THE SPT-CRANK FOR OVERPARTITIONS

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Abstract. Bringmann, Lovejoy, and Osburn [14, 15] showed that the generating functions of the spt-overpartition functions \( \text{spt}(n) \), \( \text{spt}_1(n) \), \( \text{spt}_2(n) \), and \( M_2 \text{spt}(n) \) are quasimock theta functions, and satisfy a number of simple Ramanujan-like congruences. Andrews, Garvan, and Liang [7] defined an spt-crank in terms of weighted vector partitions which combinatorially explain simple congruences \( \text{mod} \) 5 and 7 for \( \text{spt}(n) \). Chen, Ji, and Zang [17] were able to define this spt-crank in terms of ordinary partitions. In this paper we define spt-cranks in terms of vector partitions that combinatorially explain the known simple congruences for all the spt-overpartition functions as well as new simple congruences. For all the overpartition functions except \( M_2 \text{spt}(n) \) we are able to define the spt-crank purely in terms of marked overpartitions. The proofs of the congruences depend on Bailey’s Lemma and the difference formulas for the Dyson rank of an overpartition [24] and the \( M_2 \)-rank of a partition without repeated odd parts [25].

1. Introduction

Here we consider Ramanujan type congruences for various spt type functions and combinatorial interpretations of them in terms of rank and crank type functions. We recall the spt function began with Andrews in [3] defining \( \text{spt}(n) \) as the number of smallest parts in the partitions of \( n \). In the same paper he proved the following congruences.

\textbf{Theorem 1.1.} For \( n \geq 0 \) we have

\begin{align}
\text{spt}(5n+4) & \equiv 0 \pmod{5}, \\
\text{spt}(7n+5) & \equiv 0 \pmod{7}, \\
\text{spt}(13n+6) & \equiv 0 \pmod{13}.
\end{align}

These congruences are reminiscent of the Ramanujan congruences for the partition function. The proof of Theorem 1.1 relied on relating the spt function to the second moment of the rank function for partitions. With this \( \text{spt}(n) \) could be expressed in terms of rank differences. Formulas for the required rank differences are found in [9] and [26].

We recall an overpartition of \( n \) is a partition of \( n \) in which the first occurrence of a part may be overlined. In [14] Bringmann, Lovejoy, and Osburn defined \( \text{sp}_t(n) \) as the number of smallest parts in the overpartitions of \( n \). Additionally they defined \( \text{sp}_t^1(n) \) to be the number of smallest parts in the overpartitions of \( n \) with smallest part odd and \( \text{sp}_t^2(n) \) to be the number of smallest parts in the overpartitions of \( n \) with smallest part even. We alter this definition to only count the smallest parts of the overpartitions on \( n \) where the smallest part is not overlined. This simply means the count of smallest parts here is half of the count of smallest parts in [14] and in other articles. This does not have any effect on congruences unless the modulus is even. We illustrate this change with an example.

The overpartitions of 4 are

\[ 4, \, \overline{4}, \, 3+1, \, 3+\overline{1}, \, 3+\overline{1}, \, 3+\overline{1}, \, 3+\overline{1}, \, 2+2, \, 2+\overline{1}+1, \, 2+\overline{1}+1, \, 2+\overline{1}+1, \, 2+\overline{1}+1, \, 2+\overline{1}+1, \, 1+1+1+1, \, 1+1+1+1, \, 1+1+1+1, \, 1+1+1+1. \]

and so \( \text{sp}_t(4) = 13, \text{sp}_t^1(4) = 10, \) and \( \text{sp}_t^2(4) = 3. \)

Bringmann, Lovejoy, and Osburn [14] proved the following congruences for these new spt functions.
Theorem 1.2. For \( n \geq 0 \) we have

\[
\begin{align*}
\text{spt} (3n) & \equiv 0 \pmod{3}, \\
\text{spt}_1 (3n) & \equiv 0 \pmod{3}, \\
\text{spt}_1 (5n) & \equiv 0 \pmod{5}, \\
\text{spt}_2 (3n) & \equiv 0 \pmod{3}, \\
\text{spt}_2 (3n + 1) & \equiv 0 \pmod{3}, \\
\text{spt}_2 (5n + 3) & \equiv 0 \pmod{5}.
\end{align*}
\]

(1.4) - (1.9)

The proof of these congruences relied on expressing these functions in terms of the second moments of certain rank and crank functions which relate to quasi-modular forms. We will give another proof of these congruences which gives new combinatorial interpretations of these congruences. We describe this method shortly.

In [1] Ahlgren, Bringmann, and Lovejoy defined \( M_2 \text{spt} (n) \) to be the number of smallest parts in the partitions of \( n \) without repeated odd parts and with smallest part even. One congruence they proved for \( M_2 \text{spt} (n) \) is that for any prime \( \ell \geq 3 \), any integer \( m \geq 1 \), and \( n \) such that \( (\frac{\ell^2}{\ell - 1}) \equiv 1 \), we have

\[
M_2 \text{spt} \left( \frac{\ell^2mn + 1}{8} \right) \equiv 0 \pmod{\ell^m}.
\]

(1.10)

However none of the current known congruences for \( M_2 \text{spt} (n) \) appear to be of the form of the congruences we have mentioned for \( \text{spt} (n) \) and \( \text{spt} (n) \), rather they are congruences related to certain Hecke operators.

One of the results of this paper will be to prove such congruences by giving combinatorial refinements.

We will prove the following congruences for \( M_2 \text{spt} (n) \).

Theorem 1.3. For \( n \geq 0 \) we have

\[
\begin{align*}
M_2 \text{spt} (3n + 1) & \equiv 0 \pmod{3}, \\
M_2 \text{spt} (5n + 1) & \equiv 0 \pmod{5}, \\
M_2 \text{spt} (5n + 3) & \equiv 0 \pmod{5}.
\end{align*}
\]

(1.11) - (1.13)

Also we will determine the parity of \( \text{spt}_1 (n) \).

Theorem 1.4. For \( n \geq 0 \) we have \( \text{spt}_1 (n) \equiv 0 \pmod{2} \) if and only if \( n \) is an odd square.

In Theorem 1.4 it is important to note that we’re using the convention of not counting the smallest parts of overpartitions when the smallest part is overlined. Otherwise we would have \( \text{spt}_1 (n) \) is trivially always even and instead this congruence tells when \( \text{spt}_1 (n) \) is 2 modulo 4.

The generating functions for the \( \text{spt} \) functions are given as follows, these are special cases of a general SPT function due to Bringmann, Lovejoy, and Osburn [15, Section 7],

\[
\text{SPT} (d, e; q) = \frac{(-dq; q)^{\infty} (-eq; q)^{\infty}}{(deq; q)^{\infty} (q; q)^{\infty}} \sum_{n=1}^{\infty} \frac{q^n \left( q; q \right)_n \left( dq; q \right)_n \left( eq; q \right)_n}{(1 - q^n)^2 (q^{n+1}; q)^{\infty}}.
\]

(1.14)

The case \( d = 0, e = 0 \) gives a generating function for \( \text{spt} (n) \),

\[
\sum_{n=1}^{\infty} \text{spt} (n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2 (q^{n+1}; q)^{\infty}}.
\]

(1.15)

The case \( d = 1, e = 0 \) gives a generating function for \( \text{spt}_1 (n) \),

\[
\sum_{n=1}^{\infty} \text{spt}_1 (n) q^n = \sum_{n=1}^{\infty} \frac{q^n \left( -q^{n+1}; q \right)_n}{(1 - q^n)^2 (q^{n+1}; q)^{\infty}}.
\]

(1.16)
The case \( d = 1, e = 1/q, q = q^2 \) gives a generating function for \( \overline{\text{spt}}_2 (n) \),
\[
\sum_{n=1}^{\infty} \overline{\text{spt}}_2 (n) q^n = \sum_{n=1}^{\infty} \frac{q^{2n} (q^{-2n+1}; q)_\infty}{(1-q^{-2n})^2 (q^{2n+1}; q)_\infty}.
\]
(1.17)

Similar to the \( \text{spt} \) and \( \overline{\text{spt}}_2 \) we see a generating function for \( \overline{\text{spt}}_1 (n) \) is
\[
\sum_{n=1}^{\infty} \overline{\text{spt}}_1 (n) q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1} (q^{-2n+2}; q^2)_\infty}{(1-q^{2n+1})^2 (q^{2n+2}; q^2)_\infty}.
\]
(1.18)

The case \( d = 0, e = 1/q, q = q^2 \) gives a generating function for \( M_2 \text{spt} (n) \),
\[
\sum_{n=1}^{\infty} M_2 \text{spt} (n) q^n = \sum_{n=0}^{\infty} \frac{q^{2n} (q^{-2n+1}; q^2)_\infty}{(1-q^{2n})^2 (q^{2n+2}; q^2)_\infty}.
\]
(1.19)

Here we are using the product notation,
\[
(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k),
\]
(1.20)
\[
(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},
\]
(1.21)
\[
(a_1, a_2, \ldots, a_j; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_j; q)_\infty,
\]
(1.22)
\[
(a_1, a_2, \ldots, a_j; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_j; q)_n.
\]
(1.23)

We note that the three special cases of \( \text{SPT}(d, e; q) \) described above are quasimock theta functions. See [15, page 240] for a definition.

Andrews, Garvan, and Liang [7] found combinatorial interpretations of the mod 5 and 7 congruences in Theorem 1.1 in terms of weighted counts of special vector partitions called S-partitions. This was done by adding an extra variable to the generating function of the spt-function. In particular they defined
\[
S(z, q) = \sum_{n=1}^{\infty} \frac{q^n (q^{n+1}; q)_\infty}{(zq^n; q)_\infty (z^{-1}q^n; q)_\infty}
\]
(1.24)
\[
= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_S (m, n) z^m q^n.
\]
(1.25)

One then finds the congruences in (1.1) and (1.2) follow by showing the coefficients of \( q^{5n+4} \) in \( S(\zeta_5, q) \) and \( q^{7n+5} \) in \( S(\zeta_7, q) \) are zero, where \( \zeta_5 \) is a primitive fifth root of unity and \( \zeta_7 \) is a primitive seventh root of unity. This is the approach we take to prove the congruences for \( \text{spt} (n) \), \( \text{spt}_1 (n) \), \( \overline{\text{spt}}_2 (n) \), and \( M_2 \text{spt} (n) \), and their combinatorial refinements.

In the next section we give two variable generalizations of the generating functions (1.16) – (1.19), introduce various ranks and cranks, and state numerous identities for these functions. At the end of the next section we describe the plan for the remainder of the paper.

2. Statement of Results and Preliminaries

In this paper we give alternate proofs of the congruences in Theorem 1.2 and prove the congruences of Theorems 1.3 and 1.4 as well as giving combinatorial interpretations. We consider two variable generalizations of the generating functions from the introduction. We set
\[
S(z, q) = \sum_{n=1}^{\infty} \frac{q^n (q^{-n+1}; q)_\infty (q^{n+1}; q)_\infty}{(zq^n; q)_\infty (z^{-1}q^n; q)_\infty}
\]
(2.1)
\[ S_2(z, q) = \sum_{n=1}^{\infty} q^{2n} (q^{2n+2}; q^2)_{\infty} \left( -q^{2n+1}; q^2 \right)_{\infty} \]

\[ = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N S_2(m, n) z^m q^n, \]  

\[ S_2(z, q) = \sum_{n=1}^{\infty} q^{2n} (q^{2n+2}; q^2)_{\infty} \left( -q^{2n+1}; q^2 \right)_{\infty} \]

\[ = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N S_2(m, n) z^m q^n, \]  

\[ \overline{S}_2(z, q) = \sum_{n=1}^{\infty} q^{2n} \left( -q^{2n+1}; q^2 \right)_{\infty} \left( q^{2n+1}; q^2 \right)_{\infty} \]

\[ = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N \overline{S}_2(m, n) z^m q^n, \]  

\[ \overline{S}_1(z, q) = \sum_{n=0}^{\infty} q^{2n+1} \left( -q^{2n+2}; q^2 \right)_{\infty} \left( q^{2n+2}; q^2 \right)_{\infty} \]

\[ = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N \overline{S}_1(m, n) z^m q^n. \]  

Setting \( z = 1 \) then gives the generating functions from the introduction.

Furthermore we define

\[ N \overline{S}(k, t, n) = \sum_{m \equiv k \pmod{t}} N \overline{S}(m, n) \]  

so

\[ \text{spt}(n) = \sum_{m=-\infty}^{\infty} N \overline{S}(m, n) = \sum_{k=0}^{r-1} N \overline{S}(k, r, n) \]  

for any positive integer \( r \). We similarly define

\[ N \overline{S}_1(k, t, n) = \sum_{m \equiv k \pmod{t}} N \overline{S}_1(m, n), \]

\[ N \overline{S}_2(k, t, n) = \sum_{m \equiv k \pmod{t}} N \overline{S}_2(m, n), \]

\[ N \overline{S}_2(k, t, n) = \sum_{m \equiv k \pmod{t}} N \overline{S}_2(m, n). \]

We use these series to give another proof of the spt congruences.

First we consider the congruence in (1.4) of Theorem 1.2. With \( \zeta_3 \) a primitive third root of unity, we have

\[ \overline{S}(\zeta_3, q) = \sum_{n=1}^{\infty} \left( N \overline{S}(0, 3, n) + N \overline{S}(1, 3, n) \zeta_3 + N \overline{S}(2, 3, n) \zeta_3^2 \right) q^n \]

The minimal polynomial for \( \zeta_3 \) is \( 1 + x + x^2 \), and so if

\[ N \overline{S}(0, 3, n) + N \overline{S}(1, 3, n) \zeta_3 + N \overline{S}(2, 3, n) \zeta_3^2 = 0 \]

then

\[ N \overline{S}(0, 3, 3n) = N \overline{S}(1, 3, 3n) = N \overline{S}(2, 3, 3n). \]
But if (2.14) holds, then
\[
\overline{\text{spt}}(3n) = 3N_{\overline{S}}(k, 3, 3n) \quad \text{for } k = 0, 1, 2
\]
and so clearly \(\overline{\text{spt}}(3n) \equiv 0 \pmod{3}\). That is to say, if we show the coefficient of \(q^{3n}\) in \(S(\zeta_3, q)\) to be zero, then we have proved the first congruence in Theorem 1.2, and the stronger result (2.14).

In the same fashion, the congruences (1.5) and (1.6), will follow by showing the coefficients of \(q^{3n}\) in \(S_1(\zeta_3, q)\) and the coefficients of \(q^{3n}\) in \(S_2(\zeta_3, q)\) are zero. The congruences (1.7), (1.8), and (1.9) will follow by showing the coefficients of \(q^{3n}\) and \(q^{3n+1}\) in \(S_2(\zeta_3, q)\) and the coefficients of \(q^{3n+1}\) in \(S_2(\zeta_3, q)\) are zero. The congruences in Theorem 1.3 will follow by showing the coefficients of \(q^{3n+1}\) in \(S_2(\zeta_3, q)\) and the coefficients of \(q^{3n+1}\) and \(q^{3n+3}\) in \(S_2(\zeta_3, q)\) are zero.

