* $n \geq 2$.

(vi) *Let* $S \equiv \pm\{1, 2, 4, 5, 11, 13, 14, 16, 17, 19, 23, 25\} \pmod{54}$, *and* $T \equiv \pm\{1, 2, 3, 8, 10, 13, 14, 15, 17, 19, 21, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 3$.

(vii) *Let* $S \equiv \pm\{1, 3, 5, 7, 8, 15, 17, 19, 20, 21, 22, 26\} \pmod{54}$, *and* $T \equiv \pm\{1, 2, 5, 7, 13, 14, 17, 19, 20, 22, 23, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(viii) *Let* $S \equiv \pm\{3, 4, 5, 10, 11, 13, 14, 15, 16, 17, 21, 23\} \pmod{54}$, *and* $T \equiv \pm\{1, 5, 7, 10, 11, 13, 16, 17, 20, 23, 25, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n-3)$, *for all* $n \geq 3$.

(ix) *Let* $S \equiv \pm\{1, 5, 7, 8, 10, 11, 13, 14, 17, 19, 22, 25\} \pmod{54}$, *and* $T \equiv \pm\{2, 3, 5, 7, 8, 11, 15, 19, 20, 21, 22, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 1$.

Proof. The identities (i)—(ix) follow from (2.5) with the choice of parameters $[1, 2, 5, 16, 22]$, $[1, 2, 4, 8, 9]$, $[1, 3, 2, 6, 9]$, $[1, 3, 4, 18, 20]$, $[1, 5, 3, 8, 11]$, $[1, 2, 5, 7, 13]$, $[1, 2, 4, 14, 23]$, $[1, 5, 2, 7, 12]$, $[1, 3, 4, 9, 11]$ and with q replaced by q^{27} in each instance. □

Theorem 10.2.

(i) *Let* $S \equiv \pm\{1, 5, 6, 9, 10, 11, 13, 14, 21, 22, 23, 25\} \pmod{54}$, *and* $T \equiv \pm\{1, 4, 5, 9, 10, 11, 17, 19, 21, 22, 23, 24\} \pmod{54}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1, 2, 3, 4, 5, 7, 9, 12, 13, 20, 23, 25\} \pmod{54}$, *and* $T \equiv \pm\{1, 2, 3, 4, 5, 7, 9, 11, 16, 17, 24, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 11$.

(iii) *Let* $S \equiv \pm\{1, 7, 8, 9, 10, 12, 13, 15, 16, 17, 19, 23\} \pmod{54}$, *and* $T \equiv \pm\{1, 6, 7, 8, 9, 11, 15, 16, 19, 23, 25, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(iv) *Let* $S \equiv \pm\{1, 2, 6, 7, 9, 10, 11, 13, 15, 17, 25, 26\} \pmod{54}$, *and* $T \equiv \pm\{1, 2, 5, 8, 9, 11, 12, 13, 15, 17, 19, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 5$.

(v) *Let* $S \equiv \pm\{2, 3, 5, 9, 11, 12, 13, 16, 19, 22, 23, 25\} \pmod{54}$, *and* $T \equiv \pm\{1, 3, 7, 9, 11, 13, 16, 19, 20, 22, 24, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n-2)$, *for all* $n \geq 2$.

(vi) *Let* $S \equiv \pm\{4, 5, 6, 7, 8, 9, 13, 17, 21, 22, 23, 25\} \pmod{54}$, *and* $T \equiv \pm\{1, 4, 8, 9, 13, 14, 17, 19, 21, 23, 24, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n-4)$, *for all* $n \geq 4$.

(vii) *Let* $S \equiv \pm\{4, 5, 6, 7, 9, 11, 13, 14, 19, 21, 23, 26\} \pmod{54}$, *and* $T \equiv \pm\{1, 5, 7, 9, 13, 14, 17, 19, 21, 22, 24, 26\} \pmod{54}$. *Then* $p(S, n) = p(T, n-4)$, *for all* $n \geq 4$.

(viii) *Let* $S \equiv \pm\{3, 5, 7, 9, 11, 12, 13, 14, 16, 17, 20, 23\} \pmod{54}$, *and* $T \equiv \pm\{2, 3, 7, 9, 11, 14, 17, 19, 20, 23, 24, 25\} \pmod{54}$. *Then* $p(S, n) = p(T, n-3)$, *for all* $n \geq 3$.

(ix) *Let* $S \equiv \pm\{5,6,7,8,9,10,11,15,17,19,20,25\} \pmod{54}$, *and* $T \equiv \pm\{1,5,9,10,12,15,17,19,20,23,25,26\} \pmod{54}$. *Then* $p(S,n) = p(T,n-5)$, *for all* $n \geq 5$.

Proof. We prove (ii). We apply (2.8) with q replaced by q^9 and x is replaced by q^2, we find that

$$\begin{aligned}&[3,24,24:54] - q^2[15,12,6:54]\\ &= [2,3,5,7,9,11,12,13,15,16,18,20,23,24,25:54].\end{aligned} \tag{10.1}$$

We reformulate the identity in (ii) in its equivalent form

$$\begin{aligned}&[12,13,20,23:54] - [11,16,17,24:54]\\ &= q^{11}[1,2,3,4,5,7,9,11,12,13,16,17,20,23,24,25:54].\end{aligned} \tag{10.2}$$

By (10.1), the equation (10.2) is equivalent to

$$\begin{aligned}&q^{11}[1,4,17]\big\{[3,24,24:54] - q^2[6,12,15:54]\big\}\\ &= [15,18]\big\{[12,13,20,23:54] - [11,16,17,24:54]\big\}.\end{aligned} \tag{10.3}$$

After regrouping the the terms in (10.3), we are led to prove

$$\begin{aligned}&[17,24:54]\big\{[11,15,16,18:54] + q^{11}[1,3,4,24:54]\big\}\\ &= [15,12:54]\big\{[18,13,20,23:54] + q^{13}[1,4,6,17:54]\big\}.\end{aligned} \tag{10.4}$$

Next, we apply (2.2) twice with the sets of parameters $[1,2,13,16,17]$, $[1,2,15,19,21]$ and with q replaced by q^{54} in each instance to find that

$$\begin{aligned}&[11,15,16,18:54] + q^{11}[1,3,4,24:54] = [12,14,15,19:54] \text{ and}\\ &[18,13,20,23] + q^{13}[1,4,6,17:54] = [14,17,19,24].\end{aligned}$$

These two identities clearly imply (10.4). Hence, the proof of (ii) is complete. ☐

Theorem 10.3.

(i) *Let* $S \equiv \pm\{1,3,5,7,11,13,15,16,20,24,25,26\} \pmod{54}$, *and* $T \equiv \pm\{1,2,5,9,12,13,15,19,20,23,25,26\} \pmod{54}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{5,6,7,8,9,10,11,13,17,21,22,25\} \pmod{54}$, *and* $T \equiv \pm\{1,5,8,11,12,15,17,19,21,22,25,26\} \pmod{54}$. *Then* $p(S,n) = p(T,n-5)$, *for all* $n \geq 5$.

(iii) *Let* $S \equiv \pm\{3,4,5,6,7,13,17,19,20,21,22,23\} \pmod{54}$, *and* $T \equiv \pm\{1,3,7,9,13,14,17,19,20,22,24,25\} \pmod{54}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(iv) *Let* $S \equiv \pm\{1,3,4,5,7,9,11,16,17,19,22,24\} \pmod{54}$, *and* $T \equiv \pm\{1,3,4,5,6,11,13,14,16,19,21,23\} \pmod{54}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 6$.

(v) *Let* $S \equiv \pm\{1,7,8,10,11,12,13,14,15,17,19,21\} \pmod{54}$, *and* $T \equiv \pm\{1,6,7,9,10,11,13,14,21,23,25,26\} \pmod{54}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(vi) *Let* $S \equiv \pm\{1,5,7,9,10,12,13,15,16,17,20,23\} \pmod{54}$, *and* $T \equiv \pm\{2,3,5,7,10,11,15,16,17,23,24,25\} \pmod{54}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 1$.

(vii) *Let* $S \equiv \pm\{2,5,8,9,11,12,13,15,16,17,19,23\} \pmod{54}$, *and* $T \equiv \pm\{2,3,7,8,11,13,15,19,20,23,24,25\} \pmod{54}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(viii) *Let* $S \equiv \pm\{4,5,6,7,8,9,11,19,21,23,25,26\} \pmod{54}$, *and* $T \equiv \pm\{1,4,7,10,12,15,17,19,21,23,25,26\} \pmod{54}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(ix) *Let* $S \equiv \pm\{1,2,3,4,9,11,14,17,19,23,24,25\} \pmod{54}$, *and* $T \equiv \pm\{1,2,3,5,6,13,14,17,21,22,23,25\} \pmod{54}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 4$.

