CONGRUENCES FOR ANDREWS' SPT-FUNCTION MODULO POWERS OF 5, 7 AND 13

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Dedicated to my friend and mentor Michael D. Hirschhorn on the occasion of his 63rd birthday

Abstract. Congruences are found modulo powers of 5, 7 and 13 for Andrews' smallest parts partition function $\operatorname{spt}(n)$. These congruences are reminiscent of Ramanujan's partition congruences modulo powers of 5, 7 and 11. Recently, Ono proved explicit Ramanujan-type congruences for $\operatorname{spt}(n)$ modulo ℓ for all primes $\ell \ge 5$ which were conjectured earlier by the author. We extend Ono's method to handle the powers of 5, 7 and 13 congruences. We need the theory of weak Maass forms as well as certain classical modular equations for the Dedekind eta-function.

1. Introduction

Andrews [3] defined the function $\operatorname{spt}(n)$ as the number of smallest parts in the partitions of n. He related this function to the second rank moment. He also proved some surprising congruences mod 5, 7 and 13. Namely, he showed that

$$\operatorname{spt}(n) = np(n) - \frac{1}{2}N_2(n), \tag{1.1}$$

where $N_2(n)$ is the second rank moment function [4] and $p(n)$ is the number of partitions of n, and he proved that

$$\operatorname{spt}(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$

$$\operatorname{spt}(7n+5) \equiv 0 \pmod{7}, \tag{1.3}$$

$$\operatorname{spt}(13n+6) \equiv 0 \pmod{13}. \tag{1.4}$$

Bringmann [9] studied analytic, arithmetic and asymptotic properties of the generating function for the second rank moment as a quasi-weak Maass form. Further congruence properties of Andrews' spt-function were found by the author [16], Folsom and Ono [14] and Ono [22]. In particular, Ono [22] proved that if $\left(\frac{1-24n}{\ell}\right) = 1$ then

$$\operatorname{spt}(\ell^2 n - \tfrac{1}{24}(\ell^2-1)) \equiv 0 \pmod{\ell}, \tag{1.5}$$

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for any prime $\ell \geq 5$. This amazing result was originally conjectured by the author[1]. Earlier special cases were observed by Tina Garrett [17] and her students.

We prove some suprising congruences for $\operatorname{spt}(n)$ modulo powers of 5, 7 and 13. For a, b, $c \geq 3$,

$$\operatorname{spt}(5^a n + \delta_a) + 5\operatorname{spt}(5^{a-2} n + \delta_{a-2}) \equiv 0 \pmod{5^{2a-3}}, \tag{1.6}$$

$$\operatorname{spt}(7^b n + \lambda_b) + 7\operatorname{spt}(7^{b-2} n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2)\rfloor}}, \tag{1.7}$$

$$\operatorname{spt}(13^c n + \gamma_c) - 13\operatorname{spt}(13^{c-2} n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}, \tag{1.8}$$

where δ_a, λ_b and γ_c are the least nonnegative residues of the reciprocals of 24 mod 5^a, 7^b and 13^c respectively. This together with (1.2)–(1.4) implies that

$$\operatorname{spt}(5^a n + \delta_a) \equiv 0 \pmod{5^{\lfloor \frac{a+1}{2}\rfloor}}, \tag{1.9}$$

$$\operatorname{spt}(7^b n + \lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+1}{2}\rfloor}}, \tag{1.10}$$

$$\operatorname{spt}(13^c n + \gamma_c) \equiv 0 \pmod{13^{\lfloor \frac{c+1}{2}\rfloor}}, \tag{1.11}$$

for a, b, $c \geq 1$. These congruences are reminiscent of Ramanujan's partition congruences for powers of 5, 7 and 11:

$$p(5^a n + \delta_a) \equiv 0 \pmod{5^a}, \tag{1.12}$$

$$p(7^b n + \lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+2}{2}\rfloor}}, \tag{1.13}$$

$$p(11^c n + \varphi_c) \equiv 0 \pmod{11^c}, \tag{1.14}$$

for all a, b, $c \geq 1$. Here φ_c is the reciprocal of 24 mod 11^c. The congruences mod powers of 5 and 7 were proved by Watson [25], although many of the details had been worked out earlier by Ramanujan in an unpublished manuscript. The powers of 11 congruence was proved by Atkin [7].