To this end, we will express the series \(S(z, q), S_1(z, q), S_2(z, q), S_2(z, q)\) as the difference of the generating functions for certain ranks and cranks. In [7] Andrews, the first author, and Liang found that \(S(z, q)\) could be expressed in terms of the difference of the rank and crank of a partition. We recall the rank of a partition \(\pi\) is the largest parts minus the number of parts. The crank of a partition is the largest part if there are no

\[
\text{M} = 2 \text{rank} = \left\lfloor \frac{l(\pi)}{2} \right\rfloor - \#(\pi),
\]

where \(l(\pi)\) is the largest part of \(\pi\) and \(\#(\pi)\) is the number of parts of \(\pi\). The \(M_2\)-rank was introduced by Berkovich and the first author in [10]. We let \(N_2(m, n)\) denote the number of partitions \(\pi\) of \(n\) with distinct odd parts and \(M_2\)-rank \(m\). By Lovejoy and Osburn [25] the generating function for \(N_2\) is given by
\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) z^m q^n = \sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{(-q^2; q^2)_n (z^{-2}; q^2)_n (zq^2; q^2)_n}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n}.
\]
We set
\[ N(k, t, n) = \sum_{m \equiv k \pmod{t}} N(m, n), \quad (2.22) \]
\[ M(k, t, n) = \sum_{m \equiv k \pmod{t}} M(m, n), \quad (2.23) \]
\[ N2(k, t, n) = \sum_{m \equiv k \pmod{t}} N2(m, n), \quad (2.24) \]
\[ M2(k, t, n) = \sum_{m \equiv k \pmod{t}} M2(m, n). \quad (2.25) \]

We see \( \overline{N}(-m, n) = \overline{N}(m, n) \) and so \( \overline{N}(k, t, n) = \overline{N}(t-k, t, n) \). Similarly we have \( \overline{M}(k, t, n) = \overline{M}(t-k, t, n), N2(k, t, n) = N2(t-k, t, n) \), and \( M2(k, t, n) = M2(t-k, t, n) \).

We will show the following:

**Theorem 2.1.**
\[(1-z)(1-z^{-1})S(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\overline{N}(m, n) - \overline{M}(m, n)) z^m q^n. \quad (2.26)\]

**Theorem 2.2.**
\[(1-z)(1-z^{-1})S2(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (N2(m, n) - M2(m, n)) z^m q^n. \quad (2.27)\]

**Theorem 2.3.**
\[(1-z)(1-z^{-1})S_2(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left( \frac{\overline{N}(m, n)}{2} - \overline{M}(m, n) \right) z^m q^n
+ \frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^n}{1-zq^n(1-z^{-1}q^n)} \right). \quad (2.28)\]

**Theorem 2.4.**
\[(1-z)(1-z^{-1})S_1(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\overline{N}(m, n)}{2} z^m q^n
- \frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^n}{1-zq^n(1-z^{-1}q^n)} \right). \quad (2.29)\]

In [24] Lovejoy and Osburn determined formulas for the differences of \( \overline{N}(s, \ell, \ell n + d) \) for \( \ell = 3, 5 \) and in [25] they did the same for \( N2(s, \ell, \ell n + d) \). From these difference formulas, we know the 3-dissection and 5-dissection for the generating functions of \( \overline{N}(m, n) \) and \( N2(m, n) \). In particular, we will have the following.

**Theorem 2.5.**
\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m, n) \zeta_5^n q^n = \overline{N}_{0,3}(q^3) + q \overline{N}_{1,3}(q^3) + q^2 \overline{N}_{2,3}(q^3) \quad (2.30)\]

where
\[ \overline{N}_{0,3}(q) = \frac{(q^3; q^3)_\infty^4 (q^2; q^2)_\infty}{(q; q)_\infty^2 (q^6; q^6)_\infty}, \quad (2.31) \]
\[ \overline{N}_{1,3}(q) = 2 \frac{(q^3; q^3)_\infty (q^6; q^6)_\infty}{(q; q)_\infty^2}. \quad (2.32) \]

**Theorem 2.6.**
\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m, n) \zeta_5^n q^n = \overline{N}_{0,5}(q^5) + q \overline{N}_{1,5}(q^5) + q^2 \overline{N}_{2,5}(q^5) + q^3 \overline{N}_{3,5}(q^5) + q^4 \overline{N}_{4,5}(q^5) \quad (2.33)\]
where
\[
\mathcal{N}_{0,5}(q) = \frac{(q^4, q^6; q^{10})_\infty (q^5; q^5)_\infty^2}{(q^2, q^5; q^5)_\infty^2 (q^{10}; q^{10})_\infty} + 2(\zeta_5 + \zeta_5^{-1}) q \frac{(q^{10}; q^{10})_\infty}{(q^3, q^6; q^{10})_\infty},
\]
(2.34)
\[
\mathcal{N}_{3,5}(q) = \frac{2(1 - \zeta_5 - \zeta_5^{-1}) (q^{10}; q^{10})_\infty}{(q^2, q^5; q^5)_\infty}.
\]
(2.35)

**Theorem 2.7.**
\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N^2(m, n) \zeta_5^m q^n = N^2_{0,3}(q^3) + q N^2_{1,3}(q^3) + q^2 N^2_{2,3}(q^3)
\]
where
\[
N^2_{1,3}(q) = \frac{(q^6; q^6)_\infty^4}{(q^2; q^2)_\infty (q^4; q^4)_\infty (q^{12}; q^{12})_\infty}.
\]
(2.37)

**Theorem 2.8.**
\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N^2(m, n) \zeta_5^m q^n = N^2_{0,5}(q^5) + q N^2_{1,5}(q^5) + q^2 N^2_{2,5}(q^5) + q^3 N^2_{3,5}(q^5) + q^4 N^2_{4,5}(q^5)
\]
where
\[
N^2_{1,5}(q) = \frac{(-q^5; q^{10}; q^{10})_\infty}{(q^2; q^5; q^{10})_\infty},
\]
(2.39)
\[
N^2_{2,5}(q) = (\zeta_5 + \zeta_5^4)(-q^5; q^{10}; q^{10})_\infty
\]
(2.40)

The terms \(\mathcal{N}_{2,3}(q), \mathcal{N}_{4,5}(q), \mathcal{N}_{2,5}(q), \mathcal{N}_{4,5}(q), N^2_{0,3}(q), N^2_{2,3}(q), N^2_{0,5}(q), N^2_{2,5}(q), N^2_{4,5}(q), \) and \(N^2_{4,5}(q)\) are also products and series in \(q\) and follow from the difference formulas of Lovejoy and Obsurn in [24] and [25]. However we will not need to use them here.

We will determine dissections for the cranks and other series. In particular, we will prove the following.

**Theorem 2.9.**
\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{M}(m, n) \zeta_5^m q^n = \mathcal{M}_{0,3}(q^3) + q \mathcal{M}_{1,3}(q^3) + q^2 \mathcal{M}_{2,3}(q^3)
\]
where
\[
\mathcal{M}_{0,3}(q) = \frac{(q^3; q^3)_\infty^4 (q^2; q^2)_\infty}{(q^5; q^5)_\infty^2 (q^6; q^6)_\infty},
\]
(2.42)
\[
\mathcal{M}_{1,3}(q) = -\frac{(q^6; q^6)_\infty (q^3; q^3)_\infty}{(q; q)_\infty}
\]
(2.43)
\[
\mathcal{M}_{2,3}(q) = -2 \frac{(q^6; q^6)_\infty^4}{(q^3; q^3)_\infty (q^2; q^2)_\infty^2}.
\]
(2.44)

**Theorem 2.10.**
\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{M}(m, n) \zeta_5^m q^n = \mathcal{M}_{0,5}(q^5) + q \mathcal{M}_{1,5}(q^5) + q^2 \mathcal{M}_{2,5}(q^5) + q^3 \mathcal{M}_{3,5}(q^5) + q^4 \mathcal{M}_{4,5}(q^5)
\]
where
\[
\mathcal{M}_{0,5}(q) = \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q^2, q^4, q^6; q^{10})_\infty} - q(\zeta_5 + \zeta_5^{-1}) \frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q^4, q^6, q^{10})_\infty},
\]
(2.46)
\[
\mathcal{M}_{1,5}(q) = (\zeta_5 + \zeta_5^4) \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q^2, q^4, q^6; q^{10})_\infty}.
\]
(2.47)
\[ M_{2,5}(q) = - \frac{(q^{10}; q^{10})_{\infty}}{(q, q^4; q^5)_{\infty}}, \quad M_{3,5}(q) = - (\zeta_5 + \zeta_5^4) \frac{(q^{10}; q^{10})_{\infty}}{(q^2, q^4; q^5)_{\infty}}, \quad M_{4,5}(q) = - \frac{(q^2, q^8; q^{10})_{\infty}}{(q, q^4; q^5)_{\infty}}. \]

Theorem 2.11.
\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_2(m, n) \zeta_5^n q^m = M_{0,3}(q^3) + qM_{1,3}(q^3) + q^2M_{2,3}(q^3) \]

where
\[ M_{20,3}(q) = \frac{(q^6; q^6)_{10}^{10} q^4; q^4)_{\infty}}{(q^{12}; q^{12})_{\infty} (q^3; q^3)_{\infty} (q^2; q^2)_{\infty}}, \quad M_{21,3}(q) = \frac{(q^6; q^6)_{4}^{4}}{(q^{12}; q^{12})_{\infty} (q^3; q^3)_{\infty} (q^2; q^2)_{\infty}}, \quad M_{22,3}(q) = -2 \frac{(q^{12}; q^{12})_{2}^{2} (q^3; q^3)_{\infty} (q^2; q^2)_{\infty}}{(q^6; q^6)_{\infty} (q^4; q^4)_{\infty} (q; q)_{\infty}}. \]

Theorem 2.12.
\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_2(m, n) \zeta_5^n q^m = M_{0,5}(q^5) + qM_{1,5}(q^5) + q^2M_{2,5}(q^5) + q^3M_{3,5}(q^5) + q^4M_{4,5}(q^5) \]

where
\[ M_{20,5}(q) = \frac{(-q^3, -q^5, q^7, q^9, q^{10}; q^{10})_{\infty}}{(-q, q^2, q^4, q^5, q^{10})_{\infty}}, \quad M_{21,5}(q) = \frac{(-q^3, q^{10}; q^{10})_{\infty}}{(q^2; q^3; q^5)_{\infty}}, \]
\[ M_{22,5}(q) = (\zeta_5 + \zeta_5^4) \frac{(q^2, -q^3, -q^4, -q^5, q^6, -q^8, q^{10}; q^{10})_{\infty}}{(-q, q^2, q^4, q^4, -q^6, q^8, q^{10})_{\infty}} - \frac{(-q; q^4, -q^9, -q^{10}, q^{10})_{\infty}}{(q^2; q^4, q^8, q^{10})_{\infty}}, \quad M_{23,5}(q) = (\zeta_5 + \zeta_5^4) \frac{(-q^3, q^{10}; q^{10})_{\infty}}{(q^4, q^6, q^{10})_{\infty}}, \quad M_{24,5}(q) = - (\zeta_5 + \zeta_5^4) \frac{(-q, -q^5, -q^9, -q^{10}; q^{10})_{\infty}}{(q^2, -q^4, -q^4, q^8, q^{10})_{\infty}}. \]

Theorem 2.13.
\[ \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - \zeta_5)(1 - \zeta_5^{-1})(-1)^n q^n}{(1 - \zeta_5 q^n)(1 - \zeta_5^{-1} q^n)} \right) = A_0(q^3) + qA_1(q^3) + q^2A_2(q^3) \]

where
\[ A_0(q) = \frac{(q^3; q^3)_{\infty}^4 (q^2; q^2)_{\infty}}{2 (q; q)_{\infty} (q^4; q^4)_{\infty}}, \quad A_1(q) = -2 \frac{(q^6; q^6)_{\infty} (q^3; q^3)_{\infty}}{(q; q)_{\infty}}, \quad A_2(q) = 2 \frac{(q^6; q^6)_{\infty}^4}{(q^3; q^3)_{\infty}^2 (q^2; q^2)_{\infty}}. \]
Lastly the congruences for $M_2spt (3q^n)$ follow from
\[ \frac{(-q^n)_{\infty}}{(q^n)_{\infty}} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - \zeta_5)(1 - \zeta_5^{-1})(-1)^n q^n}{(1 - \zeta_5^n)(1 - \zeta_5^{-1} q^n)} \right) = B_0(q^n) + qB_1(q^n) + q^2B_2(q^n) + q^3B_3(q^n) + q^4B_4(q^n) \] \tag{2.65}

where
\[ B_0(q) = \frac{(q^5; q^5)_{\infty}^2 (q^4, q^6; q^{10})_{\infty}}{2 (q^{10}; q^{10})_{\infty} (q^2, q^4; q^5)_{\infty}^2} + (\zeta_5 + \zeta_5^{-1}) \frac{q(q^{10}; q^{10})_{\infty}}{(q^3, q^6, q^6, q^4; q^{10})_{\infty}} \] \tag{2.66}
\[ B_1(q) = (\zeta_5 + \zeta_5^{-1} - 1) \frac{(q^4, q^6; q^{10})_{\infty}}{(q^2, q^6, q^{10})_{\infty} (q^2, q^4; q^5)_{\infty}} \] \tag{2.67}
\[ B_2(q) = (1 - 2\zeta_5 - 2\zeta_5^{-1}) \frac{(q^{10}; q^{10})_{\infty}}{(q, q^6; q^{10})_{\infty} (q^4, q^6, q^{10})_{\infty}} \] \tag{2.68}
\[ B_3(q) = \frac{(q^{10}; q^{10})_{\infty}}{(q^2, q^4; q^5)_{\infty}} \] \tag{2.69}
\[ B_4(q) = (\zeta_5 + \zeta_5^{-1}) \frac{(q^2, q^6, q^{10}; q^{10})_{\infty}}{(q, q^6, q^{10})_{\infty} (q^2, q^4; q^5)_{\infty}^2} \] \tag{2.70}

With these dissections, we need only match up the appropriate terms for each congruence. The congruence for $spt (3n)$ of Theorem 1.2 follows from
\[ N_{0,3} - M_{0,3} = 0. \] \tag{2.71}

The congruences for $spt_1 (3n)$ and $spt_1 (5n)$ follow from
\[ \frac{N_{0,3}}{2} - A_0 = 0, \] \tag{2.72}
\[ \frac{N_{0,5}}{2} - B_0 = 0. \] \tag{2.73}

The congruences for $spt_2 (3n)$, $spt_2 (3n + 1)$, and $spt_2 (5n + 3)$ follow from
\[ \frac{N_{0,3}}{2} + A_0 - M_{0,3} = 0, \] \tag{2.74}
\[ \frac{N_{1,3}}{2} + A_1 - M_{1,3} = 0, \] \tag{2.75}
\[ \frac{N_{3,5}}{2} + B_3 - M_{3,5} = 0. \] \tag{2.76}

Lastly the congruences for $M2spt (3n + 1)$, $M2spt (5n + 1)$, and $M2spt (5n + 3)$ follow from
\[ N_{2,1,3} - M_{2,1,3} = 0, \] \tag{2.77}
\[ N_{2,1,5} - M_{2,1,5} = 0, \] \tag{2.78}
\[ N_{2,3,5} - M_{2,3,5} = 0. \] \tag{2.79}

For Theorem 1.4 we have to do a little better. In particular we will prove the following.

**Theorem 2.15.**
\[ \overline{S}_1 (i, q) = \sum_{n=1}^{\infty} q^{(2n-1)^2}. \] \tag{2.80}

That is to say,
\[ \sum_{n=1}^{\infty} q^{(2n-1)^2} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_{spt_2} (m, n) i^m q^n \] \tag{2.81}
\[
\sum_{n=0}^{\infty} \left( N_{S_1}(0,4,n) - N_{S_1}(2,4,n) + i(N_{S_1}(1,4,n) - N_{S_1}(3,4,n)) \right) q^n = \sum_{n=0}^{\infty} \left( N_{S_1}(0,4,n) - N_{S_1}(2,4,n) \right) q^n.
\]

But
\[
\overline{\text{spt}}_1(n) = N_{S_1}(0,4,n) + N_{S_1}(1,4,n) + N_{S_1}(2,4,n) + N_{S_1}(3,4,n)
\]
and so we see
\[
\overline{\text{spt}}_1(n) \equiv 1 \pmod{2}
\]
if and only if \( n \) is an odd square.

In [7] Andrews, the first author, and Liang also showed that \( N_{S}(m,n) \), the coefficients in \( S(z,q) \), to be nonnegative. The same phenomenon occurs here.

**Theorem 2.16.** For all \( m \) and \( n \) we have \( N_{S_1}(m,n) \), \( N_{S_1}(m,n) \), and \( N_{S_2}(m,n) \) are nonnegative.