Proof. We prove (i) by using the same identity (10.1) that we employed in our previous proof namely,

$$\begin{aligned}&[3,24,24:54] - q^2[15,12,6:54]\\ &= [2,3,5,7,9,11,12,13,15,16,18,20,23,24,25:54]. \end{aligned} \tag{10.5}$$

We start by writing (i) in its equivalent form

$$\begin{aligned}&[2,9,12,19,23:54] - q[3,7,11,16,24:54]\\ &= [1,2,3,5,7,9,11,12,13,15,16,19,20,23,24,25,26:54]. \end{aligned} \tag{10.6}$$

By (10.5), we see that (10.6) is equivalent to

$$\begin{aligned}&[18:54]\big\{[2,9,12,19,23:54] - q[3,7,11,16,24:54]\big\}\\ &= [1,19,26:54]\big\{[3,24,24:54] - q^2[6,12,15:54]\big\}. \end{aligned} \tag{10.7}$$

After rearrangement of the terms in above sum, we are led to prove

$$\begin{aligned}&[3,24:54]\big\{[1,19,24,26:54] + q[7,11,16,18:54]\big\}\\ &= [12,19:54]\big\{[2,9,18,23:54] + q^2[1,6,15,26:54]\big\}. \end{aligned} \tag{10.8}$$

The equation (10.8) follows from the pair of identities

$$[1,19,24,26:54]+q[7,11,16,18:54]=[8,12,17,19:54] \text{ and}$$
$$[2,9,18,23:54]+q^2[1,6,15,26:54]=[3,8,17,24:54],$$

which are obtained by (2.2) by the choice of parameters $[1,8,9,20,25]$, $[1,2,4,10,19]$ and q replaced by q^{54} in each instance. □

Theorem 10.4.

(i) *Let* $S \equiv \pm\{1,3,5,7,11,12,13,20,22,23,24,25\} \pmod{54}$, *and* $T \equiv \pm\{1,2,5,9,11,12,17,19,22,23,24,25\} \pmod{54}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,2,5,6,7,9,10,12,13,17,23,25\} \pmod{54}$, *and* $T \equiv \pm\{1,2,5,6,7,8,11,12,15,17,19,25\} \pmod{54}$. *Then* $p(S,n)=p(T,n)$, *for all* $n \neq 8$.

(iii) *Let* $S \equiv \pm\{1,6,7,8,9,11,13,14,19,23,24,25\} \pmod{54}$, *and* $T \equiv \pm\{1,5,6,7,8,13,17,19,21,22,23,24\} \pmod{54}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n \geq 1$.

(iv) *Let* $S \equiv \pm\{1,5,6,7,9,11,13,17,22,24,25,26\} \pmod{54}$, *and* $T \equiv \pm\{1,4,5,6,11,13,17,19,21,23,24,26\} \pmod{54}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n \geq 1$.

(v) *Let* $S \equiv \pm\{1,6,7,10,11,12,13,15,16,17,19,25\} \pmod{54}$, *and* $T \equiv \pm\{1,5,6,9,11,12,13,16,19,23,25,26\} \pmod{54}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n \geq 1$.

(vi) *Let* $S \equiv \pm\{3,4,5,7,11,12,13,16,17,23,24,25\} \pmod{54}$, *and* $T \equiv \pm\{1,4,7,9,12,13,17,19,20,23,24,25\} \pmod{54}$. *Then* $p(S,n)=p(T,n-3)$, *for all* $n \geq 3$.

(vii) *Let* $S \equiv \pm\{1,5,7,9,11,12,13,14,16,17,19,24\} \pmod{54}$, *and* $T \equiv \pm\{2,3,5,7,11,12,13,14,19,23,24,25\} \pmod{54}$. *Then* $p(S,n)=p(T,n)$, *for all* $n \neq 1$.

(viii) *Let* $S \equiv \pm\{5,6,7,8,9,11,12,13,17,19,20,23\} \pmod{54}$, *and* $T \equiv \pm\{1,6,7,11,12,15,17,19,20,23,25,26\} \pmod{54}$. *Then* $p(S,n)=p(T,n-5)$, *for all* $n \geq 5$.

(ix) *Let* $S \equiv \pm\{4,5,6,7,9,10,11,17,19,23,24,25\} \pmod{54}$, *and* $T \equiv \pm\{1,5,6,10,13,14,17,19,21,23,24,25\} \pmod{54}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[2,9,17,19:54] - q[3,7,13,20:54]\\ &= [1,2,3,5,7,9,11,12,13,17,19,20,22,23,24,25:54].\end{aligned} \tag{10.9}$$

By (10.2)

$$\begin{aligned}&[12,13,20,23:54] - [11,16,17,24:54]\\ &= q^{11}[1,2,3,4,5,7,9,11,12,13,16,17,20,23,24,25:54].\end{aligned} \tag{10.10}$$

By (10.10), the equation (10.9) is equivalent to

$$\begin{aligned}&q^{11}[4,16:54]\big\{[2,9,17,19:54] - q[3,7,13,20:54]\big\}\\ &= [19,22:54]\big\{[12,13,20,23:54] - [11,16,17,24:54]\big\}.\end{aligned} \tag{10.11}$$

After regrouping the terms in (10.11), we are led to prove

$$\begin{aligned}&[13,20:54]\big\{[12,19,22,23:54] + q^{12}[3,4,7,16:54]\big\}\\ &= [19,16:54]\big\{[11,17,22,24] + q^{11}[2,4,9,17:54]\big\}.\end{aligned} \tag{10.12}$$

Next, we apply (2.2) twice with the sets of parameters $[1,4,16,20,23]$, $[1,3,14,18,23]$ and with q replaced by q^{54} in each instance to find that

$$\begin{aligned}&[12,19,22,23:54] + q^{12}[3,4,7,16:54] = [15,16,19,26:54] \text{ and}\\ &[11,17,22,24] + q^{11}[2,4,9,17:54] = [13,15,20,26:54].\end{aligned}$$

These two identities readily imply (10.12). Hence, the proof of (i) is complete. $\square$

11. Modulus $M = 56$

There is only one class of identities under multiplication by the group $U(56)$.

Theorem 11.1.

(i) *Let* $S \equiv \pm\{1,3,4,5,7,18,21,22,23,24,25,27\} \pmod{56}$, *and* $T \equiv \pm\{1,2,3,7,11,17,20,21,24,25,26,27\} \pmod{56}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,7,9,10,12,13,15,16,19,21,25\} \pmod{56}$, *and* $T \equiv \pm\{3,4,5,6,7,9,16,19,21,22,23,25\} \pmod{56}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iii) *Let* $S \equiv \pm\{2,3,5,7,8,13,15,20,21,22,23,25\} \pmod{56}$, *and* $T \equiv \pm\{1,5,7,8,10,12,13,15,18,21,23,27\} \pmod{56}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 1$.

(iv) *Let* $S \equiv \pm\{1,7,8,9,10,12,13,15,18,19,21,27\} \pmod{56}$, *and* $T \equiv \pm\{1,6,7,8,9,11,17,19,20,21,26,27\} \pmod{56}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(v) *Let* $S \equiv \pm\{4,5,6,7,9,11,16,17,19,21,22,23\} \pmod{56}$, *and* $T \equiv \pm\{1,5,7,11,12,16,17,18,21,23,26,27\} \pmod{56}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(vi) *Let* $S \equiv \pm\{4,6,7,9,10,11,13,15,17,19,21,24\} \pmod{56}$, *and* $T \equiv \pm\{2,3,7,11,13,15,17,20,21,24,25,26\} \pmod{56}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

Proof. The identities (i)—(vi) follow from (2.5) with the following choice of parameters

$$[1,2,4,12,25],\ [1,4,6,10,13],\ [1,3,4,9,11],\ [1,3,2,9,11],\ [1,5,6,17,24],\ [1,7,3,10,11]$$

and with q replaced by q^{28} in each instance. □

12. Modulus $M = 60$

There are eight classes of identities under multiplication by the group $U(60)$. The first two identities in Theorem 12.3 below were first given by Kalvade [6].

Theorem 12.1.