Following Ono [22], we define

$$\mathbf{a}(n) := 12\operatorname{spt}(n) + (24n-1)p(n), \tag{1.15}$$

for $n \geq 0$, and define

$$\alpha(z) := \sum_{n\geq 0} \mathbf{a}(n) q^{n-\frac{1}{24}}, \tag{1.16}$$

where as usual $q = \exp(2\pi i z)$ and $\Im(z) > 0$. We note that $\operatorname{spt}(0) = 0$ and $p(0) = 1$. Bringmann [9] showed that $\alpha(24z)$ is the holomorphic part of a weight $\frac{3}{2}$ weak Maass form. Using this observation and the idea of using the weight $\frac{3}{2}$ Hecke operator $T(\ell^2)$ to annihilate the nonholomorphic part enabled Ono [22] to prove the general congruence (1.5). We use a similar idea. Instead of a Hecke operator we use Atkin's $U(\ell)$ operator to annihilate the nonholomorphic part.

We show that

$$\mathbf{a}(5^a n + \delta_a) + 5\,\mathbf{a}(5^{a-2} n + \delta_{a-2}) \equiv 0 \pmod{5^{\lfloor \frac{1}{2}(5a-7)\rfloor}}, \tag{1.17}$$

$$\mathbf{a}(7^b n + \lambda_b) + 7\,\mathbf{a}(7^{b-2} n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2)\rfloor}}, \tag{1.18}$$

$$\mathbf{a}(13^c n + \gamma_c) - 13\,\mathbf{a}(13^{c-2} n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}, \tag{1.19}$$

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[1]The congruence (1.5) was first conjectured by the author in a Colloquium given at the University of Newcastle, Australia on July 17, 2008.

for all $a, b, c \geq 3$. We note that (1.17) is a stronger congruence than (1.6). The congruences (1.6)–(1.7) follow from (1.17)–(1.18) and Ramanujan's partition congruences for powers of 5 and 7 that were first proved by Watson [25]. The congruence (1.8) follows easily from (1.19).

Let $\ell \geq 5$ be prime. In Section 2 we use results of Bringmann [9] to show how Atkin's $U(\ell)$ operator can be used to annihilate the nonholomorphic part of the weight $\frac{3}{2}$ weak Maass form that corresponds to the function $\alpha(24z)$, and prove that the function

$$\alpha_\ell(z) := \sum_{n=0}^{\infty} \left(\mathbf{a}(\ell n - \tfrac{1}{24}(\ell^2-1)) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n}{\ell}\right)\right) q^{n-\frac{\ell}{24}} \tag{1.20}$$

is a weakly holomorphic weight $\frac{3}{2}$ modular form on $\Gamma_0(\ell)$. Here χ_{12} is the character given below in (2.2), and we note $\mathbf{a}(n) = 0$ if n is not a nonnegative integer. We determine the multiplier of this form and exact information about the orders at cusps. See Theorem 2.2. This enables us to prove identities such as

$$\alpha_5(z) = \sum_{n=0}^{\infty} \left(\mathbf{a}(5n-1) + 5\,\mathbf{a}\left(\frac{n}{5}\right)\right) q^{n-\frac{5}{24}} = \frac{5}{4}\frac{(5E_2(5z) - E_2(z))}{\eta(5z)} \left(125\frac{\eta(5z)^6}{\eta(z)^6} - 1\right), \tag{1.21}$$

where $E_2(z)$ is the usual quasimodular Eisenstein series of weight 2, and $\eta(z)$ is the Dedekind eta-function. We then use Watson's [25] and Atkin's [8] method of modular equations to prove the congruences (1.17)–(1.19). These details are carried out in Section 3. In Section 4 we improve some results in [16] and [10] on $\mathrm{spt}(\ell n - \frac{1}{24}(\ell^2-1))$ and $N_2(\ell n - \frac{1}{24}(\ell^2-1))$ modulo ℓ.

Since this paper was first written Ahlgren, Bringmann and Lovejoy [2] have generalized Ono's congruence (1.5) to higher powers of ℓ. They have also obtained analogous results for other spt-like functions which were studied by Bringmann, Lovejoy and Osburn [11], [12]. We state their theorem for the spt-function. Suppose $\ell \geq 5$ is prime and $m \geq 1$. Then Ahlgren, Bringmann and Lovejoy have shown the following two congruences.