In section 3 we give combinatorial interpretations of the series \( S(z,q) \), \( S_1(z,q) \), \( S_2(z,q) \), and \( S_2(z,q) \) in terms of weighted vector partitions and then prove Theorem 2.16. For \( S(z,q) \), \( S_1(z,q) \) and \( S_2(z,q) \) we define the spt-crank in terms of marked overpartitions, see equation (3.27). In Theorem 3.8 we give a combinatorial interpretation of each of the spt-overpartition congruences in Theorem 1.2 in terms of marked overpartitions. In section 4 we prove the theorems on expressing \( S(z,q) \), \( S_1(z,q) \), \( S_2(z,q) \), and \( S_2(z,q) \) in terms of the difference between a rank and crank. In section 5 we prove the various dissections. In section 6 we conclude with remarks on the nonnegativity of the coefficients of \( S_2(z,q) \); a recent result by Andrews, Chan, Kim, and Osburn [6] on the first moments for the rank and crank of overpartitions; and the remaining spt function of [15].

### 3. Combinatorial Interpretations

In this section we provide combinatorial interpretations of the coefficients in the series \( S(z,q) \), \( S_1(z,q) \), \( S_2(z,q) \), and \( S_2(z,q) \). For all four series we provide an interpretation in terms of certain vector partitions with four components. For the three series \( S(z,q) \), \( S_1(z,q) \), and \( S_2(z,q) \) we give two additional interpretations — one in terms of pairs of partitions and finally an interpretations in terms of marked overpartitions. This final interpretation will give interpretations of the congruences for overpartitions directly in terms of the overpartitions themselves.

#### 3.1. Vector partitions and \( S \)-partitions

The coefficients in the series \( S(z,q) \), \( S_1(z,q) \), \( S_2(z,q) \), and \( S_2(z,q) \) can be interpreted in terms of cranks of vector partitions. This can be done with vectors with 4 components, each a partition with certain restrictions.

We let \( \overline{\text{V}} = \mathcal{D} \times \mathcal{P} \times \mathcal{P} \times \mathcal{D} \), where \( \mathcal{P} \) denotes the set of all partitions and \( \mathcal{D} \) denotes the set of all partitions into distinct parts. For a partition \( \pi \) we let \( s(\pi) \) denote the smallest part of \( \pi \) (with the convention that the empty partition has smallest part \( \infty \)), \( \#(\pi) \) the number of parts in \( \pi \), and \( |\pi| \) the sum of the parts of \( \pi \). For \( \overline{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4) \in \overline{\text{V}} \), we define the weight \( \omega(\overline{\pi}) = (-1)^{\#(\pi_1)-1} \), the crank(\( \overline{\pi} \)) = \#(\pi_2) - \#(\pi_3), and the norm \( |\overline{\pi}| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4| \). We say \( \overline{\pi} \) is a vector partition of \( n \) if \( |\overline{\pi}| = n \).

We then let \( \overline{\mathcal{S}} \) denote the subset of \( \overline{\text{V}} \) given by
\[
\overline{\mathcal{S}} = \{(\pi_1, \pi_2, \pi_3, \pi_4) \in \overline{\text{V}} : 1 \leq s(\pi_1) < \infty, s(\pi_1) \leq s(\pi_2), s(\pi_1) \leq s(\pi_3), s(\pi_1) < s(\pi_4) \}.
\]

We let \( \overline{\mathcal{S}_1} \) and \( \overline{\mathcal{S}_2} \) denote the subsets of \( \overline{\mathcal{S}} \) with \( s(\pi_1) \) odd and even, respectively.
We see then that the number of vector partitions of \( n \) in \( \overline{S} \) with crank \( m \) counted according to the weight \( \omega \) is exactly \( N_{\overline{S}}(m, n) \). Similarly the number of vector partitions in of \( n \) in \( \overline{S}_1 \) with crank \( m \) counted according to the weight \( \omega \) is \( N_{\overline{S}_1}(m, n) \), and the number of vector partitions of \( n \) in \( \overline{S}_2(m, n) \) with crank \( m \) counted according to the weight \( \omega \) is \( N_{\overline{S}_2}(m, n) \).

We let \( n_o(\pi) \) and \( n_e(\pi) \) denote the number of odd and even parts, respectively, of \( \pi \). We let \( S_2 \) denote the subset of \( S \) given by

\[
S_2 = \{ (\pi_1, \pi_2, \pi_3, \pi_4) \in \overline{S} : n_o(\pi_1) = 0, n_o(\pi_2) = 0, n_o(\pi_3) = 0, n_e(\pi_4) = 0 \}. \tag{3.2}
\]

Then \( N_{S_2}(m, n) \) is the number of vector partitions from \( S_2 \) of \( n \) with crank \( m \) counted according to the weight \( \omega \).

For each of the four spt functions, we give an example to illustrate a congruence.

Example 3.1.

\[
N_{\overline{S}}(0, 3, 3) = N_{\overline{S}}(1, 3, 3) = N_{\overline{S}}(2, 3, 3) = \frac{1}{3} \overline{\text{spt}}(3),
\]

\[
N_{\overline{S}_1}(0, 3, 3) = N_{\overline{S}_1}(1, 3, 3) = N_{\overline{S}_1}(2, 3, 3) = \frac{1}{3} \overline{\text{spt}}_1(3). \tag{3.4}
\]

We see the vector partitions from \( \overline{S} \) of 3, along with their weights and cranks are given as follows.

<table>
<thead>
<tr>
<th>( \overline{S} )-vector partition</th>
<th>weight</th>
<th>crank</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1, (-), (-), 2]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[1, (-), 1+1, (-)]</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>[1, (-), 2, (-)]</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>[1, 1, 1, (-)]</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>[1, 1+1, (-), (-)]</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>[1, 2, (-), (-)]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>[1+2, (-), (-)]</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>[3, (-), (-), (-)]</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Here we have used \(-\) to indicate the empty partition. We notice that in this example, all of the vector partitions are also from \( \overline{S}_1 \). We have

\[
N_{\overline{S}}(0, 3, 3) = N_{\overline{S}_1}(0, 3, 3) = 1 + 1 - 1 + 1 = 2,
\]

\[
N_{\overline{S}_1}(1, 3, 3) = N_{\overline{S}_1}(1, 3, 3) = 1 + 1 = 2,
\]

\[
N_{\overline{S}_1}(2, 3, 3) = N_{\overline{S}_1}(2, 3, 3) = 1 + 1 = 2.
\]

Example 3.2.

\[
N_{\overline{S}_2}(0, 3, 4) = N_{\overline{S}_2}(1, 3, 4) = N_{\overline{S}_2}(2, 3, 4) = \frac{1}{3} \overline{\text{spt}}_2(4).
\]

We see the vector partitions from \( \overline{S}_2 \) of 4, along with their weights and cranks are given as follows.

<table>
<thead>
<tr>
<th>( \overline{S}_2 )-vector partition</th>
<th>weight</th>
<th>crank</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2, (-), (-), (-)]</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>[2, (-), 2, (-)]</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>[4, (-), (-), (-)]</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus we have

\[
N_{\overline{S}_2}(0, 3, 4) = N_{\overline{S}_2}(1, 3, 4) = N_{\overline{S}_2}(2, 3, 4) = 1. \tag{3.5}
\]

Example 3.3.

\[
N_{S_2}(0, 5, 6) = N_{S_2}(1, 5, 6) = N_{S_2}(2, 5, 6) = N_{S_2}(3, 5, 6) = N_{S_2}(4, 5, 6) = \frac{1}{5} M_2 \text{spt}(6). \tag{3.6}
\]
We see the vector partitions from $S_2$ of 6, along with their weights and cranks are given as follows.

<table>
<thead>
<tr>
<th>S2-vector partition</th>
<th>weight</th>
<th>crank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[2, -, 4, -]$</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$[2, -, 2+2, -]$</td>
<td>1</td>
<td>$-2$</td>
</tr>
<tr>
<td>$[2, 2, 2, -]$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$[2, 4, - , -]$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$[2, 2+2, -, -]$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$[4+2, -, -, -]$</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$[6, -, -, -]$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

And so

$$N_{S_2}(0, 5, 6) = N_{S_2}(1, 5, 6) = N_{S_2}(2, 5, 6) = N_{S_2}(3, 5, 6) = N_{S_2}(4, 5, 6) = 1.$$  

3.2. $SP$-partition pairs. In this section we prove that

$$N_S(m, n) \geq 0, \ (3.7)$$

for all $m, n$ and provide a combinatorial interpretation in terms of partition pairs.

3.2.1. Proof of nonnegativity.

$$S(z, q) = \sum_{n=1}^{\infty} \sum_{m} N_S(m, n) z^m q^n \quad (3.8)$$

$$= \sum_{n=1}^{\infty} q^n (q^{n+1}; q)_{\infty} (z^{-1} q^n; q)_{\infty} (-q^{n+1}; q)_{\infty}$$

$$= \sum_{n=1}^{\infty} q^n (q^{2n}; q)_{\infty} (z^{-1} q^n; q)_{\infty} (q^{2n}; q)_{\infty}$$

$$= \sum_{n=1}^{\infty} q^n \sum_{k=0}^{\infty} (z q^{n+k}; q)_{\infty} (q)_{\infty} (1 - q^{2n}) \frac{1}{(q^{2n+1}; q^2)_{\infty}}$$

by [11, Prop. 4.1]. The inequality (3.7) clearly follows. Replacing $n$ by $2n+1$ and $2n$ in the second line of (3.8) gives $N_{S_1}(m, n) \geq 0$ and $N_{S_2}(m, n) \geq 0$, respectively.

3.2.2. The sptcrank in terms of partition pairs. We define

$$SP = \{ \vec{\lambda} = (\lambda_1, \lambda_2) \in \mathcal{P} \times \mathcal{P} : 0 < s(\lambda_1) \leq s(\lambda_2) \}$$

and all parts of $\lambda_2$ that are $\geq 2s(\lambda_1) + 1$ are odd.}

First we show that

$$spt(n) = \sum_{\vec{\lambda} \in SP \atop |\vec{\lambda}| = |\lambda_1| + |\lambda_2| = n} 1. \quad (3.10)$$

$$\sum_{n=1}^{\infty} spt(n) q^n = \sum_{n=1}^{\infty} q^n (-q^{n+1}; q)_{\infty} (1 - q^n)^2 (q^{n+1}; q)_{\infty} \quad (3.11)$$

$$= \sum_{n=1}^{\infty} q^n (1 - q^n)^2 (q^{n+1}; q)_{\infty} \times \frac{(q^{2n+2}; q^2)_{\infty}}{(q^{n+1}; q)_{\infty}}$$

$$= \sum_{n=1}^{\infty} q^n (q^n; q)_{\infty} \times \frac{1}{(1 - q^n)(1 - q^{n+1}) \cdots (1 - q^{2n})(q^{2n+1}; q^2)_{\infty}}$$
From (3.8) we have

Proof. Next we define a crank of partition pairs

We let $\overline{SP}_1$ be the set of $\bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ with $s(\lambda_1)$ odd and let $\overline{SP}_2$ be the set of $\bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ with $s(\lambda_1)$ even. Then in the same fashion we have

Next we define a crank of partition pairs $\bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ by interpreting the coefficient of $z^m q^n$ in (3.8). For $\bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}$ we define

and define

$$ \text{crank}(\bar{\lambda}) = \begin{cases} \# \text{ of parts of } \lambda_1 \geq s(\lambda_1) + k - 1, & \text{if } k > 0, \\ \# \text{ of parts of } \lambda_1 - 1 & \text{if } k = 0. \end{cases} $$

where $k = k(\bar{\lambda})$. We have

**Theorem 3.4.**

$$ N_{\overline{SP}}(m, n) = \# \text{ of } \bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP} \text{ with } |\bar{\lambda}| = n \text{ and } \text{crank}(\bar{\lambda}) = m, \tag{3.17} $$

$$ N_{\overline{SP}_1}(m, n) = \# \text{ of } \bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}_1 \text{ with } |\bar{\lambda}| = n \text{ and } \text{crank}(\bar{\lambda}) = m, \tag{3.18} $$

$$ N_{\overline{SP}_2}(m, n) = \# \text{ of } \bar{\lambda} = (\lambda_1, \lambda_2) \in \overline{SP}_2 \text{ with } |\bar{\lambda}| = n \text{ and } \text{crank}(\bar{\lambda}) = m. \tag{3.19} $$

**Proof.** From (3.8) we have

$$ S(z, q) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} N_{\overline{SP}}(m, n) z^m q^n \tag{3.20} $$

$$ = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{q^n}{(1 - q^n) \cdots (1 - q^{n+k-1}) (q^k; q)_{\infty}} \times z^{-k} q^n k^{[n+k-1]} k \times \frac{1}{(1 - q^{2n}) (q^{2n+1}; q^2)_{\infty}} $$

$$ = \sum_{n=1}^{\infty} \frac{q^n}{(z q^n; q)_{\infty}} \times \frac{1}{(1 - q^{2n}) (q^{2n+1}; q^2)_{\infty}} $$

$$ + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^n}{(1 - q^n) \cdots (1 - q^{n+k-1}) (q^k; q)_{\infty}} \times z^{-k} q^n k^{[n+k-1]} k \times \frac{1}{(1 - q^{2n}) (q^{2n+1}; q^2)_{\infty}}. $$
We note that the $q$-binomial coefficient $\frac{n+k-1}{k}$ is the generating function for partitions into parts $\leq n-1$ with number of parts $\leq k$. Thus we see that $q^{nk}\frac{n+k-1}{k}$ is the generating function for partitions into exactly $k$ parts, where $n \leq j \leq 2n - 1$. Hence

$$S(z, q) = \sum_{n=1}^{\infty} \sum_{\vec{\lambda} \in \text{SP}} z^{\text{crank}(\vec{\lambda})} q^{\mid \vec{\lambda} \mid}$$

(3.21)

$$= \sum_{\vec{\lambda} \in \text{SP}} z^{\text{crank}(\vec{\lambda})} q^{\mid \vec{\lambda} \mid}.$$  (3.22)

The result (3.17) follows. The results (3.18), (3.19), follow in a similar fashion. □

### 3.2.3. Examples

We illustrate our combinatorial interpretation of each $spt$, $spt_2$, $spt_3$ congruence in terms of the crank of $\text{SP}$-partition pairs.

**Example 3.5** ($n = 3$). The overpartitions of 3 with smallest parts not overlined are 3, 2 + 1, 2 + 1, 1 + 1 + 1 so that $spt(3) = 6$. There are 6 $\text{SP}$-partition pairs of 3.