(i) *Let* $S \equiv \pm\{1,3,4,5,7,22,23,24,25,26,27,29\} \pmod{60}$, *and* $T \equiv \pm\{1,2,3,7,11,19,23,24,26,27,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,7,9,11,12,13,14,16,17,19,21,23\} \pmod{60}$, *and* $T \equiv \pm\{2,5,7,9,11,12,19,21,23,25,26,28\} \pmod{60}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iii) *Let* $S \equiv \pm\{2,3,5,11,13,14,16,17,19,24,25,27\} \pmod{60}$, *and* $T \equiv \pm\{1,3,8,11,13,14,17,19,22,24,27,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iv) *Let* $S \equiv \pm\{1, 5, 8, 9, 12, 13, 14, 17, 21, 22, 25, 29\} \pmod{60}$, *and* $T \equiv \pm\{1, 4, 7, 9, 12, 13, 17, 21, 22, 23, 26, 29\} \pmod{60}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

Proof. The identities (i)–(iv) follow from (2.5) with the following sets of parameters

$$[1, 2, 4, 12, 27],\ [1, 5, 3, 10, 14],\ [1, 3, 4, 17, 23],\ [1, 2, 6, 22, 23],$$

and with q replaced by q^{30} in each instance. □

Theorem 12.2.

(i) *Let* $S \equiv \pm\{1, 5, 6, 8, 9, 13, 17, 21, 22, 25, 28, 29\} \pmod{60}$, *and* $T \equiv \pm\{1, 4, 5, 7, 13, 16, 17, 22, 23, 25, 26, 29\} \pmod{60}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1, 2, 5, 7, 8, 11, 19, 23, 25, 26, 28, 29\} \pmod{60}$, *and* $T \equiv \pm\{1, 3, 4, 5, 7, 16, 18, 23, 25, 26, 27, 29\} \pmod{60}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 2$.

(iii) *Let* $S \equiv \pm\{2, 4, 5, 7, 11, 13, 14, 16, 17, 19, 23, 25\} \pmod{60}$, *and* $T \equiv \pm\{2, 5, 6, 7, 8, 9, 11, 19, 21, 23, 25, 28\} \pmod{60}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 4$.

(iv) *Let* $S \equiv \pm\{3, 4, 5, 11, 13, 14, 16, 17, 18, 19, 25, 27\} \pmod{60}$, *and* $T \equiv \pm\{1, 5, 8, 11, 13, 14, 17, 19, 22, 25, 28, 29\} \pmod{60}$. *Then* $p(S, n) = p(T, n-3)$, *for all* $n \geq 3$.

Proof. The identities (i)–(iv) follow from (2.5) with the following sets of parameters

$$[1, 3, 2, 7, 10],\ [1, 2, 4, 8, 9],\ [1, 3, 7, 12, 14],\ [1, 5, 2, 7, 13],$$

and with q replaced by q^{30} in each instance. □

Theorem 12.3.

(i) *Let* $S \equiv \pm\{1, 6, 7, 11, 12, 13, 16, 17, 18, 19, 23, 29\} \pmod{60}$, *and* $T \equiv \pm\{1, 5, 6, 11, 12, 13, 17, 18, 19, 25, 28, 29\} \pmod{60}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1, 6, 7, 8, 11, 13, 17, 18, 19, 23, 24, 29\} \pmod{60}$, *and* $T \equiv \pm\{1, 5, 6, 7, 13, 16, 17, 18, 23, 24, 25, 29\} \pmod{60}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(iii) *Let* $S \equiv \pm\{1,5,6,7,8,11,12,18,19,23,25,29\} \pmod{60}$, *and* $T \equiv \pm\{1,4,6,7,11,12,13,17,18,19,23,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 4$.

(iv) *Let* $S \equiv \pm\{4,5,6,7,11,13,17,18,19,23,24,25\} \pmod{60}$, *and* $T \equiv \pm\{1,6,7,11,13,17,18,19,23,24,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

Proof. The identities (i)–(iv) follow from (2.5) with the following sets of parameters

$$[1,3,2,8,13],\ [1,2,7,21,24],\ [1,2,6,12,13],\ [1,5,6,18,25],$$

and with q replaced by q^{30} in each instance. □

Theorem 12.4.

(i) *Let* $S \equiv \pm\{1,5,6,9,13,14,16,17,20,21,25,29\} \pmod{60}$, *and* $T \equiv \pm\{1,4,5,11,13,14,17,19,20,25,26,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,3,5,7,8,18,20,22,23,25,27,29\} \pmod{60}$, *and* $T \equiv \pm\{1,2,5,7,13,17,20,22,23,25,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(iii) *Let* $S \equiv \pm\{4,5,6,7,9,11,19,20,21,23,25,26\} \pmod{60}$, *and* $T \equiv \pm\{1,5,7,11,14,16,19,20,23,25,26,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(iv) *Let* $S \equiv \pm\{2,5,7,8,11,13,17,19,20,22,23,25\} \pmod{60}$, *and* $T \equiv \pm\{2,3,5,11,13,17,18,19,20,25,27,28\} \pmod{60}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i)–(iv) follow from (2.5) with the following sets of parameters

$$[1,2,6,20,23],\ [1,2,4,14,26],\ [1,5,6,17,26],\ [1,3,6,19,25],$$

and with q replaced by q^{30} in each instance. □

Theorem 12.5.

(i) *Let* $S \equiv \pm\{1,7,8,10,11,13,14,17,19,20,23,29\} \pmod{60}$, *and* $T \equiv \pm\{1,6,7,9,10,13,17,20,21,23,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1,4,7,10,11,13,17,19,20,22,23,29\} \pmod{60}$, *and* $T \equiv \pm\{1,3,7,10,11,16,18,19,20,23,27,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(iii) *Let* $S \equiv \pm\{6,7,8,9,10,11,13,17,19,20,21,23\} \pmod{60}$, *and* $T \equiv \pm\{1,7,10,11,13,17,19,20,23,26,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-6)$, *for all* $n \geq 6$.

(iv) *Let* $S \equiv \pm\{1,3,4,10,11,13,17,18,19,20,27,29\} \pmod{60}$, *and* $T \equiv \pm\{1,2,7,10,11,13,16,17,19,20,23,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

Proof. The identities (i)–(iv) follow from (2.5) with the following sets of parameters

$$[1,3,2,9,12],\ [1,2,5,19,24],\ [1,8,2,10,12],\ [1,2,4,8,14],$$

and with q replaced by q^{30} in each instance. □

Theorem 12.6.

(i) *Let* $S \equiv \pm\{1,8,10,11,12,13,14,15,17,19,21,23\} \pmod{60}$, *and* $T \equiv \pm\{1,7,9,10,11,12,13,15,23,26,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,7,10,11,15,16,17,19,23,24,29\} \pmod{60}$, *and* $T \equiv \pm\{1,4,7,10,13,15,17,19,22,24,27,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iii) *Let* $S \equiv \pm\{1,3,4,10,11,13,15,17,22,23,24,29\} \pmod{60}$, *and* $T \equiv \pm\{1,2,7,10,11,13,15,16,19,23,24,27\} \pmod{60}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iv) *Let* $S \equiv \pm\{7,8,9,10,11,12,13,14,15,17,19,29\} \pmod{60}$, *and* $T \equiv \pm\{1,7,10,12,15,17,19,21,23,26,28,29\} \pmod{60}$. *Then* $p(S,n) = p(T,n-7)$, *for all* $n \geq 7$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[7,9,26,28,29:60] - q[8,14,17,19,21:60]\\ &= [1,7,8,9,10,11,12,13,14,15,17,19,21,23,26,28,29:60].\end{aligned} \tag{12.1}$$

Similarly by the first part of Theorem (12.5), we have

$$\begin{aligned}&[6,9,21,28:60] - q[8,11,14,19:60]\\ &= [1,6,7,8,9,10,11,13,14,17,19,20,21,23,28,29:60].\end{aligned} \tag{12.2}$$

By (12.2), the equation (12.1) is equivalent to

$$\begin{aligned}&[6,20:60]\big\{[7,9,26,28,29:60]-q[8,14,17,19,21:60]\big\}\\&=[12,15,26:60]\big\{[6,9,21,28:60]-q[8,11,14,19:60]\big\}.\end{aligned}\tag{12.3}$$

After regrouping the the terms in (12.3), we are led to prove

$$\begin{aligned}&[6,9,28:60]\big\{[7,20,26,29:60]-[12,15,21,26:60]\big\}\\&=q[8,14,19:60]\big\{[6,17,20,21:60]-[11,12,15,26:60]\big\}.\end{aligned}\tag{12.4}$$

Next, we apply (2.2) twice with the sets of parameters $[1,6,13,21,27]$, $[1,6,12,18,21]$ and with q replaced by q^{60} in each instance to find that

$$\begin{aligned}&[7,20,26,29:60]-[12,15,21,26:60]=-q^7[5,8,14,19:60]\text{ and}\\&[6,17,20,21:60]-[11,12,15,26:60]=-q^6[5,6,9,28:60].\end{aligned}$$

These two identities readily imply (12.4). Hence, the proof of (i) is complete. □

Theorem 12.7.