(i) If $\left(\frac{-23-24n}{\ell}\right) = 1$ then

$$\mathrm{spt}(\ell^{2m} n + d_{\ell,2m}) \equiv 0 \pmod{\ell^m}. \tag{1.22}$$

(ii) If $n \geq 0$ then

$$\mathrm{spt}(\ell^{2m+1} n + d_{\ell,2m+1}) \equiv \chi_{12}(\ell)\,\mathrm{spt}(\ell^{2m-1} n + d_{\ell,2m-1}) \pmod{\ell^m}. \tag{1.23}$$

Here $d_{\ell,a}$ is the least positive integer such that $24 d_{\ell,a} \equiv 1 \pmod{\ell^a}$. The congruences (1.22)–(1.23) are truly amazing results. For the cases $\ell = 5$, 7 and 13 these congruences follow from (1.9)–(1.11). Our congruences (1.6)–(1.8) do not follow from Ahlgren, Bringmann and Lovejoy's results. Ahlgren, Bringmann and Lovejoy's proof of (1.22)–(1.23) is an extension of Ono's proof of (1.5). Their proofs utilize the Hecke operators $T(\ell^{2m})$. Our results and proofs are different in that they involve the Atkin operators $U(\ell^a)$.

2. The Atkin operator U_ℓ^*

In this section we prove that the function $\alpha_\ell(z)$, which is defined in (1.20) is a weakly holomorphic weight $\frac{3}{2}$ modular form on $\Gamma_0(\ell)$ when $\ell \geq 5$ is prime. The proof uses results of Bringmann [9] and the idea of using the Atkin operator U_ℓ to annihilate the nonholomorphic part of a certain weak Maass form.

Following Bringmann [9] and Ono [22] we define

$$\mathcal{M}(z) := \alpha(24z) - \frac{3i}{\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)\, d\tau}{(-i(\tau+z))^{\frac{3}{2}}}, \tag{2.1}$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta-function and $\alpha(z)$ is defined in (1.16). Then $\mathcal{M}(z)$ is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_0(576)$ with Nebentypus χ_{12} where

$$\chi_{12}(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Let

$$\mathcal{N}(z) = -\frac{3i}{\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)\, d\tau}{(-i(\tau+z))^{\frac{3}{2}}} = \frac{3}{\pi\sqrt{2}} \int_{y}^{\infty} \frac{\eta(24(-x+it))\, dt}{(y+t)^{3/2}}, \tag{2.3}$$

where $z = x + iy$, $y > 0$, so that

$$\mathcal{M}(z) = \alpha(24z) + \mathcal{N}(z). \tag{2.4}$$

We define

$$\mathcal{A}(z) := \mathcal{M}\left(\frac{z}{24}\right). \tag{2.5}$$

The following theorem follows in a straightforward way from the work of Bringmann [9].

Theorem 2.1.

$$\mathcal{A}\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^{3/2}}{\nu_\eta(A)} \mathcal{A}(z),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, *and* $\nu_\eta(A)$ *is the eta-multiplier.*

Remark. When defining $z^{3/2}$ we use the principal branch; i.e. for $z = re^{i\theta}$, $r > 0$, $-\pi \le \theta < \pi$, we take $z^{3/2} = r^{3/2} e^{3i\theta/2}$.

Proof. We note that

$$\sum_{n=0}^{\infty} (24n-1) p(n) q^{n-\frac{1}{24}} = -\frac{E_2(z)}{\eta(z)}, \tag{2.6}$$

where $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma(n) q^n$ is a quasi-modular form that satisfies

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) - \frac{6iz}{\pi} c(cz+d). \tag{2.7}$$

Using (2.7) and Corollary 4.3 and Lemma 4.4 in [9],

$$\mathcal{M}\left(-\frac{1}{z}\right) = \frac{-(-iz)^{3/2}}{48\sqrt{6}} \mathcal{M}\left(\frac{z}{576}\right),$$

and hence

$$\mathcal{A}\left(-\frac{1}{z}\right) = -(-iz)^{3/2} \mathcal{A}(z) = e^{\pi i/4} z^{3/2} \mathcal{A}(z).$$

Therefore,

$$\mathcal{A}(Sz) = \frac{z^{3/2}}{\nu_\eta(S)}\mathcal{A}(z),$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From (1.16), (2.3) and (2.4)

$$\begin{aligned} \mathcal{M}(z + \tfrac{1}{24}) &= e^{-\pi i/12}\mathcal{M}(z), \\ \mathcal{N}(z + \tfrac{1}{24}) &= e^{-\pi i/12}\mathcal{N}(z), \\ \mathcal{A}(z+1) &= e^{-\pi i/12}\mathcal{A}(z), \\ \mathcal{A}(Tz) &= \frac{1}{\nu_\eta(T)}\mathcal{A}(z), \end{aligned}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since S, T generate $\mathrm{SL}_2(\mathbb{Z})$ the result follows. ☐