<table>
<thead>
<tr>
<th>$\text{SP}$-partition pair</th>
<th>$k$</th>
<th>crank</th>
<th>(mod 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[3, −]</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>[2 + 1, −]</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>[1 + 1 + 1, −]</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>[1 + 1, 1]</td>
<td>1</td>
<td>−1</td>
<td>2</td>
</tr>
<tr>
<td>[1, 1 + 1]</td>
<td>2</td>
<td>−2</td>
<td>1</td>
</tr>
<tr>
<td>[1, 2]</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that

$$N_{\text{SP}}(0, 3, 3) = N_{\text{SP}}(1, 3, 3) = N_{\text{SP}}(2, 3, 3) = 2 = \frac{spt(3)}{3}.$$  

**Example 3.6** ($n = 5$). There are 10 overpartitions of 5 with smallest parts odd and not overlined:

$$5, \quad 4 + 1, \quad \overline{4} + 1, \quad 3 + 1 + 1, \quad \overline{3} + 1 + 1$$

$$2 + 2 + 1, \quad \overline{2} + 2 + 1, \quad 2 + 1 + 1 + 1, \quad \overline{2} + 1 + 1 + 1, \quad 1 + 1 + 1 + 1 + 1,$$

so that $spt_2(5) = 20$. There are 20 $\text{SP}_2$-partitions pairs of 5.
\[
\begin{array}{ccc}
\text{SP}_2\text{-partition pair} & k & \text{crank (mod 5)} \\
[1, 1+1+1+1] & 4 & -4 \\
[1, 2+1+1] & 2 & -2 \\
[1, 2+2] & 0 & 0 \\
[1, 3+1] & 1 & -1 \\
[1+1, 1+1+1] & 3 & -3 \\
[1+1, 2+1] & 1 & -1 \\
[1+1, 3] & 0 & 1 \\
[1+1+1, 1+1] & 2 & -2 \\
[1+1+1, 2] & 0 & 2 \\
[2+1, 1+1] & 2 & -2 \\
[2+1, 2] & 0 & 1 \\
[1+1+1+1, 1] & 1 & -1 \\
[2+1+1, 1] & 1 & 0 \\
[3+1, 1] & 1 & 0 \\
[1+1+1+1+1, -] & 0 & 4 \\
[2+1+1+1, -] & 0 & 3 \\
[2+2+1, -] & 0 & 2 \\
[3+1+1, -] & 0 & 2 \\
[4+1, -] & 0 & 1 \\
[5, -] & 0 & 0 \\
\end{array}
\]

We see that
\[
N_{S_2}(0, 5, 5) = N_{S_2}(1, 5, 5) = N_{S_2}(2, 5, 5) = N_{S_2}(3, 5, 5) = N_{S_2}(4, 5, 5) = 4 = \frac{\text{spt}_2(5)}{5}.
\]

**Example 3.7** \((n = 8)\). There are 9 overpartitions of 8 with smallest parts even and not overlined:

\[
\begin{align*}
8, & \quad 6 + 2, & \quad 6 + 2, & \quad 3 + 3 + 2, & \quad 3 + 3 + 2 \\
4 + 4, & \quad 4 + 2 + 2, & \quad 4 + 2 + 2, & \quad 2 + 2 + 2 + 2,
\end{align*}
\]

so that \(\text{spt}_2(8) = 15\). There are 15 \(\text{SP}_2\)-partitions pairs of 8.

\[
\begin{array}{ccc}
\text{SP}_2\text{-partition pair} & k & \text{crank (mod 5)} \\
[2, 2+2+2] & 3 & -3 \\
[2, 3+3] & 2 & -2 \\
[2, 4+2] & 1 & -1 \\
[2+2, 2+2] & 2 & -2 \\
[2+2, 4] & 0 & 1 \\
[4, 4] & 1 & -1 \\
[3+2, 3] & 1 & 0 \\
[2+2+2, 2] & 1 & -1 \\
[4+2, 2] & 1 & 0 \\
[2+2+2+2, -] & 0 & 3 \\
[3+3+2, -] & 0 & 2 \\
[4+2+2, -] & 0 & 2 \\
[4+4, -] & 0 & 1 \\
[6+2, -] & 0 & 1 \\
[8, -] & 0 & 0 \\
\end{array}
\]

We see that
\[
N_{S_2}(0, 5, 8) = N_{S_2}(1, 5, 8) = N_{S_2}(2, 5, 8) = N_{S_2}(3, 5, 8) = N_{S_2}(4, 5, 8) = 3 = \frac{\text{spt}_2(8)}{5}.
\]
3.3. SPT-crank for marked overpartitions. Andrews, Dyson and Rhoades [2] defined a marked partition as a pair \((\lambda, k)\) where \(\lambda\) is a partition and \(k\) is an integer identifying one of its smallest parts; i.e. \(k = 1, 2, \ldots, \nu(\lambda)\), where \(\nu(\lambda)\) is the number of smallest parts of \(\lambda\). They asked for a statistic like the crank which would divide the relevant marked partitions into \(t\) equal classes for \(t = 5, 7, 13\) thus explaining the congruences (1.1), (1.2), (1.3). This problem was solved by Chan, Ji and Zang [17] for the cases \(t = 5, 7\). They defined an spt-crank for double marked partitions and found a bijection between double marked partitions and marked partitions. It is an open problem to define the spt-crank directly in terms of marked partitions. In this section we solve the analogous problem for overpartitions.

3.3.1. Definition of sptcrank for marked overpartitions. We define a marked overpartition of \(n\) as a pair \((\pi, j)\) where \(\pi\) is an overpartition of \(n\) in which the smallest part is not overlined and \(j\) is an integer \(1 \leq j \leq \nu(\pi)\), where as above \(\nu(\pi)\) is the number of smallest parts of \(\pi\). It is clear that \(\text{spt}(n) = \# \text{ of marked overpartitions } (\pi, j) \text{ of } n\). (3.23)

For example, there are 6 marked overpartitions of 3:

\[
(2 + 1, 1), \quad (2 + 1, 1), \quad (3, 1),
(1 + 1 + 1, 1), \quad (1 + 1 + 1, 2), \quad (1 + 1 + 1, 3),
\]

so that \(\text{spt}(3) = 6\).

To define the sptcrank of a marked overpartition we first need to define a function \(k(m, n)\). For positive integers \(m, n\) such that \(m \geq n + 1\) we write

\[m = b2^j,\]

where \(b\) is odd and \(j \geq 0\). For a given odd integer \(b\) and a positive integer \(n\) we define \(j_0 = j_0(b, n)\) to be the smallest nonnegative integer \(j_0\) such that \(b2^{j_0} \geq n + 1\).

We define

\[k(m, n) = \begin{cases} 
0, & \text{if } b \geq 2n, \\
2^{j - j_0}, & \text{if } b2^{j_0} < 2n, \\
0, & \text{if } b2^{j_0} = 2n.
\end{cases} \quad (3.24)

We note that if \(j_0 \geq 1\) then \(b2^{j_0} \leq 2n\) so that the function \(k(m, n)\) is well-defined. For a partition

\[\pi : m_1 + m_2 + \cdots + m_a \]

into distinct parts

\[m_1 > m_2 > \cdots > m_a \geq n + 1\]

we define the function

\[k(\pi, n) = \sum_{j=1}^{a} k(m_j, n) = \sum_{m \in \pi} k(m, n). \quad (3.25)\]

For a marked overpartition \((\pi, j)\) we let \(\pi_1\) be the partition formed by the non-overlined parts of \(\pi\), \(\pi_2\) be the partition (into distinct parts) formed by the overlined parts of \(\pi\), so that

\[s(\pi_2) > s(\pi_1)\]

We define a function

\[\overline{k}(\pi, j) = \nu(\pi_1) - j + k(\pi_2, s(\pi_1)). \quad (3.26)\]

Finally we can now define

\[\text{sptcrank}(\pi, j) = \begin{cases} 
\# \text{ of parts of } \pi_1 \geq s(\pi_1) + \overline{k}, & \text{if } \overline{k} = \overline{k}(\pi, j) > 0, \\
\# \text{ of parts of } \pi_1 - 1 & \text{if } \overline{k} = \overline{k}(\pi, j) = 0.
\end{cases} \quad (3.27)\]

We state our main theorem.

**Theorem 3.8.** (i) The residue of the sptcrank mod 3 divides the marked overpartitions of \(3n\) into 3 equal classes.
(ii) The residue of the \( \text{sptcrank} \mod 3 \) divides the marked overpartitions of \( 3n \) with smallest part odd into 3 equal classes.

(iii) The residue of the \( \text{sptcrank} \mod 5 \) divides the marked overpartitions of \( 5n \) with smallest part odd into 5 equal classes.

(iv) The residue of the \( \text{sptcrank} \mod 3 \) divides the marked overpartitions of \( 3n \) and of \( 3n+1 \) with smallest part even into 3 equal classes.

(v) The residue of the \( \text{sptcrank} \mod 5 \) divides the marked overpartitions of \( 5n+3 \) with smallest part even into 5 equal classes.

3.3.2. Examples.

Example 3.9 \((n = 3)\). There are 6 marked overpartitions of 3 so that \( \text{spt}(3) = 6 \).

<table>
<thead>
<tr>
<th>((\pi, j))</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>(\nu(\pi_1))</th>
<th>(k(\pi_2; s(\pi_1)))</th>
<th>(\text{sptcrank} \mod 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2 + 1, 1))</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((1 + 1 + 1, 1))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((1 + 1 + 1, 2))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((1 + 1 + 1, 3))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>((2 + 1, 1))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((3, 1))</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that the residue of the \( \text{sptcrank} \mod 3 \) divides the marked overpartitions of 3 into 3 equal classes. This illustrates Theorem 3.8(i).

Example 3.10 \((n = 5)\). There are 20 marked overpartitions of 5 with smallest part odd so that \( \text{spt}_1(5) = 20 \).

<table>
<thead>
<tr>
<th>((\pi, j))</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>(\nu(\pi_1))</th>
<th>(k(\pi_2; s(\pi_1)))</th>
<th>(\text{sptcrank} \mod 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((4 + 1, 1))</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((3 + 1 + 1, 1))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((3 + 1 + 1, 2))</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 1))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 2))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 3))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 4))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>((2 + 2 + 1, 1))</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((1 + 1 + 1 + 1 + 1, 1))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>((1 + 1 + 1 + 1 + 1, 2))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>((1 + 1 + 1 + 1 + 1, 3))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>((1 + 1 + 1 + 1 + 1, 4))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 1))</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 2))</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((2 + 1 + 1 + 1, 3))</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((2 + 2 + 1, 1))</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>((3 + 1 + 1, 1))</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>((3 + 1 + 1, 2))</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>((4 + 1, 1))</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((5, 1))</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that the residue of the \( \text{sptcrank} \mod 5 \) divides the marked overpartitions of 5 with odd smallest part into 5 equal classes. This illustrates Theorem 3.8(iii).
Example 3.11 \((n = 5)\). There are 15 marked overpartitions of 8 with smallest part even so that \(\overline{spt_2}(8) = 15\).

<table>
<thead>
<tr>
<th>((\pi,j))</th>
<th>(\pi_1)</th>
<th>(\pi_2)</th>
<th>(\nu(\pi_1))</th>
<th>(k(\pi_2,s(\pi_1)))</th>
<th>(K)</th>
<th>(\overline{sptcrank}) (mod 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5 + 2, 1)</td>
<td>2 (;;;;;6)</td>
<td>1 (;;;;;4)</td>
<td>(;;;;;0)</td>
<td>(;;;;;2)</td>
<td>(-2)</td>
<td>(-3)</td>
</tr>
<tr>
<td>(\bar{A} + 2, 2, 1)</td>
<td>2 (;;;;;6)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(\bar{A} + 2, 2)</td>
<td>2 (;;;;;2)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(;;;;;1)</td>
<td>(-3)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(\bar{A} + 3, 2, 1)</td>
<td>3 (;;;;;3)</td>
<td>1 (;;;;;1)</td>
<td>(;;;;;0)</td>
<td>(;;;;;0)</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(2 + 2 + 2 + 2 + 2)</td>
<td>2 (;;;;;2)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(2 + 2 + 2 + 2 + 2)</td>
<td>2 (;;;;;2)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(;;;;;1)</td>
<td>(-3)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(2 + 2 + 2 + 2 + 2)</td>
<td>2 (;;;;;2)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(;;;;;1)</td>
<td>(-3)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(3 + 3 + 2, 1)</td>
<td>3 (;;;;;3)</td>
<td>1 (;;;;;1)</td>
<td>(;;;;;0)</td>
<td>(;;;;;0)</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(4 + 2 + 2 + 2)</td>
<td>4 (;;;;;2)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(;;;;;1)</td>
<td>(-3)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(4 + 2 + 2 + 2)</td>
<td>4 (;;;;;2)</td>
<td>1 (;;;;;2)</td>
<td>(;;;;;0)</td>
<td>(;;;;;1)</td>
<td>(-3)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(6 + 2, 1)</td>
<td>6 (;;;;;2)</td>
<td>1 (;;;;;1)</td>
<td>(;;;;;0)</td>
<td>(;;;;;0)</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(8, 1)</td>
<td>8 (;;;;;2)</td>
<td>1 (;;;;;1)</td>
<td>(;;;;;0)</td>
<td>(;;;;;0)</td>
<td>(-2)</td>
<td>(-1)</td>
</tr>
</tbody>
</table>

We see that the residue of the \(\overline{sptcrank}\) (mod 5) divides the marked overpartitions of 8 with even smallest part into 5 equal classes. This illustrates Theorem 3.8(v).

3.3.3. Proof of the main result.

Bijection 3.12. Let \(M\) denote the set of marked overpartitions. There is a weight-preserving bijection

\[ \Phi : M \rightarrow SP \]

such that,

\[ K(\pi,j) = k(\bar{\lambda}), \]

and

\[ \overline{sptcrank}(\pi,j) = \overline{crank}(\bar{\lambda}), \]

where

\[ \bar{\lambda} = (\lambda_1, \lambda_2) = \Phi(\pi,j). \]

Once this theorem is proved, the main result Theorem 3.8 will then follow from Theorems 3.4, 2.1, 2.4, and 2.3 and the appropriate dissections listed in Section 2.

Before we can construct the bijection \(\Phi\), we need to extend Euler’s Theorem that the number of partitions of \(n\) into distinct parts equals the number of partitions of \(n\) into odd parts. Let \(n\) be a nonnegative integer.

Let \(D_n\) denote the set of partitions into distinct parts \(\geq n + 1\). Let \(P_n\) denote the set of partitions into parts \(\geq n + 1\) in which all parts > \(2n\) are odd. Then we have

Theorem 3.13. Let \(n \geq 0\) and \(\ell \geq 1\). Then the number of partitions of \(\ell\) from \(D_n\) equals the number of partitions of \(\ell\) from \(P_n\).

Remark 3.14. Euler’s Theorem is the case \(n = 0\).

Proof.

\[ 1 + \sum_{\pi \in P_n} q^{s(\pi)} = \prod_{j=n+1}^\infty (1 + q^j) \]

\[ = \prod_{j=n+1}^\infty \frac{(1 + q^j)(1 - q^j)}{(1 - q^j)} = \prod_{j=n+1}^\infty \frac{(1 - q^{2j})}{(1 - q^j)} \]

\[ = \frac{1}{(1 - q^{n+1})(1 - q^{n+2}) \cdots (1 - q^{2n})(q^{2n+1}; q^2)_{\infty}} \]
\[= 1 + \sum_{\pi \in \mathcal{P}_n} q^{\vert \pi \vert}. \quad (3.31)\]

The result follows by considering the coefficient of \(q^k\) on both sides of this identity. \(\square\)

We require a bijective proof of this theorem. Glaisher [28, p.23] has a well-known straightforward bijective proof of Euler’s Theorem. We extend this in a natural way to obtain a bijective proof of our theorem.

**Bijection 3.15.** Let \(n \geq 1\). There is a weight-preserving bijection
\[
\Psi_n : \mathcal{D}_n \rightarrow \mathcal{P}_n,
\]
such that
\[k(\pi, n) = \# \text{ of parts of } \lambda \leq 2n - 1. \quad (3.32)\]

We define the weight-preserving bijection \(\Psi_n\) as follows. Let \(\pi \in \mathcal{D}_n\). We describe the image of each part \(m\) of \(\pi\). We note that \(m \geq n + 1\) and as before we write
\[m = \beta 2^j,\]
where \(\beta\) is odd and \(j \geq j_0 = j_0(\beta, n)\). We map
\[m \mapsto \underbrace{2^{j-j_0}}_{2j_0 \text{ times}} \beta, 2^{j_0} \beta, \ldots, 2^{j_0} \beta, \quad (3.33)\]
which preserves the weight since
\[m = (2^{j-j_0}) (2^{j_0} \beta) .\]
Recall that \(j_0 = j_0(\beta, n)\) is the smallest nonnegative integer \(j_0\) such that
\[\beta 2^{j_0} \geq n + 1,\]
and we see that each image part is \(\geq n + 1\). If an image part \(2^{j_0} \beta\) is even then \(j_0 \geq 1\) and
\[2^{j_0} \beta \leq 2n,\]
as noted before so that each even image part is \(\leq 2n\). Also any odd image part \(2^{j_0} \beta = \beta \geq n + 1\). This induces a well-defined map
\[
\Psi_n : \mathcal{D}_n \rightarrow \mathcal{P}_n.
\]
We show this map is onto. Let \(\lambda\) be a partition in \(\mathcal{P}_n\) with part \(p\) and multiplicity \(\mu\). Then we write
\[p = \beta 2^{j_0} \geq n + 1,\]
where \(\beta\) is odd and \(j_0 = j_0(\beta, n)\). Now we write \(\mu\) in binary
\[\mu = \sum_a 2^a.\]
This part \(p\) with multiplicity \(\mu\) arises from a partition in \(\mathcal{D}_n\) with parts \(\beta 2^{j_0+\mu_0}\) under the action of \(\Psi_n\). We see that \(\Psi_n\) is onto and Theorem 3.13 implies that it is a weight-preserving bijection.