(i) *Let* $S\equiv\pm\{1,4,7,10,11,13,17,18,19,24,25,27\}\pmod{60}$, *and* $T\equiv\pm\{1,3,7,10,11,16,17,18,23,24,25,29\}\pmod{60}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(ii) *Let* $S\equiv\pm\{1,5,6,7,8,10,11,12,17,19,21,23\}\pmod{60}$, *and* $T\equiv\pm\{1,5,6,7,9,10,11,12,13,17,28,29\}\pmod{60}$. *Then* $p(S,n)=p(T,n)$, *for all* $n\neq 8$.

(iii) *Let* $S\equiv\pm\{6,7,8,9,10,11,12,13,19,23,25,29\}\pmod{60}$, *and* $T\equiv\pm\{1,6,10,12,13,17,19,21,23,25,28,29\}\pmod{60}$. *Then* $p(S,n)=p(T,n-6)$, *for all* $n\geq 6$.

(iv) *Let* $S\equiv\pm\{3,4,5,10,11,13,17,18,19,23,24,29\}\pmod{60}$, *and* $T\equiv\pm\{1,5,7,10,13,16,18,19,23,24,27,29\}\pmod{60}$. *Then* $p(S,n)=p(T,n-3)$, *for all* $n\geq 3$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[3,16,23,29:60]-q[4,13,19,27:60]\\&=[1,3,4,7,10,11,13,16,17,18,19,23,24,25,27,29:60].\end{aligned}\tag{12.5}$$

Similarly by the second part of Theorem (12.5), we have

$$\begin{aligned}&[3,16,18,27:60]-q[4,13,17,22:60]\\&=[1,3,4,7,10,11,13,16,17,18,19,20,22,23,27,29:60].\end{aligned}\tag{12.6}$$

Therefore, by (12.6), it suffices to prove that

$$\begin{aligned}&[20,22:60]\big\{[3,16,23,29:60]-q[4,13,19,27:60]\big\}\\&=[24,25:60]\big\{[3,16,18,27:60]-q[4,13,17,22:60]\big\}.\end{aligned}\tag{12.7}$$

After regrouping the the terms in (12.7), we are led to prove

$$\begin{aligned}&[3,16:60]\big\{[20,22,23,29:60]-[18,24,25,27:60]\big\}\\&=-q[4,13:60]\big\{[17,22,24,25:60]-[19,20,22,27:60]\big\}.\end{aligned}\tag{12.8}$$

Next, we apply (2.2) twice with the sets of parameters $[1,3,21,25,26]$, $[1,3,20,23,25]$ and with q replaced by q^{60} in each instance to find that

$$\begin{aligned}&[20,22,23,29:60]-[18,24,25,27:60]=q^{18}[2,4,5,13:60]\text{ and}\\&[17,22,24,25:60]-[19,20,22,27:60]=-q^{17}[2,3,5,16:60].\end{aligned}$$

From these two identities (12.8) trivially follows and so the proof of (i) is complete. □

Theorem 12.8.

(i) *Let* $S\equiv\pm\{1,6,7,8,11,13,18,19,20,21,23,25\} \pmod{60}$, *and* $T\equiv\pm\{1,5,6,7,13,17,18,19,20,21,28,29\} \pmod{60}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(ii) *Let* $S\equiv\pm\{4,5,6,7,11,13,17,18,19,20,27,29\} \pmod{60}$, *and* $T\equiv\pm\{1,6,7,11,13,16,18,20,23,25,27,29\} \pmod{60}$. *Then* $p(S,n)=p(T,n-4)$, *for all* $n\geq 4$.

(iii) *Let* $S\equiv\pm\{5,6,7,8,9,11,17,18,19,20,23,29\} \pmod{60}$, *and* $T\equiv\pm\{1,6,9,11,13,17,18,20,23,25,28,29\} \pmod{60}$. *Then* $p(S,n)=p(T,n-5)$, *for all* $n\geq 5$.

(iv) *Let* $S\equiv\pm\{1,3,4,6,11,13,17,18,19,20,23,25\} \pmod{60}$, *and* $T\equiv\pm\{1,3,5,6,7,16,17,18,19,20,23,29\} \pmod{60}$. *Then* $p(S,n)=p(T,n)$, *for all* $n\neq 4$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[5,17,28,29:60]-q[8,11,23,25:60]\\&=[1,5,6,7,8,11,13,17,18,19,20,21,23,25,28,29:60].\end{aligned}\tag{12.9}$$

Similarly by the second part of Theorem (12.3), we have

$$\begin{aligned}&[5,16,25:60]-q[8,11,19:60]\\&=[1,5,6,7,8,11,13,16,17,18,19,23,24,25,29:60].\end{aligned}\tag{12.10}$$

Therefore, we need to prove that

$$\begin{aligned}&[20,21,28:60]\big\{[5,16,25:60]-q[8,11,19:60]\big\}\\&=[16,24:60]\big\{[5,17,28,29:60]-q[8,11,23,25:60]\big\}.\end{aligned}\tag{12.11}$$

After regrouping the the terms in (12.11), we are led to prove

$$\begin{aligned}&[5,16:60]\big\{[20,21,25,28:60]-[17,24,28,29:60]\big\}\\&=-q[8,11:60]\big\{[16,23,24,25:60]-[19,20,21,28:60]\big\}.\end{aligned}\tag{12.12}$$

Next, we apply (2.2) twice with the sets of parameters $[1,4,21,25,29]$, $[1,4,20,24,25]$ and with q replaced by q^{60} in each instance to find that

$$[20,21,25,28:60]-[17,24,28,29:60]=q^{17}[3,4,8,11:60] \text{ and}$$
$$[16,23,24,25:60]-[19,20,21,28:60]=-q^{16}[3,4,5,16:60].$$

From these two identities (12.12) trivially follows and so the proof of (i) is complete. $\square$

13. Modulus $M=62$

There is only one class of identities under multiplication by the group $U(62)$.

Theorem 13.1.

(i) *Let* $S\equiv\pm\{1,6,7,11,12,15,16,17,19,20,25,29\} \pmod{62}$, *and* $T\equiv\pm\{1,5,6,11,13,14,17,18,21,25,29,30\} \pmod{62}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(ii) *Let* $S\equiv\pm\{2,3,5,11,13,14,17,18,21,25,26,29\} \pmod{62}$, *and* $T\equiv\pm\{1,3,8,11,13,15,18,20,23,25,28,29\} \pmod{62}$. *Then* $p(S,n)=p(T,n-2)$, *for all* $n\geq 2$.

(iii) *Let* $S\equiv\pm\{1,3,5,7,8,19,21,23,25,26,28,30\} \pmod{62}$, *and* $T\equiv\pm\{1,2,5,7,13,18,21,23,24,27,29,30\} \pmod{62}$. *Then* $p(S,n)=p(T,n-1)$, *for all* $n\geq 1$.

(iv) *Let* $S\equiv\pm\{5,7,9,11,12,13,15,16,17,19,20,22\} \pmod{62}$, *and* $T\equiv\pm\{2,5,7,11,15,17,20,23,24,26,27,29\} \pmod{62}$. *Then* $p(S,n)=p(T,n-5)$, *for all* $n\geq 5$.

(v) *Let* $S\equiv\pm\{2,3,7,8,9,13,17,22,23,24,25,29\} \pmod{62}$, *and* $T\equiv\pm\{1,6,8,9,11,13,15,16,20,23,25,29\} \pmod{62}$. *Then* $p(S,n)=p(T,n)$, *for all* $n\neq 1$.

(vi) *Let* $S \equiv \pm\{1,3,4,7,9,11,12,17,19,20,27,30\} \pmod{62}$, *and* $T \equiv \pm\{1,3,4,8,9,10,11,15,21,23,27,28\} \pmod{62}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 7$.