In what follows $\ell \geq 5$ is prime. We let d_ℓ denote the least nonnegative residue of the reciprocal of 24 mod ℓ so that $24d_\ell \equiv 1 \pmod{\ell}$. We define

$$r_\ell := \frac{24d_\ell - 1}{\ell}, \qquad r_\ell^* := \frac{24d_\ell + \ell^2 - 1}{24\ell}, \qquad s_\ell := \frac{(\ell^2-1)}{24}. \tag{2.8}$$

so that

$$\begin{aligned} \alpha_\ell(z) &:= \sum_{n=-r_\ell^*}^{\infty} \left(\mathbf{a}(\ell n + d_\ell) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n + r_\ell^*}{\ell}\right)\right) q^{n+\frac{r_\ell}{24}} \\ &= \sum_{n=0}^{\infty} \left(\mathbf{a}(\ell n - s_\ell) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n}{\ell}\right)\right) q^{n-\frac{\ell}{24}}. \end{aligned} \tag{2.9}$$

For a function $G(z)$ we define the Atkin-type operator U_ℓ^* by

$$U_\ell^*(G) := \frac{1}{\ell}\sum_{k=0}^{\ell-1} G\left(\frac{z+24k}{\ell}\right), \tag{2.10}$$

so that

$$\alpha_\ell(z) = U_\ell^*(\alpha) - \chi_{12}(\ell)\,\ell\,\alpha(\ell z).$$

The usual Atkin operator U_ℓ is defined by

$$U_\ell(G) := \frac{1}{\ell}\sum_{k=0}^{\ell-1} G\left(\frac{z+k}{\ell}\right). \tag{2.11}$$

We need U_ℓ^* since $\alpha(z)$ has fractional powers of q, and we note that

$$U_\ell^*(G) = U_\ell(G^*)(z/24),$$

where $G^*(z) = G(24z)$. For a congruence subgroup Γ we let $M_k(\Gamma)$ denote the space of entire modular forms of weight k with respect to the group Γ, and we let $M_k(\Gamma, \chi)$ denote the space of entire modular forms of weight k and character χ with respect to the group Γ. Then we have

Theorem 2.2. *If $\ell \geq 5$ is prime, then*

$$G_\ell(z) := \alpha_\ell(z)\,\frac{\eta^{2\ell}(z)}{\eta(\ell z)} \in M_{\ell+1}(\Gamma_0(\ell)). \tag{2.12}$$

In other words, the function $G_\ell(z)$ is an entire modular form of weight $\ell+1$ with respect to the group $\Gamma_0(\ell)$.

Proof. We assume $\ell \ge 5$ is prime. We divide the proof into four parts:

(i) $U_\ell^*(\mathcal{A}) - \ell\,\chi_{12}(\ell)\,\mathcal{A}(\ell z) = \alpha_\ell(z)$ and $G_\ell(z)$ is holomorphic for $\Im(z) > 0$.

(ii) $G_\ell(Az) = (cz+d)^{\ell+1} G_\ell(z)$ for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$.

(iii) $G_\ell(z)$ is holomorphic at $i\infty$.

(iv) $G_\ell(z)$ is holomorphic at the cusp 0.

Part (i). It is well-known (and an easy exercise) to show that

$$U_\ell(\eta(24z)) = \chi_{12}(\ell)\,\eta(24\ell z). \tag{2.13}$$

Using (2.3) and (2.13) we easily find that

$$U_\ell(\mathcal{N}(z)) = \ell\,\chi_{12}(\ell)\,\mathcal{N}(\ell z).$$

It follows that

$$U_\ell(\mathcal{M}) - \ell\,\chi_{12}(\ell)\,\mathcal{M}(\ell z)$$

is holomorphic for $\Im(z) > 0$. By replacing z by $\frac{z}{24}$ we see that

$$U_\ell^*(\mathcal{A}) - \ell\,\chi_{12}(\ell)\,\mathcal{A}(\ell z) = U_\ell^*(\alpha) - \ell\,\chi_{12}(\ell)\,\alpha(\ell z) = \alpha_\ell(z)$$

and it is clear that $G_\ell(z)$ is holomorphic for $\Im(z) > 0$.