Next we prove (3.32). We let
\[\tilde{k} = \# \text{ of parts } p \text{ of } \lambda \text{ where } p \leq 2n - 1.\]

We note that if \(m\) is a part of \(\pi\) then as before
\[m = \beta 2^j \geq n + 1,\]
where \(\beta\) is odd and \(j \geq j_0\). Under the map \(\Psi_n\) the image of \(m\) is given by (3.33). This contributes \(2^{j-j_0}\) to \(\tilde{k}\) provided \(\beta 2^{j_0} < 2n\), and (3.32) follows.
Example 3.16 \((n = 3)\). We illustrate the bijection \(\Psi_n\) when \(n = 3\). There are 6 partitions of 16 in \(D_3\), the set of partitions into distinct parts \(\geq 4\):

\[
\begin{align*}
7 + 5 + 4 & \rightarrow 7 \cdot 2^0, 5 \cdot 2^0, 1 \cdot 2^2 & \rightarrow 7 \cdot 2^0, 5 \cdot 2^0, 1 \cdot 2^2 & \rightarrow 7 + 5 + 4 \\
9 + 7 & \rightarrow 9 \cdot 2^0, 7 \cdot 2^0 & \rightarrow 9 \cdot 2^0, 7 \cdot 2^0 & \rightarrow 9 + 7 \\
10 + 6 & \rightarrow 5 \cdot 2^1, 3 \cdot 2^1 & \rightarrow 5 \cdot 2^0, 5 \cdot 2^0, 3 \cdot 2^1 & \rightarrow 6 + 5 + 5 \\
11 + 5 & \rightarrow 11 \cdot 2^0, 5 \cdot 2^0 & \rightarrow 11 \cdot 2^0, 5 \cdot 2^0 & \rightarrow 11 + 5 \\
12 + 4 & \rightarrow 3 \cdot 2^2, 1 \cdot 2^2 & \rightarrow 3 \cdot 2^1, 3 \cdot 2^1, 1 \cdot 2^2 & \rightarrow 6 + 6 + 4 \\
16 & \rightarrow 1 \cdot 2^4 & \rightarrow 1 \cdot 2^2, 1 \cdot 2^2, 1 \cdot 2^2, 1 \cdot 2^2 & \rightarrow 4 + 4 + 4 + 4
\end{align*}
\]

Each partition has been mapped into \(\mathcal{P}_3\), the set of partitions with smallest part \(\geq 4\) and all parts \(> 6\) are odd.

We are now ready to construct our weight-preserving bijection \(\Phi: \mathcal{M} \rightarrow \mathfrak{SF}\). Suppose \((\pi, j)\) is a marked overpartition with \(1 \leq j \leq \nu(\pi)\). As described before we let \(\pi_1\) be the partition formed by the non-overlined parts of \(\pi\), \(\pi_2\) be the partition (into distinct parts) formed by the overlined parts of \(\pi\), so that

\[s(\pi_2) > s(\pi_1) = s(\pi) = n.\]

We let

\[
\begin{align*}
\pi_1 &= (n, n, \ldots, n, n_2, n_3, \ldots, n_a), \\
\pi_2 &= (m_1, m_2, \ldots, m_b),
\end{align*}
\]

where

\[
\begin{align*}
n < n_2 & \leq n_3 \leq \cdots \leq n_a, \\
n < m_1 & < m_2 < \cdots < m_b.
\end{align*}
\]

Define

\[\Phi(\pi, j) = \lambda = (\lambda_1, \lambda_2),\]

where

\[
\begin{align*}
\lambda_1 &= (n, n, \ldots, n, n_2, n_3, \ldots, n_a), \\
\lambda_2 &= (n, n, \ldots, n, \Psi_n(\pi_2)).
\end{align*}
\]

The map \(\Phi\) is clearly weight-preserving. We see that \(s(\lambda_1) = n\) and \(\lambda_1 \in \mathcal{P}\). In addition, \(\Psi_n(\pi_2)\) is a partition into parts \(\geq n + 1\) with all parts \(\geq 2n + 1\) being odd so that \(\lambda_2 \in \mathfrak{SF}\) and the map \(\Phi\) is well-defined. By (3.10) and (3.23) we need only show that \(\Phi\) is onto.

Let

\[\lambda = (\lambda_1, \lambda_2) \in \mathfrak{SF}.
\]

Let \(n = s(\lambda_1)\) so that \(\lambda_1, \lambda_2 \in \mathcal{P}\), \(s(\lambda_2) \geq s(\lambda_1) = n\) and all parts of \(\lambda_2 \geq 2n + 1\) are odd. Let \(j = \nu(\lambda_1)\), and let \(\ell\) denote the number of parts of \(\lambda_2\) that are equal to \(n\), so that \(j \geq 1\) and \(\ell \geq 0\). Remove any parts of \(\lambda_2\) equal to \(n\) to form the partition \(\tilde{\lambda}_2\) and add the parts removed from \(\lambda_2\) to \(\lambda_1\) to form the partition \(\pi_1\). Now let \(\pi_2 = \Psi_n^{-1}(\tilde{\lambda}_2)\) so that \(\pi_2\) is a partition into distinct parts \(\geq n + 1\). Form the partition \(\pi\) by overlining the parts of \(\pi_2\) and adding them to \(\pi_1\). We see that \((\pi, j) \in \mathcal{M}\), \(1 \leq j \leq \nu(\pi) = j + \ell\) and

\[\Phi(\pi, j) = \lambda = (\lambda_1, \lambda_2).
\]

The map \(\Phi\) is onto and hence a bijection.

Now we prove (3.28), (3.29). As before we let

\[\Phi(\pi, j) = \lambda = (\lambda_1, \lambda_2),\]

where \(\lambda_1, \lambda_2\) are given in (3.35), so that \(s(\lambda_1) = n\), and \(1 \leq j \leq \nu(\pi) = \nu(\pi_1)\). Then

\[
k(\lambda) = \nu - j + (\text{# of parts of } \Psi_n(\pi_2) \leq 2n - 1) \\
= \nu(\pi_1) - j + k(\pi_2, n) \quad \text{(by (3.32))}
\]
\[ = \nu(\pi_1) - j + k(\pi_2, s(\pi_1)) \]
\[ = \kappa(\pi, j), \]
which proves (3.28). Finally, from (3.16) we have
\[
\text{crank}(\lambda) = \begin{cases} 
\text{(# of parts of } \pi_1 \geq s(\pi_1) + \kappa) - \kappa & \text{if } \kappa > 0 \\
\text{(# of parts of } \pi_1) - 1 & \text{if } \kappa = 0, 
\end{cases}
\]
(3.37)
where \( \kappa = \kappa(\pi, j) \), since \( \kappa(\pi, j) = k(\lambda) \) and if \( \kappa = \kappa(\pi, j) = 0 \), then \( \nu(\pi_1) = j \) and \( k(\pi_2, s(\pi_1)) = 0 \) in which case, the number of parts of \( \pi_1 \) equals the number of parts of \( \lambda_1 \). Hence we have
\[
\text{crank}(\lambda) = \text{spcrank}(\pi, j),
\]
which is (3.29). This completes the proof of our main result.

4. Proofs of Theorems 2.1, 2.2, 2.3, 2.4

These four proofs all follow the same method. The generating function for the rank series is rewritten using Watson’s transformation and then the two variable series matches the difference of a rank and crank by Bailey’s Lemma.

We recall a pair of sequences of functions, \((\alpha_n, \beta_n)\), forms a Bailey for \((a, q)\) if
\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}.
\]
(4.1)
The limiting case of Bailey’s Lemma gives for a Bailey pair \((\alpha_n, \beta_n)\) that
\[
\sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \frac{aq}{p_1 p_2} \beta_n = \frac{(aq/\rho_1, aq/\rho_2; q)_\infty}{(aq, aq/p_1 p_2; q)_\infty} \sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \left( \frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n.
\]
(4.2)

Proof of Theorem 2.1. Letting \( a = 1, b = 1, c = -1, d \to 0 \) in Theorem 8 of [22], we have a Bailey pair for \((1, q)\) given by
\[
\alpha_n = \begin{cases} 
1 & n = 0 \\
(-1)^n 2q^n & n \geq 1 
\end{cases},
\]
\[
\beta_n = \frac{1}{(q^2; q^2)_n}.
\]

Then by Bailey’s Lemma we have that
\[
\sum_{n=0}^{\infty} \frac{(z, z^{-1}; q)_n q^n}{(-q; q)_n} = \frac{(zq, z^{-1}; q)_\infty}{(q, q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(z, z^{-1}; q)_n (-1)^n 2q^{n+2} q^n}{(zq, z^{-1}; q)_n} \right) = \frac{(zq, z^{-1}; q)_\infty}{(q, q)_\infty} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n+2} q^n}{(1-qz^n)(1-z^{-1}q^n)} \right).
\]

But then
\[
S(z, q) = \sum_{n=1}^{\infty} \frac{q^n (-q^{n+1}, q^{n+1}; q)_\infty}{(z, z^{-1}; q)_\infty} = \frac{(-q, q; q)_\infty}{(z, z^{-1}; q)_\infty} \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q)_n q^n}{(-q, q)_n} = \frac{(-q, q; q)_\infty}{(z, z^{-1}; q)_\infty} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n+2} q^n}{(1-qz^n)(1-z^{-1}q^n)} \right) = \frac{(-q, q; q)_\infty}{(z, z^{-1}; q)_\infty}
\]

Proof of Theorem 2.2. Before using a Bailey pair, we will apply a limiting case of Watson’s transformation to the generating function of \(N2(m,n)\). We recall Watson’s transformation gives

\[
\sum_{n=0}^{\infty} \frac{(aq/bc, d, e; q)_n (q_{2z}^2)_n}{(q, ac/b, ac/e; q)_n}\n\]

\[
= \frac{(aq/d, aq/e; q)_\infty}{(aq, aq/d; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, \sqrt{aq}, -\sqrt{aq}, b, c, d, e; q)_n (aq)^{2n}(-1)^{n}q^{n(n-1)/2}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e; q)_n (bcde)_n}.
\]

Applying this with \(q \mapsto q^2, a = 1, b = z, c = z^{-1}, d = -q\) and \(e \to \infty\) we get the following.

\[
\sum_{n=0}^{\infty} q^{n^2} \frac{(-q; q^2)_n}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} = \lim_{e\to\infty} \sum_{n=0}^{\infty} \frac{(q^2, -q, e; q^2)_n (-1)^n e^{-n} q^n}{(q^2, z^{-1}q^2, zq^2; q^2)_n}.
\]

\[
= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 + \lim_{a\to1, e\to\infty} \sum_{n=1}^{\infty} \frac{(1-a) (-q^2, z, z^{-1}, c; q^2)_n q^{2n+2n}}{(1-\sqrt{a})(-1, z^{-1}q^2, zq^2; q^2)_n e^n}\right)
\]

\[
= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1+q^{2n})(1-z)(1-z^{-1})(-1)^n q^{2n^2+n}}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right).
\]

By [31, page 468] we have, after replacing \(q\) by \(q^2\), a Bailey pair for \((1, q^2)\) given by

\[
\alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^{n}q^{2n}(q^n + q^{-n}) & n \geq 1 \end{cases}, \quad \beta_n = \frac{1}{(-q, q^2; q^2)_n}.
\]

Then by Bailey’s Lemma we have that

\[
\sum_{n=0}^{\infty} \frac{(z, z^{-1}; q^2)_n q^{2n}}{(-q, q^2; q^2)_n} = \frac{(zq^2, z^{-1}q^2; q^2)_\infty}{(q^2, q^2; q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(z, z^{-1}; q^2)_n (-1)^n q^{2n^2+2n}(q^n + q^{-n})}{(zq^2, z^{-1}q^2; q^2)_n}\right)
\]

\[
= \frac{(zq^2, z^{-1}q^2; q^2)_\infty}{(q^2, q^2; q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+2n}(q^n + q^{-n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right).
\]

But then

\[
S_2(z, q) = \sum_{n=1}^{\infty} \frac{q^{2n} (q^{2n+2} - q^{2n+1}; q^2)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty}
\]

\[
= \frac{(-q, q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q^2)_n q^{2n}}{(-q, q^2; q^2)_n} - \frac{(-q, q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty}.
\]
Thus from page 468 of [31], we have the Bailey pair

\[
(\frac{-q, q^2, zq^2, z^{-1}q^2, q^2}{(z, z^{-1}, q^2, q^2; q^2)} \infty, 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^nq^{2n+1}(q^n + q^{-n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}) - (\frac{-q, q^2; q^2}{(z, z^{-1}; q^2)} \infty)
\]

This proves the theorem.

\[\square\]

**Proof of Theorem 2.3.** We have

\[
\mathcal{S}_2(z, q) = \sum_{n=1}^{\infty} q^{2n} \left( -q^{2n+1}, q^{2n+1}; q \right) \infty
\]

\[
= \frac{(-q, q; q) \infty}{(z, z^{-1}; q) \infty} \sum_{n=1}^{\infty} q^{2n} \left( z, z^{-1}; q \right)_{2n}
\]

\[
= \frac{(-q, q; q) \infty}{(z, z^{-1}; q) \infty} \sum_{n=0}^{\infty} \frac{(-q, q; q)_{2n}}{(-q, q; q)_{2n}} = \frac{(-q, q; q) \infty}{(z, z^{-1}; q) \infty}.
\]

Using the Bailey pair in proof of Theorem 2.1 along with the Bailey pair for (1, q)

\[
\alpha_n = \begin{cases} 
1 & n = 0 \\
(-1)^n & n \geq 1
\end{cases}
\]

\[
\beta_n = \frac{(-1)^n}{(-q, q; q)_n}
\]

from page 468 of [31], we have the Bailey pair

\[
\alpha_n = \begin{cases} 
1 & n = 0 \\
(-1)^n(1 + q^{n^2}) & n \geq 1
\end{cases}
\]

\[
\beta_n = \begin{cases} 
\frac{1}{2(q^2; q^2)_n} + \frac{(-1)^n}{2(q^2; q^2)_n} & n \equiv 0 \pmod{2} \\
\frac{1}{(q^2; q^2)_n} & n \equiv 1 \pmod{2}
\end{cases}
\]

Thus

\[
\sum_{n=0}^{\infty} \frac{q^{2n} (z, z^{-1}; q)_{2n}}{(-q, q; q)_{2n}}
\]

\[
= \sum_{n=0}^{\infty} (z, z^{-1}; q)_n q^n \beta_n
\]

\[
= \frac{(zq, z^{-1}q; q) \infty}{(q, q; q) \infty} \sum_{n=0}^{\infty} (z, z^{-1}; q)_n q^n \alpha_n
\]

\[
= \frac{(zq, z^{-1}q; q) \infty}{(q, q; q) \infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^nq^n(1+q^{n^2})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right).
\]
By Bailey’s Lemma we have then

\[ S_2(z, q) \]
\[ = \frac{(-q; q)_{\infty}}{(1 - z)(1 - z^{-1})(q; q)_{\infty}} \left( 1 + \sum_{n=1}^{\infty} \frac{(1 - z)(1 - z^{-1})(-1)^n q^n (1 + q^{n^2})}{(1 - zq^n)(1 - z^{-1}q^n)} \right) \]
\[ - \frac{(q^2; q^2)_{\infty}}{(1 - z)(1 - z^{-1})(zq, z^{-1}; q; q)_{\infty}}. \]

This proves the theorem.