(vii) *Let* $S \equiv \pm\{3,4,5,13,14,15,16,17,18,19,25,27\} \pmod{62}$, *and* $T \equiv \pm\{1,5,9,12,13,15,16,19,22,27,29,30\} \pmod{62}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(viii) *Let* $S \equiv \pm\{1,3,5,7,9,13,15,16,21,22,24,28\} \pmod{62}$, *and* $T \equiv \pm\{1,3,6,7,8,10,15,19,21,23,25,28\} \pmod{62}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 5$.

(ix) *Let* $S \equiv \pm\{1,3,4,9,10,14,15,17,21,22,23,27\} \pmod{62}$, *and* $T \equiv \pm\{1,3,5,7,9,13,17,18,21,22,24,30\} \pmod{62}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 4$.

(x) *Let* $S \equiv \pm\{6,7,8,9,10,11,13,19,20,21,23,25\} \pmod{62}$, *and* $T \equiv \pm\{1,7,10,12,13,18,19,21,23,27,29,30\} \pmod{62}$. *Then* $p(S,n) = p(T,n-6)$, *for all* $n \geq 6$.

(xi) *Let* $S \equiv \pm\{2,6,7,10,11,15,16,17,19,21,25,29\} \pmod{62}$, *and* $T \equiv \pm\{2,4,5,11,14,15,17,21,23,26,27,29\} \pmod{62}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(xii) *Let* $S \equiv \pm\{5,8,9,11,12,13,14,15,17,19,20,23\} \pmod{62}$, *and* $T \equiv \pm\{3,4,5,14,15,17,19,23,25,26,27,28\} \pmod{62}$. *Then* $p(S,n) = p(T,n-5)$, *for all* $n \geq 5$.

(xiii) *Let* $S \equiv \pm\{3,4,5,9,10,11,19,21,25,26,27,28\} \pmod{62}$, *and* $T \equiv \pm\{1,5,6,9,15,16,19,22,25,26,27,29\} \pmod{62}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(xiv) *Let* $S \equiv \pm\{4,6,7,9,10,11,13,21,23,24,25,27\} \pmod{62}$, *and* $T \equiv \pm\{2,3,7,13,14,17,18,23,24,25,27,29\} \pmod{62}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(xv) *Let* $S \equiv \pm\{2,3,5,9,11,12,19,21,26,27,28,29\} \pmod{62}$, *and* $T \equiv \pm\{1,3,7,9,12,17,19,22,24,27,29,30\} \pmod{62}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i)–(xv) follow from (2.5) with the following sets of parameters

$$[1,3,2,8,13],\ [1,3,4,17,24],\ [1,2,4,14,27],\ [1,6,8,24,25],\ [1,3,4,10,12],$$
$$[1,2,9,12,13],\ [1,5,2,7,14],\ [1,2,7,10,14],\ [1,2,6,9,15],\ [1,8,2,10,12],$$
$$[1,3,7,22,24], [1,6,9,24,26], [1,5,2,7,11], [1,7,3,10,11], [1,4,2,6,11],$$

and with q replaced by q^{31} in each instance. □

14. Modulus $M = 64$

There is only one class of identities under multiplication by the group $U(64)$.

Theorem 14.1.

(i) *Let* $S \equiv \pm\{4,5,6,7,11,13,19,21,22,25,27,28\} \pmod{64}$, *and* $T \equiv \pm\{1,6,7,11,15,17,20,21,25,26,28,31\} \pmod{64}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(ii) *Let* $S \equiv \pm\{1,2,7,11,12,15,17,18,20,21,25,31\} \pmod{64}$, *and* $T \equiv \pm\{1,3,4,11,13,14,18,19,20,21,29,31\} \pmod{64}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(iii) *Let* $S \equiv \pm\{2,3,5,9,11,12,21,23,27,28,29,30\} \pmod{64}$, *and* $T \equiv \pm\{1,3,7,9,12,18,20,23,25,29,30,31\} \pmod{64}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iv) *Let* $S \equiv \pm\{4,7,9,10,12,13,15,17,19,22,23,25\} \pmod{64}$, *and* $T \equiv \pm\{3,4,5,13,15,17,19,22,26,27,28,29\} \pmod{64}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(v) *Let* $S \equiv \pm\{1,3,4,6,10,11,13,19,21,28,29,31\} \pmod{64}$, *and* $T \equiv \pm\{1,3,4,7,9,10,12,22,23,25,29,31\} \pmod{64}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 6$.

(vi) *Let* $S \equiv \pm\{2,7,9,12,13,14,15,17,19,20,23,25\} \pmod{64}$, *and* $T \equiv \pm\{2,5,7,11,12,13,19,21,25,27,28,30\} \pmod{64}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(vii) *Let* $S \equiv \pm\{3,4,5,13,14,15,17,18,19,20,27,29\} \pmod{64}$, *and* $T \equiv \pm\{1,5,9,12,14,15,17,20,23,27,30,31\} \pmod{64}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(viii) *Let* $S \equiv \pm\{3,4,5,9,10,11,21,23,26,27,28,29\} \pmod{64}$, *and* $T \equiv \pm\{1,5,6,9,15,17,20,23,26,27,28,31\} \pmod{64}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

Proof. The identities (i)–(viii) follow from (2.5) with the following sets of parameters

$$[1,5,6,18,27],\ [1,2,4,8,15],\ [1,4,2,6,11],\ [1,5,8,23,27],\ [1,2,8,11,12],$$
$$[1,5,3,10,15],\ [1,5,2,7,14], [1,5,2,7,11].$$

and with q replaced by q^{32} in each instance. ☐

15. Modulus $M = 66$

There are two classes of identities under multiplication by the group $U(66)$.

Theorem 15.1.

(i) *Let* $S \equiv \pm\{7,8,9,10,11,12,13,19,21,22,23,25\} \pmod{66}$, *and* $T \equiv \pm\{1,9,11,12,14,19,21,22,25,29,31,32\} \pmod{66}$. *Then* $p(S,n) = p(T,n-7)$, *for all* $n \geq 7$.

(ii) *Let* $S \equiv \pm\{4,5,6,7,11,13,21,22,23,27,28,29\} \pmod{66}$, *and* $T \equiv \pm\{1,6,7,11,16,17,21,22,26,27,29,31\} \pmod{66}$. *Then* $p(S,n) = p(T,n-4)$, *for all* $n \geq 4$.

(iii) *Let* $S \equiv \pm\{1,3,5,7,11,15,18,19,22,23,26,32\} \pmod{66}$, *and* $T \equiv \pm\{1,3,4,10,11,15,17,18,22,23,25,29\} \pmod{66}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 4$.

(iv) *Let* $S \equiv \pm\{5,7,9,11,13,15,16,17,19,20,22,24\} \pmod{66}$, *and* $T \equiv \pm\{2,5,9,11,15,17,22,24,25,28,29,31\} \pmod{66}$. *Then* $p(S,n) = p(T,n-5)$, *for all* $n \geq 5$.

(v) *Let* $S \equiv \pm\{2,3,5,11,13,14,19,22,23,27,30,31\} \pmod{66}$, *and* $T \equiv \pm\{1,3,8,11,13,17,20,22,25,27,30,31\} \pmod{66}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

Proof. The identities (i)–(v) follow from (2.5) with the following sets of parameters

$$[1,9,2,11,13],\ [1,5,6,18,28],\ [1,2,6,9,16],\ [1,6,8,25,26],\ [1,3,4,17,26],$$

and with q replaced by q^{33} in each instance. ☐

Theorem 15.2.

(i) *Let* $S \equiv \pm\{1, 10, 11, 12, 13, 14, 15, 16, 19, 21, 23, 25\} \pmod{66}$, *and* $T \equiv \pm\{1, 9, 10, 11, 12, 13, 14, 15, 25, 29, 31, 32\} \pmod{66}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1, 4, 5, 6, 7, 9, 11, 13, 16, 21, 23, 28\} \pmod{66}$, *and* $T \equiv \pm\{1, 4, 5, 6, 7, 9, 11, 14, 16, 17, 27, 29\} \pmod{66}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 13$.

(iii) *Let* $S \equiv \pm\{3, 4, 5, 7, 11, 18, 19, 23, 25, 26, 27, 32\} \pmod{66}$, *and* $T \equiv \pm\{1, 4, 7, 11, 15, 18, 20, 23, 25, 27, 29, 32\} \pmod{66}$. *Then* $p(S, n) = p(T, n-3)$, *for all* $n \geq 3$.