Proof of Theorem 2.4. With \( S(z, q) \) and \( S_2(z, q) \) known, we also know \( S_1(z, q) \). However we can also derive the result from a Bailey pair as we have for the other series.

We have

\[ S_1(z, q) = \sum_{n=0}^{\infty} q^{2n+1} \frac{(-q^{2n+2}, q^{2n+2}; q)_{\infty}}{(z q^{2n+1}, z^{-1} q^{2n+1}; q)_{\infty}} \]
\[ = \frac{(-q, q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q)_{2n+1} q^{2n+1}}{(-q, q; q)_{2n+1}}. \]

By combining Bailey pairs as we did for \( S_2(z, q) \), we have a Bailey pair for \((1, q)\) given by

\[ \alpha_n = \begin{cases} 0 & n = 0 \\ (-1)^n(q^{n^2} - 1) & n \geq 1 \end{cases}, \]

\[ \beta_n = \frac{1}{2(q^2, q^2)_n} - \frac{(-1)^n}{2(q^2, q^2)_n} = \begin{cases} 0 & n \equiv 0 \pmod{2} \\ \frac{1}{(q, q^2)_n} & n \equiv 1 \pmod{2} \end{cases}. \]

By Bailey’s Lemma we have then

\[ \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q)_{2n+1} q^{2n+1}}{(-q, q; q)_{2n+1}} \]
\[ = \frac{(zq, z^{-1}; q)_{\infty}}{(q, q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q)_{n} q^n \alpha_n}{(zq, z^{-1}; q)_{n}} \]
\[ = \frac{(zq, z^{-1}; q)_{\infty}}{(q, q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{(1 - z)(1 - z^{-1}) q^n (-1)^n (q^{n^2} - 1)}{(1 - zq^n)(1 - z^{-1}q^n)}. \]

This gives

\[ S_1(z, q) = \frac{(-q; q)_{\infty}}{(1 - z)(1 - z^{-1})(q; q)_{\infty}} \sum_{n=1}^{\infty} \frac{(1 - z)(1 - z^{-1}) q^n (-1)^n (q^{n^2} - 1)}{(1 - zq^n)(1 - z^{-1}q^n)} \]

and completes the proof.

5. Dissections

Proofs of Theorems 2.5 and 2.7. We are to show

\[ \sum_{n=0}^{\infty} \sum_{r=0}^{2} N(r, 3n) \zeta_3^r q^n = \frac{(q^3; q^3)_\infty}{(q; q^2)_\infty} \frac{(q^2; q^2)_\infty}{(q^3; q^3)_\infty}, \]

(5.1)
\[
\sum_{n=0}^{\infty} \sum_{r=0}^{2} \eta(r, 3, 3n + 1) \zeta_5^n q^n = 2 \frac{(q^3; q^3)^{\infty}_{\infty}}{(q^6; q^6)_{\infty}}, \quad (5.2)
\]

\[
\sum_{n=0}^{\infty} \sum_{r=0}^{2} \eta^{2}(r, 3, 3n + 1) \zeta_5^n q^n = \frac{(q^6; q^6)^{\infty}_{\infty}}{(q^2; q^2)_{\infty} (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}. \quad (5.3)
\]

For (5.1) we have
\[
\sum_{n=0}^{\infty} \left( \eta(0, 3, 3n) + \eta(1, 3, 3n) \zeta_5 + \eta(2, 3, 3n) \zeta_5^2 \right) q^n
\]
\[
= \sum_{n=0}^{\infty} \left( \eta(0, 3, 3n) - \eta(1, 3, 3n) \right) q^n
\]
\[
= \frac{(q^3; q^3)^{\infty}_{\infty} (-q; q)_{\infty}}{(q^3; q^3)^{\infty}_{\infty} (-q^3; q^3)_{\infty}}
\]
\[
= \frac{(q^3; q^3)^{\infty}_{\infty} (q^2; q^2)_{\infty}}{(q^3; q^3)^{\infty}_{\infty} (q^6; q^6)_{\infty}}. \quad (5.4)
\]

The penultimate equality in (5.4) is the first part of Theorem 1.1 of [24], although we’ve omitted their –1 term. The –1 is due to how one interprets the empty overpartition and its rank. We use the convention that the empty overpartition has rank 0 and don’t adjust the q0 term of the generating function.

Equations (5.2) and (5.3) are also just restatements of results in [24] and [25], respectively.

**Proofs of Theorems 2.6 and 2.8.** We see we are to prove
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{4} \eta(k, 5, 5n) \zeta_k^n q^n = \frac{(q^3; q^3; q^{10})^{\infty}_{\infty} (q^5; q^5)_{\infty}}{(q^2; q^2; q^5)^{\infty}_{\infty} (q^{10}; q^{10})_{\infty}} + 2(\zeta_5 + \zeta_5^{-1})q \frac{(q^{10}; q^{10})_{\infty}}{(q^5; q^5; q^5)_{\infty}}, \quad (5.5)
\]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{4} \eta(k, 5, 5n + 2) \zeta_k^n q^n = \frac{2(1 - \zeta_5 - \zeta_5^{-1}) (q^{10}; q^{10})_{\infty}}{(q^2; q^2; q^5)_{\infty}}, \quad (5.6)
\]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{4} \eta(k, 5, 5n + 1) \zeta_k^n q^n = \frac{(-q^2; q^2; q^{10})_{\infty}}{(q^2; q^2; q^5)_{\infty}}, \quad (5.7)
\]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{4} \eta(k, 5, 5n + 1) \zeta_k^n q^n = (\zeta_5 + \zeta_5^{-1}) \frac{(-q^5; q^5; q^{10})_{\infty}}{(q^2; q^2; q^5)_{\infty}}. \quad (5.8)
\]

But we see that
\[
\eta(0, 5, 5n) + \eta(1, 5, 5n) \zeta_5 + \eta(2, 5, 5n) \zeta_5^2 + \eta(3, 5, 5n) \zeta_5^3 + \eta(4, 5, 5n) \zeta_5^4 \quad (5.9)
\]
\[
= \eta(0, 5, 5n) + \eta(1, 5, 5n) (\zeta_5 + \zeta_5^2) + \eta(2, 5, 5n) (\zeta_5^2 + \zeta_5^3) \quad (5.10)
\]
\[
= \eta(0, 5, 5n) - \eta(2, 5, 5n) + (\zeta_5 + \zeta_5^2) (\eta(1, 5, 5n) - \eta(2, 5, 5n)). \quad (5.11)
\]

By the difference formulas in [24] we have then
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{4} \eta(k, 5, 5n) \zeta_k^n q^n = \frac{(-q^2; q^2; q^5)_{\infty}}{(q^2; q^2; q^5)_{\infty}} \frac{(-q^5; q^5)_{\infty}}{(q^2; q^2; q^5)_{\infty}} + 2(\zeta_5 + \zeta_5^{-1})q \frac{(q^{10}; q^{10})_{\infty}}{(q^5; q^5; q^5)_{\infty}}
\]
\[
= \frac{(q^2; q^2; q^10)^{\infty}_{\infty} (q^5; q^5)^{\infty}_{\infty}}{(q^2; q^2; q^5)^{\infty}_{\infty} (q^{10}; q^{10})_{\infty}} + 2(\zeta_5 + \zeta_5^{-1})q \frac{(q^{10}; q^{10})_{\infty}}{(q^5; q^5; q^5)_{\infty}}. \quad (5.12)
\]

Equations (5.6), (5.7), and (5.8) are also just restatements of the results in [24] and [25].
**Proof of Theorem 2.9.** By definition we have

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{M}(m, n) \zeta_3^m q^n = \frac{(q^2; q^2)_\infty}{(\zeta_3 q; q)\zeta_3(q; q)_\infty} = \frac{(q^2; q^2)_\infty (q; q)\zeta_3}{(q^3; q^3)_\infty}.
\]

(5.13)

We see we are to show

\[
\frac{(q^2; q^2)_\infty (q; q)_\infty}{(q^3; q^3)_\infty} = \frac{(q^0; q^0)_4 (q^6; q^6)_\infty q^{18} (q^9; q^9)_\infty}{(q^3; q^3)\zeta_3^4 (q^9; q^9)\zeta_3^4 (q^6; q^6)_\infty} - 2q^2 \frac{(q^18; q^18)_\infty (q^9; q^9)_\infty}{(q^3; q^3)_\infty}.
\]

(5.14)

Replacing \(q\) by \(q^{1/3}\) and multiplying by \(\frac{(q; q)_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty}\), the proposition is equivalent to

\[
\frac{(q^{1/3}; q^{1/3})_\infty (q^{2/3}; q^{2/3})_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty} = \frac{(q^3; q^3)3 (q^2; q^2)_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty} - q^{1/3} 2q^{2/3} \frac{(q; q)_\infty (q^6; q^6)_\infty}{(q^3; q^3)_\infty(q^6; q^6)_\infty}.
\]

(5.15)

If we let \(v\) be the infinite continued fraction

\[
v = \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^2 + q^4}{1 + \frac{q^3 + q^5}{1 + \ldots}}}}
\]

(5.16)

then by Entry 3.3.1(a) of Ramanujan’s Lost notebook part I [4] we have

\[
v = q^{1/3} \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty}.
\]

(5.17)

Thus with \(x(q) = q^{-1/3} v\) we have

\[
x(q) = \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty} = \frac{(q; q)_\infty (q^6; q^6)_\infty}{(q^3; q^6)_\infty(q^2; q^2)_\infty}.
\]

(5.18)

(5.19)

But now (5.15) is exactly Theorem 2 of [16].

**Proof of Theorem 2.10.** We have

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{M}(m, n) \zeta_5^m q^n = \frac{(q^2; q^2)_\infty}{(\zeta_5 q, \zeta_5^{-1} q; q)\zeta_5}
\]

(5.20)

and so we find a dissection for this product.

By Lemma 3.9 of [19] we have

\[
1 = \frac{1}{(\zeta_5 q, \zeta_5^{-1} q; q)_\infty} = \frac{1}{(q^2, q^{20}; q^{25})_\infty} + \frac{(\zeta_5 + \zeta_5^{-1})q}{(q^{10}, q^{15}; q^{25})_\infty}.
\]

(5.21)

Replacing \(q\) by \(q^2\) in Lemma 3.18 in [19] we have

\[
(q^2; q^2)_\infty = (q^{50}; q^{50})_\infty \left(\frac{(q^{20}, q^{30}; q^{50})_\infty}{(q^{10}, q^{40}; q^{50})_\infty} - q^2 - \frac{(q^{10}, q^{40}; q^{50})_\infty}{(q^{20}, q^{30}; q^{50})_\infty}\right).
\]

(5.22)

Expanding the product of these two expressions then gives the result.
Proof of Theorem 2.11. We see

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M2(m, n) \zeta_3^n q^n = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q^6; q^6)_\infty}. \tag{5.23}$$

We are then to show

$$\frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q^6; q^6)_\infty} = \frac{(q^{18}; q^{18})^{10}_\infty (q^{12}; q^{12})^{4}_\infty (q^3; q^3)_\infty + q (q^{46}; q^{46})^{4}_\infty (q^9; q^9)_\infty (q^6; q^6)_\infty}{(q^{36}; q^{36})^{4}_\infty (q^9; q^9)_\infty (q^6; q^6)_\infty} - 2q^2 \frac{(q^{36}; q^{36})^{2}_\infty (q^9; q^9)_\infty (q^6; q^6)_\infty}{(q^{18}; q^{18})^{2}_\infty (q^{12}; q^{12})^{2}_\infty (q^3; q^3)_\infty}. \tag{5.24}$$

Noting \((-q; q^2)_\infty (q^2; q^2)_\infty = (-q; -q)_\infty (q^2; q^2)_\infty\), in equation (5.14) of the proof of Theorem 2.9 we replace $q$ by $-q$ and multiply by \(\frac{(-q^3; q^3)_\infty}{(-q^3; q^3)_\infty}\) to get

$$\frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q^6; q^6)_\infty} = \frac{(-q^3; q^3)_\infty}{(-q^3; q^3)_\infty} \frac{(q^{18}; q^{18})^{3}_\infty (q^9; q^9)_\infty}{(q^{36}; q^{36})^{3}_\infty (q^9; q^9)_\infty} - 2q^2 \frac{(q^{18}; q^{18})^{4}_\infty (q^9; q^9)_\infty}{(q^6; q^6)_\infty}. \tag{5.25}$$

But we have

$$(-q^3; q^3)_\infty = \frac{(q^6; q^6)_\infty^3}{(q^{12}; q^{12})^{3}_\infty (q^3; q^3)_\infty}, \tag{5.26}$$

Equations (5.25) and (5.26) with (5.24) then give the theorem.

Proof of Theorem 2.12. We have

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M2(m, n) \zeta^n q^n = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(\zeta_5 q^2, \zeta_5^{-1} q^2; q^2)_\infty}. \tag{5.27}$$

and so we find a dissection for this product.

Replacing $q$ by $q^2$ in Lemma 3.9 of [19] we have

$$\frac{1}{(\zeta_5 q^2, \zeta_5^{-1} q^2; q^2)_\infty} = \frac{1}{(q^{10}; q^{40}; q^{50})_\infty} + \frac{1}{(\zeta_5 + \zeta_5^{-1}) q^2 (q^{20}; q^{40}; q^{50})_\infty}. \tag{5.28}$$

Next we note that \((-q; q^2)_\infty (q^2; q^2)_\infty = (-q; -q)_\infty\) and so replacing $q$ by $-q$ in Lemma 3.18 in [19] we have

$$(-q; q^2)_\infty (q^2; q^2)_\infty = \left(\frac{q_{10}; q_{15}; -q_{25}}{-q_{25}; q_{20}; -q_{25}}\right)_\infty + q^2 \frac{(-q^5; q^{20}; -q^{25})_\infty}{(q_{10}; -q_{15}; -q_{25})_\infty}. \tag{5.29}$$

$$(-q^2; q^{50}; q^{50})_\infty = \left(\frac{q_{10}; -q_{15}; -q_{35}; q^{40}; -q^{50}}{-q_{25}; q_{20}; -q_{35}; -q^{45}; q^{50}}\right)_\infty + q^2 \frac{(-q^5; q^{20}; q^{30}; -q^{45}; q^{50})_\infty}{(q_{10}; -q_{15}; -q_{35}; q^{40}; q^{50})_\infty}. \tag{5.30}$$

Multiplying out these two 5-dissections then gives

$$\frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(\zeta_5 q, \zeta_5^{-1} q; q)_\infty}. \tag{5.31}$$

27
Then equation (5.35) is given by equation (32.64) of [18] and (5.36) follows from (5.34).

Similarly we have

This proves the proposition.

\[ \varphi(q) = f(q, q) = \sum_{k=0}^{\infty} q^{k^2}. \]  

By Entry 19 of [12] we have

\[ f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \]  

Also we have

\[ \varphi(-q) = \frac{(q; q)_\infty}{(-q; q)_\infty} = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}. \]  

Proposition 1.

\[ \sum_{n=1}^{\infty} \frac{(-1)^n q^n (1 - q^n)}{1 - q^{3n}} = -\frac{1}{6} \left( 1 - \frac{(q; q)_\infty^6}{(q^2; q^2)_\infty^3 (q^3; q^3)_\infty^2} \right) = -\frac{1}{6} \left( 1 - \frac{\phi(-q)^2}{\phi(-q^3)} \right). \]  

Proof. As in [18] we let

\[ E_r(N; m) = \sum_{d \equiv r \pmod{m}} 1 - \sum_{d \equiv -r \pmod{m}} 1. \]  

Thus

\[ \sum_{N=1}^{\infty} q^N E_r(N; m) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{kn + rn} - q^{kn + (m-r)n} \]

\[ = \sum_{n=1}^{\infty} q^{rn} - q^{(m-r)n} \]

\[ = \sum_{n=1}^{\infty} \frac{q^{rn} - q^{(m-r)n}}{1 - q^{mn}}. \]

Similarly we have

\[ \sum_{N=1}^{\infty} q^N (E_r(N; m) - 2E_r(N/2; m)) = \sum_{n=1}^{\infty} \frac{(-1)^n (q^{rn} - q^{(m-r)n})}{1 - q^{mn}}. \]  

Then equation (5.35) is given by equation (32.64) of [18] and (5.36) follows from (5.34).
With this we then have
\[
\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - \zeta_3)(1 - \zeta_3^{-1})(-1)^n q^n}{(1 - \zeta q^n)(1 - \zeta_3^{-1} q^n)} \right) = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left( \frac{1}{2} + 3 \sum_{n=1}^{\infty} \frac{(-1)^n q^n (1 - q^{18})}{(1 - q^{18})} \right) \quad (5.41)
\]
\[
= \frac{(-q; q)_{\infty}}{2 (q; q)_{\infty}} \varphi(-q) \quad (5.42)
\]
\[
= \frac{(q; q)_{\infty}^2}{2 (-q; q)_{\infty}^2} \frac{(-q^3; q^3)_{\infty}}{(q^3; q^3)_{\infty}} \quad (5.43)
\]
\[
= \frac{(q; q)_{\infty}^2 (q^6; q^6)_{\infty}}{2 (-q; q)_{\infty}^2 (q^3; q^3)_{\infty}}. \quad (5.44)
\]

**Proposition 2.**
\[
\frac{(q; q)_{\infty}^2}{(-q; q)_{\infty}^2} = \frac{(q^9; q^9)_{\infty}^4}{(q^{18}; q^{18})_{\infty}^2} - 4q (q^{18}; q^{18})_{\infty} (q^9; q^9)_{\infty} (q^3; q^3)_{\infty} + 4q^2 (q^{18}; q^{18})_{\infty}^4 (q^3; q^3)_{\infty}^2. \quad (5.45)
\]

**Proof.** By the Corollary (page 49) to Entry 31 of [12], we have
\[
\phi(q) = \phi(q^9) + 2q f(q^3, q^{15}). \quad (5.46)
\]
Replacing \( q \) by \( -q \) we find that
\[
\phi(-q) = \frac{(q^9; q^9)_{\infty}^2}{(q^{18}; q^{18})_{\infty}^2} - 2q (q^3, q^{15}, q^{18}; q^{18})_{\infty}
\]
\[
= \frac{(q^9; q^9)_{\infty}^2}{(q^{18}; q^{18})_{\infty}^2} - 2q (q^{18}; q^{18})_{\infty}^2 (q^3; q^3)_{\infty}. \quad (5.47)
\]
Thus
\[
\frac{(q; q)_{\infty}^2}{(-q; q)_{\infty}^2} = \frac{(q^9; q^9)_{\infty}^4}{(q^{18}; q^{18})_{\infty}^2} - 4q (q^{18}; q^{18})_{\infty} (q^9; q^9)_{\infty} (q^3; q^3)_{\infty} + 4q^2 (q^{18}; q^{18})_{\infty}^4 (q^3; q^3)_{\infty}^2. \quad (5.48)
\]
This proves the proposition. \( \square \)

With equation (5.44) and Proposition 2, we have finished the proof of Theorem 2.13. \( \square \)

**Proof of Theorem 2.15.** We can determine \( S_1(i, q) \) from formulas about \( \phi \):
\[
\phi(q) - \phi(-q) = 4 \sum_{n=0}^{\infty} q^{(2n-1)^2}, \quad (5.49)
\]
\[
\phi(q)\phi(-q) = \phi(-q^2)^2, \quad (5.50)
\]
\[
\phi(-q^2)^2 = 1 + 4 \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{1 + q^{2n}}, \quad (5.51)
\]
\[
\phi(-q)^2 = 1 + 4 \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1 + q^{2n}}. \quad (5.52)
\]
First we define a primitive Dirichlet character modulo 5 by

\[ \chi_5(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{5} \\ i & \text{if } n \equiv 2 \pmod{5} \\ -i & \text{if } n \equiv 3 \pmod{5} \\ -1 & \text{if } n \equiv 4 \pmod{5} \\ 0 & \text{otherwise.} \end{cases} \]  

(5.59)

By (4.17) we have

\[ \mathcal{S}_1(i,q) = \frac{1}{\phi(-q)} \sum_{n=1}^{\infty} \frac{(-1)^n q^n (q^{n^2} - 1)}{1 + q^{2n}} \]  

(5.53)

\[ = \frac{1}{\phi(-q)} \left( \frac{\phi(-q)^2}{4} - \phi(-q)^2 \right) \]  

(5.54)

\[ = \frac{\phi(-q)}{\phi(-q)} \left( \phi(q) - \phi(-q) \right) \]  

(5.55)

\[ = \sum_{n=0}^{\infty} q^{(2n+1)^2}. \]  

(5.56)

**Proof of Theorem 2.14.** To start we set

\[ C(\tau) = 3 + 10 \sum_{n=1}^{\infty} \frac{(-1)^n (q^n - q^{4n})}{1 - q^{5n}}, \]  

(5.57)

\[ D(\tau) = 1 + 10 \sum_{n=1}^{\infty} \frac{(-1)^n (q^{2n} - q^{3n})}{1 - q^{5n}}. \]  

(5.58)

We claim \( C(\tau) \) and \( D(\tau) \) are elements of \( M_1(\Gamma_1(10)) \).

First we define a primitive Dirichlet character modulo 5 by

\[ \chi_5(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{5} \\ i & \text{if } n \equiv 2 \pmod{5} \\ -i & \text{if } n \equiv 3 \pmod{5} \\ -1 & \text{if } n \equiv 4 \pmod{5} \\ 0 & \text{otherwise.} \end{cases} \]  

As in [21] and [20] we set

\[ V_{\chi_5,1}(\tau) = \frac{3 + i}{10} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi_5(n) q^{mn} \]  

(5.59)

\[ = \frac{3 + i}{10} + \sum_{m=1}^{\infty} \frac{q^m - q^{4m}}{1 - q^{5m}} + \frac{i}{10} \sum_{m=1}^{\infty} \frac{q^{2m} - q^{3m}}{1 - q^{5m}} \]  

(5.60)

and

\[ V_{\chi_5,1}(\tau) = \frac{3 + i}{10} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \chi_5(n) q^{mn} \]  

(5.61)

\[ = \frac{3 + i}{10} + \sum_{m=1}^{\infty} \frac{q^m - q^{4m}}{1 - q^{5m}} - \frac{i}{10} \sum_{m=1}^{\infty} \frac{q^{2m} - q^{3m}}{1 - q^{5m}}. \]  

(5.62)

Then \( V_{\chi_5,1}(\tau) \in M_1(\Gamma_0(5), \chi_5) \) and \( V_{\chi_5,1}(\tau) \in M_1(\Gamma_0(10), \chi_5) \). Thus we have \( V_{\chi_5,1}(2\tau) \in M_1(\Gamma_0(10), \chi_5) \) and \( V_{\chi_5,1}(2\tau) \in M_1(\Gamma_0(10), \chi_5) \). Here \( M_k(\Gamma, \chi) \) is the vector space of holomorphic modular forms of weight \( k \) with respect to the subgroup \( \Gamma \) of \( \Gamma_0 \) and with character \( \chi \).

We see

\[ C(\tau) = 5 \left( \frac{V_{\chi_5,1}(2\tau) - V_{\chi_5,1}(\tau) + 2V_{\chi_5,1}(2\tau) - V_{\chi_5,1}(\tau)}{30} \right), \]  

(5.64)
but since the characters are different, we must move from $\Gamma_0$ to $\Gamma_1$. That is to say we have $C(\tau), D(\tau) \in M_1(\Gamma_1(10))$. Noting $\frac{\eta(2\tau)^2}{\eta(\tau)^2}$ is a modular form of weight $-1$ for $\Gamma_1(8)$, we have then that $C(\tau) \frac{\eta(2\tau)^2}{\eta(\tau)^2}$ and $D(\tau) \frac{\eta(2\tau)^2}{\eta(\tau)^2}$ are modular functions with respect to $\Gamma_1(40)$. By modular function, we mean a modular form of weight zero.

We use the following generalized eta notation as in [30],

$$\eta_{8,g}(\tau) = e^{\pi i P_g(\frac{\tau}{8}) \delta \tau} \prod_{m \equiv g \pmod{\delta}} (1 - q^m) \prod_{m \equiv -g \pmod{\delta}} (1 - q^m)$$

(5.66)

where

$$P_g(t) = \{t\}^2 - \{t\} + \frac{1}{6}.$$  

(5.67)

So for $g = 0$ we have

$$\eta_{8,0}(\tau) = q^{\frac{\tau}{2}} (q^g; q^{8})_\infty = \eta(\delta \tau)^2$$

(5.68)

and for $0 < g < \delta$ we have

$$\eta_{8,m}(\tau) = q^{\frac{P_2(\frac{\tau}{8} + m)}{2}} (q^g, q^{8-g}; q^{8})_\infty.$$  

(5.69)

**Proposition 3.**

$$C(\tau) \frac{\eta(2\tau)}{\eta(\tau)^2} = C_0(q^5) + qC_1(q^5) + q^2C_2(q^5) + q^3C_3(q^5) + q^4C_4(q^5)$$

(5.70)

where

$$C_0(q) = \frac{(q^5; q^5)_\infty^2 (q^2; q^2)_\infty^2}{(q^{10}; q^{10})_\infty (q; q)_\infty^4} C(\tau),$$

(5.71)

$$C_1(q) = -4 \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q^2, q^8, q^{10}; q^{10})_\infty},$$

(5.72)

$$C_2(q) = 2 \frac{(q^{10}; q^{10})_\infty}{(q^4, q^6, q^{10}; q^{10})_\infty},$$

(5.73)

$$C_3(q) = -6 \frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q^4, q^6, q^{10}; q^{10})_\infty},$$

(5.74)

$$C_4(q) = 2 \frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q^4, q^6, q^{10}; q^{10})_\infty}.$$  

(5.75)

**Proof.** Multiplying both sides of (5.70) by $\frac{\eta(2\tau)}{\eta(\tau)^2}$ and noting the powers of $q$ from $\eta_{8,g}$ really do match, we see this proposition is equivalent to

$$C(\tau) \frac{\eta(2\tau)^2}{\eta(\tau)^4} = \frac{\eta(2\tau)^2 \eta(25\tau)^2 \eta(10\tau)^2}{\eta(\tau)^2 \eta(50\tau)^2 \eta(5\tau)^4} C(5\tau)$$

$$- 4 \frac{\eta_{2,0}(\tau)^{1/2} \eta_{50,0}(\tau)^{1/2} \eta_{50,20}(\tau)}{\eta_{1,0}(\tau) \eta_{50,10}(\tau)^2 \eta_{50,15}(\tau)}$$

$$+ 2 \frac{\eta_{2,0}(\tau)^{1/2} \eta_{50,0}(\tau)^{1/2}}{\eta_{1,0}(\tau) \eta_{50,5}(\tau) \eta_{50,20}(\tau)}$$

$$- 6 \frac{\eta_{2,0}(\tau)^{1/2} \eta_{50,0}(\tau)^{1/2}}{\eta_{1,0}(\tau) \eta_{50,10}(\tau) \eta_{50,15}(\tau)}$$

(5.75)
However we have \( \eta(2\tau)\eta(25\tau)^2\eta(10\tau)^2 \) is a weight \(-1\) modular form for \( \Gamma_1(200) \) by Theorem 1.64 of [27] and so
\[
\frac{\eta(2\tau)\eta(25\tau)^2\eta(10\tau)^2}{\eta(\tau)^2\eta(50\tau)\eta(5\tau)^4} C(5\tau)
\]
is a modular function for \( \Gamma_1(200) \). By Theorem 3 of [30], the four other generalized eta quotients on the right hand side of (5.76) are also modular functions on \( \Gamma_1(200) \).

We recall some facts about modular functions as in [29] and use the notation in [12]. Suppose \( f \) is a modular function with respect to the subgroup \( \Gamma \) of \( \Gamma_0(1) \) of finite index. For \( A \in \Gamma_0(1) \) we have a cusp given by \( \zeta = A^{-1}\infty \). The width of the cusp \( N = N(\Gamma, \zeta) \) is given by
\[
N(\Gamma, \zeta) = \min\{k > 0 : A^{-1}TKA \in \Gamma_0(1)\}, \quad \text{(5.76)}
\]
where \( T \) is the translation matrix
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]
If
\[
f(A^{-1}\tau) = \sum_{m=m_0}^{\infty} b_m q^{m/N} \quad \text{(5.77)}
\]
and \( b_m \neq 0 \), then we say \( m_0 \) is the order of \( f \) at \( \zeta \) with respect to \( \Gamma \) and we denote this value by \( \text{Ord}_\Gamma(f; \zeta) \). By \( \text{ord}(f; \zeta) \) we mean the invariant order of \( f \) at \( \zeta \) given by
\[
\text{ord}(f; \zeta) = \frac{\text{Ord}_\Gamma(f; \zeta)}{N}. \quad \text{(5.78)}
\]
For \( z \) in the upper half plane \( \mathcal{H} \), we write \( \text{ord}(f; z) \) for the order of \( f \) at \( z \) as an analytic function in \( z \). We define the order of \( f \) at \( z \) with respect to \( \Gamma \) by
\[
\text{Ord}_\Gamma(f; z) = \frac{\text{ord}(f; z)}{m}, \quad \text{(5.79)}
\]
where \( m \) is the order of \( z \) as a fixed point of \( \Gamma \).

We then have the well known valence formula for modular functions as the weight zero case of the valence formula for modular forms, which is Theorem 4.1.4 of [29]. Suppose a subset \( \mathcal{F} \) of \( \mathcal{H} \cup \{\infty\} \cup \mathbb{Q} \) is a fundamental region for the action of \( \Gamma \) along with a complete set of inequivalent cusps, if \( f \) is not the zero function then
\[
\sum_{z \in \mathcal{F}} \text{Ord}_\Gamma(f; z) = 0. \quad \text{(5.80)}
\]
To prove (5.76), we use the valence formula with \( f \) being the difference of the two sides of (5.76). We note the only poles of \( f \) can be at the cusps corresponding to \( \Gamma_1(200) \) and so
\[
\sum_{z \in \mathcal{F}} \text{Ord}_\Gamma(f; z) \geq \sum_{\zeta \in C} \text{Ord}_\Gamma(f; \zeta) \quad \text{(5.81)}
\]
where \( C \) is a set of inequivalent cusps.