(iv) *Let* $S \equiv \pm\{2, 3, 5, 7, 11, 13, 15, 16, 19, 20, 24, 29\} \pmod{66}$, *and* $T \equiv \pm\{2, 3, 5, 9, 10, 11, 13, 16, 17, 24, 29, 31\} \pmod{66}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 7$.

(v) *Let* $S \equiv \pm\{5, 6, 7, 8, 9, 11, 17, 23, 26, 27, 28, 29\} \pmod{66}$, *and* $T \equiv \pm\{1, 6, 9, 11, 16, 17, 21, 23, 26, 28, 29, 31\} \pmod{66}$. *Then* $p(S, n) = p(T, n-5)$, *for all* $n \geq 5$.

(vi) *Let* $S \equiv \pm\{2, 5, 8, 11, 13, 14, 17, 19, 21, 23, 27, 30\} \pmod{66}$, *and* $T \equiv \pm\{2, 3, 8, 11, 13, 17, 19, 21, 25, 26, 30, 31\} \pmod{66}$. *Then* $p(S, n) = p(T, n-2)$, *for all* $n \geq 2$.

(vii) *Let* $S \equiv \pm\{7, 8, 9, 10, 11, 12, 13, 15, 19, 23, 31, 32\} \pmod{66}$, *and* $T \equiv \pm\{1, 8, 11, 12, 15, 19, 21, 23, 25, 28, 31, 32\} \pmod{66}$. *Then* $p(S, n) = p(T, n-7)$, *for all* $n \geq 7$.

(viii) *Let* $S \equiv \pm\{3, 4, 5, 11, 13, 14, 19, 20, 21, 25, 30, 31\} \pmod{66}$, *and* $T \equiv \pm\{1, 5, 8, 11, 14, 17, 20, 21, 25, 27, 30, 31\} \pmod{66}$. *Then* $p(S, n) = p(T, n-3)$, *for all* $n \geq 3$.

(ix) *Let* $S \equiv \pm\{1, 3, 4, 10, 11, 17, 18, 19, 25, 26, 27, 29\} \pmod{66}$, *and* $T \equiv \pm\{1, 2, 7, 10, 11, 15, 18, 19, 23, 26, 27, 29\} \pmod{66}$. *Then* $p(S, n) = p(T, n-2)$, *for all* $n \geq 2$.

(x) *Let* $S \equiv \pm\{3,5,7,9,11,13,17,20,24,28,31,32\} \pmod{66}$, *and* $T \equiv \pm\{2,3,7,11,15,17,20,24,25,28,29,31\} \pmod{66}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned} &[9,29,31,32:66] - q[16,19,21,23:66] \\ &= [1,9,10,11,12,13,14,15,16,19,21,23,25,29,31,32:66]. \end{aligned} \tag{15.1}$$

Our proof uses the first part of Theorem (15.1) that is equivalent to

$$\begin{aligned} &[1,14,29,31,32:66] - q^7[7,8,10,13,23:66] \\ &= [1,7,8,9,10,11,12,13,14,19,21,22,23,25,29,31,32:66]. \end{aligned} \tag{15.2}$$

Therefore, it suffices to prove that

$$\begin{aligned} &[15,16:66]\big\{[1,14,29,31,32:66] - q^7[7,8,10,13,23:66]\big\} \\ &= [7,8,22:66]\big\{[9,29,31,32:66] - q[16,19,21,23:66]\big\}. \end{aligned} \tag{15.3}$$

After regrouping the the terms in (15.3), we are led to prove

$$\begin{aligned} &[29,31,32:66]\big\{[1,14,15,16:66] - [7,8,9,22:66]\big\} \\ &= -q[7,8,23:66]\big\{[16,19,21,22:66] - q^6[10,13,15,16:66]\big\}. \end{aligned} \tag{15.4}$$

Next, we apply (2.2) twice with the sets of parameters $[1,2,24,8,9]$, $[1,11,17,30,32]$ and with q replaced by q^{66} in both instance to find that

$$\begin{aligned} &[1,14,15,16:66] - [7,8,9,22:66] = -q[6,7,8,23:66] \text{ and} \\ &[16,19,21,22:66] - q^6[10,13,15,16:66] = [6,29,31,32:66]. \end{aligned}$$

From these two identities (15.4) readily follows and so the proof of (i) is complete. □

16. Modulus $M = 68$

There is only one class of identities under multiplication by the group $U(68)$.

Theorem 16.1.

(i) *Let* $S \equiv \pm\{1,7,8,12,13,15,18,19,21,22,27,33\} \pmod{68}$, *and* $T \equiv \pm\{1,6,7,11,14,15,19,20,23,27,32,33\} \pmod{68}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,3,5,11,13,14,21,23,24,29,31,32\} \pmod{68}$, *and* $T \equiv \pm\{1,3,8,11,13,18,21,23,26,28,31,33\} \pmod{68}$. *Then* $p(S,n) = p(T,n-2)$, *for all* $n \geq 2$.

(iii) *Let* $S \equiv \pm\{1,3,5,7,8,22,26,27,28,29,31,33\} \pmod{68}$, *and* $T \equiv \pm\{1,2,5,7,13,21,24,27,29,30,32,33\} \pmod{68}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(iv) *Let* $S \equiv \pm\{3,7,10,11,12,15,16,18,19,23,27,31\} \pmod{68}$, *and* $T \equiv \pm\{3,4,7,9,15,19,20,25,26,27,30,31\} \pmod{68}$. *Then* $p(S,n) = p(T,n-3)$, *for all* $n \geq 3$.

(v) *Let* $S \equiv \pm\{1,4,5,6,9,15,19,25,26,28,29,33\} \pmod{68}$, *and* $T \equiv \pm\{1,3,5,9,10,14,16,24,25,29,31,33\} \pmod{68}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 3$.

(vi) *Let* $S \equiv \pm\{4,5,6,7,9,11,20,23,25,27,29,30\} \pmod{68}$, *and* $T \equiv \pm\{2,5,9,11,12,15,16,18,19,23,25,29\} \pmod{68}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

(vii) *Let* $S \equiv \pm\{7,8,9,10,11,12,13,21,22,23,25,27\} \pmod{68}$, *and* $T \equiv \pm\{1,9,11,13,14,20,21,23,25,30,32,33\} \pmod{68}$. *Then* $p(S,n) = p(T,n-7)$, *for all* $n \geq 7$.

(viii) *Let* $S \equiv \pm\{2,3,9,10,13,15,16,19,21,24,25,31\} \pmod{68}$, *and* $T \equiv \pm\{3,4,5,6,13,15,19,21,22,28,29,31\} \pmod{68}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

Proof. The identities (i)–(viii) follow from (2.5) with the following sets of parameters

$$[1,3,2,9,14],\ [1,3,4,17,27],\ [1,2,4,14,30],\ [1,4,8,26,27],\ [1,2,5,10,11],$$
$$[1,5,7,12,16],\ [1,9,2,11,13],\ [1,4,6,10,16]$$

and with q replaced by q^{34} in each instance. □

17. Modulus $M = 70$

There is one class of identities under multiplication by the group $U(70)$. The identity given in Theorem 17.1 below is equivalent to (1.3).

Theorem 17.1.

Let

$S \equiv \pm\{1,3,4,5,6,7,9,11,13,14,15,16,17,19,23,24,25,26,27,28,29,31,33,34\} \pmod{70}$,

and

$T \equiv \pm\{1,2,3,5,8,9,11,12,13,14,15,17,18,19,21,22,23,25,27,28,29,31,32,33\} \pmod{70}$.

Then $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

18. Modulus $M = 72$

There are three classes of identities under multiplication by the group $U(72)$.

Theorem 18.1.

(i) *Let* $S \equiv \pm\{1, 3, 5, 7, 8, 26, 28, 29, 30, 31, 33, 35\} \pmod{72}$, *and* $T \equiv \pm\{1, 2, 5, 7, 13, 23, 28, 29, 31, 32, 34, 35\} \pmod{72}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{1, 4, 5, 6, 11, 14, 15, 21, 25, 31, 32, 35\} \pmod{72}$, *and* $T \equiv \pm\{1, 4, 5, 7, 10, 11, 16, 25, 26, 29, 31, 35\} \pmod{72}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 6$.

(iii) *Let* $S \equiv \pm\{1, 7, 8, 13, 14, 17, 19, 20, 22, 23, 29, 35\} \pmod{72}$, *and* $T \equiv \pm\{1, 6, 7, 13, 15, 16, 20, 21, 23, 29, 34, 35\} \pmod{72}$. *Then* $p(S, n) = p(T, n-1)$, *for all* $n \geq 1$.

(iv) *Let* $S \equiv \pm\{2, 3, 5, 11, 16, 17, 19, 20, 25, 30, 31, 33\} \pmod{72}$, *and* $T \equiv \pm\{1, 5, 8, 11, 14, 17, 19, 20, 22, 25, 31, 35\} \pmod{72}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 1$.

(v) *Let* $S \equiv \pm\{4, 7, 10, 11, 13, 16, 17, 19, 23, 25, 26, 29\} \pmod{72}$, *and* $T \equiv \pm\{3, 4, 7, 13, 17, 19, 22, 23, 29, 30, 32, 33\} \pmod{72}$. *Then* $p(S, n) = p(T, n-4)$, *for all* $n \geq 4$.

(vi) *Let* $S \equiv \pm\{6, 8, 10, 11, 13, 15, 17, 19, 21, 23, 25, 28\} \pmod{72}$, *and* $T \equiv \pm\{2, 5, 11, 13, 17, 19, 23, 25, 28, 31, 32, 34\} \pmod{72}$. *Then* $p(S, n) = p(T, n-6)$, *for all* $n \geq 6$.

Proof. The identities (i)–(vi) follow from (2.5) with the following sets of parameters

$$[1, 2, 4, 14, 32],\ [1, 2, 8, 12, 13],\ [1, 3, 2, 9, 15],\ [1, 3, 4, 9, 15],\ [1, 5, 8, 25, 30],\ [1, 9, 3, 13, 14],$$

and with q replaced by q^{36} in each instance. □

Theorem 18.2.

Let

$S \equiv \pm\{1, 3, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 23, 25, 27, 29, 31, 32, 33, 34, 35\} \pmod{72}$,

and

$T \equiv \pm\{1, 2, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 18, 19, 21, 22, 23, 25, 26, 27, 29, 31, 32, 35\} \pmod{72}$.

Then $$p(S, n) = p(T, n-1), \text{ for all } n \geq 1. \tag{18.1}$$

Proof. Our proof relies heavily on the additive properties of theta functions, and so we adopt the notation of [3]. We first recall Ramanujan's definition for a general theta function. Let

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{18.2}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{18.3}$$

Observe that

$$[a:q] = \frac{f(-a,-q/a)}{(q:q)_\infty}.$$

By separating the sum into even and odd index terms, we obtain from (18.2) that [3, p. 48, Entry 31]

$$f(a,b) = f(a^3b, ab^3) + af(ab^{-1}, b^3a^5). \tag{18.4}$$

We are now ready to prove Theorem 18.2. First, we express (18.1) in its equivalent form

$$[2,15,21,22,26:72] - q[3,10,14,33,34:72] = \frac{[1,2,\cdots,35:72]}{[4,6,20,24,28,30:72]}. \tag{18.5}$$

For the left hand side of (18.5),

$$\begin{aligned} L(q) &:= [2,15,21,22,26:72] - q[3,10,14,33,34:72] \\ &= [2:24][15:36] - q[10:24][3:36] \\ &= \frac{1}{(q^{24};q^{24})_\infty(q^{36};q^{36})_\infty}\{f(-q^2,-q^{22})f(-q^{15},-q^{21}) - qf(-q^{10},-q^{14})f(-q^3,-q^{33})\}. \end{aligned} \tag{18.6}$$

By (18.4),

$$f(q,-q^2) = f(-q^5,-q^7) + qf(-q,-q^{11}) \text{ and } f(-q,q^2) = f(-q^5,-q^7) - qf(-q,-q^{11}). \tag{18.7}$$

In (18.6), we use (18.7) twice with q replaced by q^2 and q^3, we find that

$$\begin{aligned} 4q^2L(q) &= \frac{1}{(q^{24};q^{24})_\infty(q^{36};q^{36})_\infty}\Big\{(f(q^3,-q^6)+f(-q^3,q^6))(f(q^2,-q^4)-f(-q^2,q^4)) \\ &\quad - (f(q^3,-q^6)-f(-q^3,q^6))(f(q^2,-q^4)+f(-q^2,q^4))\Big\} \\ &= \frac{2}{(q^{24};q^{24})_\infty(q^{36};q^{36})_\infty}\{f(-q^3,q^6)f(q^2,-q^4) - f(q^3,-q^6)f(-q^2,q^4)\}. \end{aligned} \tag{18.8}$$

From (18.3), after some elementary product manipulations, we find that

$$f(-q,q^2) = \frac{f(q^3,q^3)}{(-q;q^2)_\infty} \text{ and } f(q,-q^2) = (-q,-q)_\infty = \frac{f(q,q)}{(-q;q^2)_\infty}.$$

From these two equations we conclude that

$$2q^2L(q) = \frac{1}{(q^{24};q^{24})_\infty(q^{36};q^{36})_\infty(-q^2;q^4)_\infty(-q^3;q^6)_\infty}\big\{f(q^2,q^2)f(q^9,q^9) - f(q^3,q^3)f(q^6,q^6)\big\}. \tag{18.9}$$

While for the right hand side of (18.5), we find that

$$\begin{aligned} R(q) &:= \frac{[1,2,\cdots,35:72]}{[4,6,20,24,28,30:72]} \\ &= \frac{(q;q)_\infty}{(q^{36};q^{72})_\infty(q^{72};q^{72})_\infty[6:36][4:24][24:72]} \\ &= \frac{(q;q)_\infty(q^{18};q^{36})_\infty(q^{12};q^{24})_\infty(q^{72};q^{72})_\infty}{(q^{36};q^{36})_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty} \\ &= \frac{(q;q)_\infty f(q^6,q^{30})}{(q^{36};q^{36})_\infty(q^{24};q^{24})_\infty}, \end{aligned} \tag{18.10}$$

after several applications of (18.3). By (18.8) and (18.10), we see that (18.5) is equivalent to

$$\begin{aligned} &f(q^2,q^2)f(q^9,q^9) - f(q^3,q^3)f(q^6,q^6) \\ &= 2q^3(-q^2;q^4)_\infty(-q^3;q^6)_\infty(q;q)_\infty f(q^6,q^{30}) = f(-q,-q^3)f(q^6,q^{30})(-q^3;q^6)_\infty, \end{aligned} \tag{18.11}$$

where in the last step (18.3) is used.

Employing (2.7) with q replaced by q^6 and x replaced by q, we find after some algebra that

$$f(q^9,q^9) - qf(q^3,q^{15}) = f(-q,-q^3)(-q^3;q^6)_\infty. \tag{18.12}$$

We need two more identities to establish (18.11) namely,

$$f(q,q) = f(q^9,q^9) + 2qf(q^3,q^{15}) \text{ and} \tag{18.13}$$

$$f(q,q)f(q^2,q^2) = f(q^3,q^3)f(q^6,q^6) + 2qf(q,q^5)f(q^2,q^{10}). \tag{18.14}$$

The identity (18.13) is the 3-dissection of the classical theta function $\varphi(q) = f(q,q)$ (see the corollary in [3, p. 49]) and the identity (18.14) can be obtained from the Theorem on page 73 of [3] with the following choice of parameters
$\epsilon_1 = \epsilon_2 = 0$, $\alpha = \beta = 1$, $a = b = q^2$, $c = d = q$, and $m = 3$.

We return to the left hand side of (18.11) and use (18.14) with q replaced by q^3, (18.13) with q replaced by q^2 and (18.12), we conclude that

$$\begin{aligned} &f(q^2,q^2)f(q^9,q^9) - f(q^3,q^3)f(q^6,q^6) \\ &= f(q^2,q^2)f(q^9,q^9) - f(q^9,q^9)f(q^{18},q^{18}) - 2q^3f(q^3,q^{15})f(q^6,q^{30}) \\ &= f(q^9,q^9)\big\{f(q^2,q^2) - f(q^{18},q^{18})\big\} - 2q^3f(q^3,q^{15})f(q^6,q^{30}) \\ &= 2q^3f(q^9,q^9)f(q^6,q^{30}) - 2q^3f(q^3,q^{15})f(q^6,q^{30}) \\ &= 2q^2f(q^6,q^{30})\big\{f(q^9,q^9) - qf(q^3,q^{15})\big\} \\ &= 2q^2f(q^6,q^{30})f(-q,-q^3)(-q^3;q^6)_\infty. \end{aligned}$$

This is (18.11) and so the proof of (18.5) is complete. ∎

Theorem 18.3.