But if we have a lower bound on the cusps not equivalent to \( \infty \), say
\[
\sum_{\zeta \in C} \text{Ord}_\Gamma(f; \zeta) \geq -M, \quad \text{(5.82)}
\]
and we knew \( \text{Ord}_\Gamma(f; \infty) > M \), then by the valence formula \( f \) must be identically zero. That is to say to prove (5.76) we would need only verify the \( q \)-series expansions agree past \( q^M \).
Noting $C(\tau)$ is a holomorphic modular form, in terms of getting a lower bound on the sum of the orders, we may ignore it. Using Theorem 4 of [30], we can compute the order of the generalized eta quotients at the cusps. Including $\infty$, there are 336 inequivalent cusps for $\Gamma_1(200)$. To get a lower bound on the sum of orders at cusps not equivalent to $\infty$, at each cusp we take the minimum order of the six generalized eta quotients in (5.76). Using Maple for the calculations, we find

\[ \sum_{\zeta \in C \setminus \{\infty\}} \text{Ord}_f(f; \zeta) \geq -1840. \] (5.83)

However we also verify in Maple that $f$ vanishes past $q^{2000}$ and so the equality holds.

\[ \square \]

**Proposition 4.**

\[ D(\tau) \frac{\eta(2\tau)}{\eta(\tau)^2} = D_0(q^5) + qD_1(q^5) + q^2D_2(q^5) + q^3D_3(q^5) + q^4D_4(q^5) \] (5.84)

where

\[ D_0(q) = \frac{(q^5; q^5) \cdot (q^2; q^2)^2}{(q^{10}; q^{10}) (q^4; q^4) (q^6; q^6)^2} D(\tau), \] (5.85)

\[ D_1(q) = 2 \frac{(q^4, q^6, q^{10}; q^{10})}{(q^2, q^8; q^4)^2 (q^3, q^7; q^4)^2}, \] (5.86)

\[ D_2(q) = -6 \frac{(q^{10}; q^{10})}{(q^4, q^8; q^{10}) (q^3, q^7; q^{10})}, \] (5.87)

\[ D_3(q) = -2 \frac{(q^{10}; q^{10})}{(q^2, q^8; q^{10}) (q^3, q^7; q^{10})}, \] (5.88)

\[ D_4(q) = 4 \frac{(q^2, q^6, q^{10}; q^{10})}{(q, q^9; q^{10}) (q^4, q^6; q^{10})^2}. \] (5.89)

**Proof.** Since $D$ is also a weight 1 form for $\Gamma_1(10)$ and these are the same products as in the previous proposition, we also need only verify the corresponding equality between modular functions holds past $q^{1840}$. This verification is done in Maple.

\[ \square \]

**Proposition 5.**

\[ (2C(\tau) - D(\tau)) \frac{(q^2; q^2) \cdot (q^5; q^5)^2}{(q; q)_\infty} = 5 \frac{(q^4, q^6; q^{10})}{(q^2, q^4; q^5)^2}. \] (5.90)

**Proof.** We see this proposition is equivalent to

\[ (2C(\tau) - D(\tau)) \frac{\eta(2\tau)^2}{\eta(\tau)^4} = 5 \frac{\eta_{10,4}(\tau)}{\eta_{5,2}(\tau)^2}. \] (5.91)

However we know the left hand side of (5.91) to be a modular function for $\Gamma_1(40)$. Using Theorem 3 of [30] we find that the right hand side is as well. Comparing the orders at cusps as we did in the proof of Proposition 3, we find a lower bound for the sum of orders at the cusps other than $\infty$ to be $-48$. However we verify in Maple that (5.91) holds past $q^{30}$ and so the equality must hold.

\[ \square \]

**Proposition 6.**

\[ (3D(\tau) - C(\tau)) \frac{(q^2; q^2)^2}{(q; q)_\infty^4} = 10 \frac{q}{(q^3, q^4, q^6, q^7; q^{10}) (q^5; q^{10})^2}. \] (5.92)
Proof. We see this proposition is equivalent to

$$(3D(\tau) - C(\tau)) \frac{\eta(2\tau)^2}{\eta(\tau)^4} = \frac{1}{\eta_{10,3}(\tau) \eta_{10,4}(\tau) \eta_{10,5}(\tau)}.$$  \hspace{1cm} (5.93)$$

Again both sides are modular functions for $\Gamma_1(40)$ and taking the minimum of orders gives that we need only verify the equality in (5.93) holds past $q^{40}$.

With these propositions we can complete the proof of Theorem 2.14. We have

\[
\begin{align*}
\frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - \zeta_5)(1 - \zeta_5^{-1})(-1)^n q^n}{(1 - \zeta_5 q^n)(1 - \zeta_5^{-1} q^n)} \right) \\
= \frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - \zeta_5)(1 - \zeta_5^{-1})(-1)^n q^n (1 - q^n)(1 - \zeta_5^2 q^n)(1 - \zeta_5^3 q^n)}{(1 - q^{5n})} \right) \\
= \frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} + (3 + \zeta_5 + \zeta_5^2) \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n}}{1 - q^{5n}} - (4 + 3\zeta_5 + 3\zeta_5^2) \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{1 - q^{5n}} \\
+ (4 + 3\zeta_5 + 3\zeta_5^2) \sum_{n=1}^{\infty} \frac{(-1)^n q^{4n}}{1 - q^{5n}} - (3 + \zeta_5 + \zeta_5^2) \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n}}{1 - q^{5n}} \right) \right)
\end{align*}
\]

\[
\begin{align*}
= \frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} + (3 + \zeta_5 + \zeta_5^2) \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n} - q^{3n}}{1 - q^{5n}} - (4 + 3\zeta_5 + 3\zeta_5^2) \sum_{n=1}^{\infty} \frac{(-1)^n (q^{2n} - q^{3n})}{1 - q^{5n}} \right) \\
= \frac{(-q; q)_\infty}{(q; q)_\infty} \left( \frac{1}{2} - \frac{3(3 + \zeta_5 + \zeta_5^2)}{10} + \frac{4 + 3\zeta_5 + 3\zeta_5^2}{10} + \frac{3 + \zeta_5 + \zeta_5^2}{10} C(\tau) - \frac{4 + 3\zeta_5 + 3\zeta_5^2}{10} D(\tau) \right) \\
= \frac{(-q; q)_\infty}{10(q; q)_\infty} \left( (3 + \zeta_5^2 + \zeta_5^3) C(\tau) - (4 + 3\zeta_5^2 + 3\zeta_5^3) D(\tau) \right) \\
= B_0(q^5) + qB_1(q^5) + q^2 B_2(q^5) + q^3 B_3(q^5) + q^4 B_4(q^5) \hspace{1cm} (5.94)
\end{align*}
\]

where

\[
\begin{align*}
B_0(q) &= \frac{(q^5; q^5)_\infty (q^2; q^2)_\infty}{10(q^{10}; q^{10})_\infty (q; q)_\infty} \left( (3 + \zeta_5^2 + \zeta_5^3) C(\tau) - (4 + 3\zeta_5^2 + 3\zeta_5^3) D(\tau) \right) \\
&= \frac{(q^5; q^5)_\infty (q^2; q^2)_\infty}{10(q^{10}; q^{10})_\infty (q; q)_\infty} \left( (2 - \zeta_5 - \zeta_5^{-1}) C(\tau) - (1 - 3\zeta_5 - 3\zeta_5^{-1}) D(\tau) \right) \\
&= \frac{(q^5; q^5)_\infty (q^2; q^2; q^5 q^5)_\infty}{2(q^{10}; q^{10})_\infty (q^2; q^2; q^5)_\infty} + \frac{(\zeta_5 + \zeta_5^{-1})}{(q^{10}; q^{10})_\infty (q^3; q^4; q^6; q^7; q^{10})_\infty (q^5; q^{10})_\infty q(q^5; q^5)_\infty}, \hspace{1cm} (5.95)
\end{align*}
\]

\[
\begin{align*}
B_1(q) &= (\zeta_5 + \zeta_5^{-1} - 1) \frac{(q^4; q^4; q^6; q^{10})_\infty}{(q^2; q^6; q^{10})_\infty (q^4; q^6; q^{10})_\infty}, \hspace{1cm} (5.96)
\end{align*}
\]

\[
\begin{align*}
B_2(q) &= (1 - 2\zeta_5 - 2\zeta_5^{-1}) \frac{(q^4; q^6; q^{10})_\infty (q^3; q^4; q^{10})_\infty}{(q^{10}; q^{10})_\infty (q^4; q^6; q^{10})_\infty}, \hspace{1cm} (5.97)
\end{align*}
\]

\[
\begin{align*}
B_3(q) &= - \frac{(q^{10}; q^{10})_\infty}{(q^2; q^2; q^{10})_\infty (q^4; q^6; q^{10})_\infty}, \hspace{1cm} (5.98)
\end{align*}
\]

\[
\begin{align*}
B_4(q) &= (\zeta_5 + \zeta_5^{-1}) \frac{(q^2; q^2; q^8; q^{10})_\infty}{(q^2; q^6; q^{10})_\infty (q^4; q^6; q^{10})_\infty}, \hspace{1cm} (5.99)
\end{align*}
\]

This finished the proof of Theorem 2.14.
In section 3 we proved the coefficients of $\mathcal{S}(z, q)$, $\mathcal{S}_1(z, q)$, and $\mathcal{S}_2(z, q)$ are nonnegative by showing each summand $\frac{q^n(1-z)^n}{(zq^n, z^{-1}q^n;q)_{\infty}^2}$ has nonnegative coefficients. Numerical evidence suggests $\mathcal{S}_2(z, q)$ also has nonnegative coefficients. However, the corresponding individual summands for $\mathcal{S}_2(z, q)$ do not have nonnegative coefficients themselves. In particular we find the coefficient of $q^{10n}$ in $q^{5} \left(\frac{1-z}{zq^n} \right)^{n} / \left(1-zq^n \right)\left(1-z^{-1}q^n \right)$ to be $z^{-1} + z - 1$. Thus for $\mathcal{S}_2(z, q)$ a more complicated argument is required.

**Conjecture 1.** For all $m$ and $n$ we have $N_{\mathcal{S}_2}(m, n)$ is nonnegative.

Related to the nonnegativity of these coefficients is the difference between the first rank and crank moment. If we let $N(m, n)$ denote the number of partitions of $n$ with rank $m$ and $M(m, n)$ denote the number of partitions of $n$ with crank $m$, then for $k \geq 1$ the $k$th rank moment $N_k(n)$ and $k$th crank moment $M_k(n)$ are given by

$$N_k(n) = \sum_{m \in \mathbb{Z}} m^k N(m, n), \quad (6.1)$$

$$M_k(n) = \sum_{m \in \mathbb{Z}} m^k M(m, n). \quad (6.2)$$

These rank and crank moments were introduced by Atkin and the first author in [8]. To allow for non-trivial odd moments Andrews, Chan, and Kim in [5] defined the modified rank and crank moments by

$$N_k^+(n) = \sum_{n=1}^{\infty} m^k N(m, n), \quad (6.3)$$

$$M_k^+(n) = \sum_{n=1}^{\infty} m^k M(m, n). \quad (6.4)$$

In the same paper they proved for all positive integers $n$ that $M_k^+(n) > N_k^+(n)$. This was done by manipulating the generating function for $M_k^+(n) - N_k^+(n)$ and carefully grouping the terms in such a way that it is clear the coefficients are positive. However it turns out that $M_k^+(n) - N_k^+(n) = N_S(0, n)$, the latter was proved to be nonnegative in [7] and so it is immediate that $M_k^+(n) \geq N_k^+(n)$.

Recently Andrews, Chan, Kim, and Osburn in [6] considered the moments for the rank and crank of overpartitions,

$$\overline{N}_k^+(n) = \sum_{n=1}^{\infty} m^k \overline{N}(m, n), \quad (6.5)$$

$$\overline{M}_k^+(n) = \sum_{n=1}^{\infty} m^k \overline{M}(m, n). \quad (6.6)$$

In that paper they prove $\overline{M}_k^+(n) > \overline{N}_k^+(n)$, As we’ll prove shortly, it also turns out that $\overline{M}_k^+(n) - \overline{N}_k^+(n) = N_S(0, n)$. Thus the nonnegativity of the coefficients of $\mathcal{S}(z, q)$ gives $\overline{M}_k^+(n) \geq \overline{N}_k^+(n)$.

To begin we use [19, equation (7.15)] that

$$\frac{(q; q)_\infty}{(zq, z^{-1}q, q)_\infty} = \frac{1}{(q; q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right), \quad (6.7)$$

so we have

$$\frac{(-q,q; q)_\infty}{(zq, z^{-1}; q)_\infty} = \frac{(-q,q; q)_\infty}{(q; q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right). \quad (6.8)$$
With this we can express \( \mathcal{S}(z, q) \) as follows.

\[
\mathcal{S}(z, q) = \frac{(-q; q)_{\infty}}{(1-z)(1-z^{-1}) (q; q_{\infty})} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-z q^n)(1-z^{-1} q^n)} \right) = \frac{(-q; q)_{\infty}}{(z, z^{-1}; q_{\infty})} \quad \text{(6.9)}
\]

\[
= \frac{(-q; q)_{\infty}}{(1-z)(1-z^{-1}) (q; q_{\infty})} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-z q^n)(1-z^{-1} q^n)} \right)
\]

\[
= \frac{(-q; q)_{\infty}}{(1-z)(1-z^{-1}) (q; q_{\infty})} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-z q^n)(1-z^{-1} q^n)} \right) \quad \text{(6.10)}
\]

\[
= \frac{(-q; q)_{\infty}}{(q; q_{\infty})} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-z q^n)(1-z^{-1} q^n)} - 2 \frac{(-q; q)_{\infty}}{(q; q_{\infty})} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1-z q^n)(1-z^{-1} q^n)} \quad \text{(6.11)}
\]

\[
= \frac{(-q; q)_{\infty}}{(q; q_{\infty})} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1-q^n)} \left( \sum_{m=0}^{\infty} z^m q^{nm} + \sum_{m=1}^{\infty} z^{-m} q^{nm} \right)
\]

\[
- 2 \frac{(-q; q)_{\infty}}{(q; q_{\infty})} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1-q^{2n})} \left( \sum_{m=0}^{\infty} z^m q^{nm} + \sum_{m=1}^{\infty} z^{-m} q^{nm} \right). \quad \text{(6.12)}
\]

In the last equality we have used that

\[
\frac{1 - q^{2n}}{(1-z q^n)(1-z^{-1} q^n)} = \frac{1}{1 - z q^n} + \frac{1}{1 - z^{-1} q^n} = 1. \quad \text{(6.13)}
\]

But \( \sum_{n=0}^{\infty} N_{\mathcal{S}}(0, n) q^n \) is the coefficient of \( z^0 \) in \( \mathcal{S}(z, q) \). From the above we see that

\[
\sum_{n=0}^{\infty} N_{\mathcal{S}}(0, n) q^n = \frac{(-q; q)_{\infty}}{(q; q_{\infty})} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1-q^n)} - 2 \frac{(-q; q)_{\infty}}{(q; q_{\infty})} \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1-q^{2n})}. \quad \text{(6.14)}
\]

With (6.14) and Proposition 2.1 of [6] we have

\[
\overline{M}_1(q) - \overline{R}_1(q) = \sum_{n=1}^{\infty} N_{\mathcal{S}}(0, n) q^n. \quad \text{(6.15)}
\]

As explained earlier, we know each \( N_{\mathcal{S}}(0, n) \) to be nonnegative and so this is another proof that \( \overline{M}_1^+(n) \geq \overline{N}_1^+(n) \).

There is also the \( d = e = 1 \) case for the general \( \overline{p}(n) \) function, which as noted in [15] reduces to \( \overline{pp}(n)/4 \), where \( \overline{pp}(n) \) is the number of overpartition pairs of \( n \). The methods in this paper do not give a new proof of the congruences for \( \overline{pp}(n) \). Using Bailey’s lemma on a two variable generating function and applying Watson’s transformation to the generating function for the rank of overpartition pairs does at first appear to give a difference between the rank of overpartition pairs and some residual crank. However, the resulting crank is

\[
\frac{(-q; q)_{\infty}^2}{(z q, z^{-1} q; q_{\infty})}, \quad \text{(6.16)}
\]

which can be written in terms of the rank for overpartition pairs as in equation (2.1) of [13]. In particular, the generating function for the rank of overpartition pairs is

\[
\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1, -1; q)_{n} q^n}{(z q, z^{-1} q; q)_{n}}. \quad \text{(6.17)}
\]

\[
\frac{4}{(1+z)(1+z^{-1})} + \sum_{n=1}^{\infty} \frac{(-1, -1; q)_{n} q^n}{(z q, z^{-1} q; q)_{n}} = \frac{4 (-q; q)_{\infty}^2}{(z q, z^{-1} q; q_{\infty})}. \quad \text{(6.18)}
\]

Thus the method of proving congruences in this paper only gives the proofs already given by Bringmann and Lovejoy in [13].
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