(i) *Let* $S \equiv \pm\{1,5,9,12,13,14,16,23,27,31,34,35\} \pmod{72}$, *and* $T \equiv \pm\{1,4,9,11,13,14,23,25,27,30,32,35\} \pmod{72}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(ii) *Let* $S \equiv \pm\{2,5,6,7,9,16,17,19,20,27,29,31\} \pmod{72}$, *and* $T \equiv \pm\{2,5,7,8,9,11,12,25,26,27,29,31\} \pmod{72}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 6$.

(iii) *Let* $S \equiv \pm\{5,6,7,8,9,17,19,26,27,28,29,31\} \pmod{72}$, *and* $T \equiv \pm\{1,7,9,12,17,19,22,26,27,29,32,35\} \pmod{72}$. *Then* $p(S,n) = p(T,n-5)$, *for all* $n \geq 5$.

(iv) *Let* $S \equiv \pm\{1,9,10,11,12,14,17,19,25,27,32,35\} \pmod{72}$, *and* $T \equiv \pm\{1,8,9,10,11,13,23,25,27,28,30,35\} \pmod{72}$. *Then* $p(S,n) = p(T,n-1)$, *for all* $n \geq 1$.

(v) *Let* $S \equiv \pm\{7,8,9,10,11,12,13,23,25,27,29,34\} \pmod{72}$, *and* $T \equiv \pm\{1,9,11,13,16,20,23,25,27,30,34,35\} \pmod{72}$. *Then* $p(S,n) = p(T,n-7)$, *for all* $n \geq 7$.

(vi) *Let* $S \equiv \pm\{4,5,6,7,9,17,19,22,27,29,31,32\} \pmod{72}$, *and* $T \equiv \pm\{2,5,9,12,13,16,17,19,22,23,27,31\} \pmod{72}$. *Then* $p(S,n) = p(T,n)$, *for all* $n \neq 2$.

Proof. We prove (i) which is equivalent to

$$\begin{aligned}&[4,11,25,30,32:72] - q[5,12,16,31,34:72]\\&= [1,4,5,9,11,12,13,14,16,23,25,27,30,31,32,34,35:72].\end{aligned} \tag{18.15}$$

From (2.5) with q replaced by q^{36} and $a,\ b,\ c,\ x,\ y$, replaced by $q,\ q^2,\ q^6,\ q^{12},\ q^{15}$, respectively, we find that

$$\frac{[8,28,34:72]}{[1,4,6,9,11,12,14,17,19,24,25,27,30,32,35:72]} - \frac{[32:72]}{[1,5,6,9,12,13,16,23,24,27,30,31,35:72]} = q^4.$$

After clearing denominators, we arrive at

$$\begin{aligned}&[5,8,13,16,23,28,31,34:72] - [4,11,14,17,19,25,32,32:72]\\&= q^4[1,4,5,6,9,11,12,13,14,16,17,19,23,24,25,27,30,31,32,35:72].\end{aligned} \tag{18.16}$$

By (18.16), we see that (18.15) is equivalent to

$$\begin{aligned}&[34:72]\big\{[5,8,13,16,23,28,31,34:72]-[4,11,14,17,19,25,32,32:72]\big\}\\&=q^4[6,17,19,24:72]\big\{[4,11,25,30,32:72]-q[5,12,16,31,34:72]\big\}.\end{aligned}\tag{18.17}$$

After rearrangement of the terms of (18.17), we are led the prove

$$\begin{aligned}&[4,11,17,19,25,32:72]\big\{[14,32,34:72]+q^4[6,24,30:72]\big\}\\&=[5,16,31,34:72]\big\{[8,13,23,28,34:72]+q^5[6,12,17,19,24:72]\big\}.\end{aligned}\tag{18.18}$$

From (2.2), with the parameters $[1,5,9,15,33]$ and with q replaced by q^{72}, we find that

$$[4:72]\big\{[14,32,34:72]+q^4[6,24,30:72]\big\}=[8,10,28,34:72].\tag{18.19}$$

Returning (18.18) and substituting (18.19), we arrive at

$$\begin{aligned}&[8,10,11,17,19,25,28,32:72]\\&=[5,16,31:72]\big\{[8,13,23,28,34:72]+q^5[6,12,17,19,24:72]\big\}.\end{aligned}\tag{18.20}$$

By switching to base q^{36}, we have from (18.20) that

$$\begin{aligned}&[5,8,11,16,17:36](5,16:36)\\&=[5,8:36](8:36)\big\{[8,13,17:36](17:36)+q^5[3,12,17:36](3:36)\big\}.\end{aligned}$$

After cancellation, we are led to prove

$$[11,16:36](5,16:36)=(8:36)\big\{[8,13:36](17:36)+q^5[3,12:36](3:36)\big\}.\tag{18.21}$$

Now, (18.21) follows from (2.2) with q replaced by q^{36} and a, b, c, x, y replaced by q^{17}, $-q^{20}$, q^4, q q^{28} $-q^{12}$, respectively. □

19. Modulus $M = 80$

There is only one class of identities under multiplication by the group $U(80)$.

Theorem 19.1.

(i) *Let* $S \equiv \pm\{4,5,6,7,15,17,23,25,26,33,35,36\} \pmod{80}$, *and* $T \equiv \pm\{2,5,11,12,15,18,19,21,25,28,29,35\} \pmod{80}$. *Then* $p(S,n)=p(T,n)$, *for all* $n \neq 2$.

(ii) *Let* $S \equiv \pm\{3,4,5,13,14,15,25,27,34,35,36,37\} \pmod{80}$, *and* $T \equiv \pm\{1,5,9,12,15,22,25,28,31,35,38,39\} \pmod{80}$. *Then* $p(S,n)=p(T,n-3)$, *for all* $n \geq 3$.

Proof. The identities (i) and (ii) follow from (2.5) with the choice of variables $[1,5,7,12,19]$, $[1,5,2,7,14]$ and with q replaced by q^{40} in both instances. □

20. Modulus $M = 82$

There is only one class of identities under multiplication by the group $U(82)$.

Theorem 20.1.

(i) *Let* $S \equiv \pm\{6, 8, 10, 11, 13, 15, 17, 28, 29, 31, 33, 35\} \pmod{82}$, *and* $T \equiv \pm\{2, 5, 11, 17, 18, 23, 24, 30, 31, 33, 37, 39\} \pmod{82}$. *Then* $p(S, n) = p(T, n - 6)$, *for all* $n \geq 6$.

(ii) *Let* $S \equiv \pm\{1, 3, 8, 9, 10, 14, 21, 25, 27, 31, 33, 38\} \pmod{82}$, *and* $T \equiv \pm\{1, 3, 7, 9, 11, 17, 19, 24, 27, 30, 32, 40\} \pmod{82}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 7$.

(iii) *Let* $S \equiv \pm\{3, 4, 5, 13, 14, 15, 27, 29, 35, 36, 37, 38\} \pmod{82}$, *and* $T \equiv \pm\{1, 5, 9, 12, 15, 23, 26, 29, 32, 37, 39, 40\} \pmod{82}$. *Then* $p(S, n) = p(T, n - 3)$, *for all* $n \geq 3$.

(iv) *Let* $S \equiv \pm\{2, 3, 7, 13, 18, 19, 22, 23, 27, 34, 35, 39\} \pmod{82}$, *and* $T \equiv \pm\{1, 6, 9, 13, 16, 20, 21, 23, 25, 28, 35, 39\} \pmod{82}$. *Then* $p(S, n) = p(T, n)$, *for all* $n \neq 1$.

(v) *Let* $S \equiv \pm\{7, 11, 12, 15, 16, 17, 19, 20, 21, 25, 26, 29\} \pmod{82}$, *and* $T \equiv \pm\{4, 5, 7, 19, 21, 22, 25, 31, 33, 34, 36, 37\} \pmod{82}$. *Then* $p(S, n) = p(T, n - 7)$, *for all* $n \geq 7$.

Proof. The identities (i)–(v) follow from (2.5) with the choice of variables $[1, 9, 3, 13, 14]$, $[1, 2, 9, 12, 18]$, $[1, 5, 2, 7, 14]$, $[1, 3, 4, 10, 17]$, $[1, 8, 12, 32, 34]$, and with q replaced by q^{41} in each instance. $\square$

Acknowledgement: We would like to thank the referee for a careful reading of the paper and for a number of suggestions.

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E-mail address: frank@math.ufl.edu

E-mail address: hamza@math.ufl.edu

Department of Mathematics, University of Florida, Gainesville, Florida 32611