Part (ii). Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$. We must show that

$$G_\ell(Az) = (cz+d)^{\ell+1} G_\ell(z).$$

Since it is well-known that

$$\left(\frac{\eta^\ell(z)}{\eta(\ell z)}\right)^2 \in M_{\ell-1}(\Gamma_0(\ell)),$$

it suffices to show that

$$\alpha_\ell(Az)\,\eta(\ell Az) = (cz+d)^2 \alpha_\ell(z)\eta(\ell z).$$

We need to show that

$$f_\ell(Az) = (cz+d)^2 f_\ell(z), \tag{2.14}$$

$$g_\ell(Az) = (cz+d)^2 g_\ell(z), \tag{2.15}$$

where

$$f_\ell(z) = U_\ell^*(\mathcal{A})\,\eta(\ell z), \qquad g_\ell(z) = \mathcal{A}(\ell z)\,\eta(\ell z).$$

Let

$$A^* = \begin{pmatrix} a & \ell b \\ c/\ell & d \end{pmatrix}.$$

Then $A^* \in \mathrm{SL}_2(\mathbb{Z})$ and (2.15) follows from Theorem 2.1 and the fact that

$$\mathcal{A}(\ell Az)\,\eta(\ell Az) = \mathcal{A}(A^*\ell z)\,\eta(A^*\ell z).$$

Now,

$$f_\ell(z) = U_\ell^*(\mathcal{A})\eta(\ell z) = U_\ell^*(\mathcal{A}(z)\eta(\ell^2 z)).$$

We define

$$F_\ell(z) := \mathcal{A}(z)\eta(\ell^2 z) = \mathcal{A}(z)\eta(z)\,\frac{\eta(\ell^2 z)}{\eta(z)}. \tag{2.16}$$

Using Theorem 2.1 and the fact that $\frac{\eta(\ell^2 z)}{\eta(z)}$ is a modular function on $\Gamma_0(\ell^2)$ we have

$$F_\ell(Cz) = (c_1 z + d_1)^2 F_\ell(z),$$

for $C = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_0(\ell^2)$.

Now for $0 \le k \le \ell - 1$, let

$$B_k = \begin{pmatrix} 1 & 24k \\ 0 & \ell \end{pmatrix}$$

so that

$$f_\ell(z) = U_\ell^*(F_\ell(z)) = \frac{1}{\ell}\sum_{k=0}^{\ell-1} F_\ell(B_k z).$$

Since $A \in \Gamma_0(\ell)$, $(a,\ell) = 1$ and we can choose unique $0 \le k^* \le \ell - 1$ such that

$$24ak^* \equiv b + 24kd \pmod{\ell}.$$

Then

$$B_k A = A_k^* B_{k^*},$$

where $A_k^* \in \Gamma_0(\ell^2)$. We have

$$f_\ell(Az) = \frac{1}{\ell}\sum_{k=0}^{\ell-1} F_\ell(B_k A z) = \frac{1}{\ell}\sum_{k^*=0}^{\ell-1} F_\ell(A_k^* B_{k^*} z) = \frac{(cz+d)^2}{\ell}\sum_{k^*=0}^{\ell-1} F_\ell(B_{k^*} z) = (cz+d)^2 f_\ell(z),$$

which is (2.14).

Part (iii). First we note that r_ℓ^* is a positive integer. We have

$$G_\ell(z) = \alpha_\ell(z)\,\frac{\eta^{2\ell}(z)}{\eta(\ell z)} = \sum_{n=-r_\ell^*} \left(\mathbf{a}(\ell n + d_\ell) - \chi_{12}(\ell)\,\ell\,\mathbf{a}\left(\frac{n + r_\ell^*}{\ell}\right)\right) q^{n + r_{\ell^*}}\,\frac{E(q)^{2\ell}}{E(q^\ell)}$$

where

$$E(q) = \prod_{n=1}^{\infty}(1 - q^n).$$

We see that $G_\ell(z)$ is holomorphic at $i\infty$.

Part (iv). We need to find $G_\ell\left(\frac{-1}{\ell z}\right)$.

$$U_\ell^*(\mathcal{A}) = \frac{1}{\ell}\sum_{k=0}^{\ell-1}\mathcal{A}\left(\frac{z+24k}{\ell}\right) = \frac{1}{\ell}\mathcal{A}\left(\frac{z}{\ell}\right) + \frac{1}{\ell}\sum_{k=1}^{\ell-1}\mathcal{A}\left(\frac{z+24k}{\ell}\right) = \frac{1}{\ell}\mathcal{A}\left(\frac{z}{\ell}\right) + \frac{1}{\ell}\sum_{k=1}^{\ell-1}\mathcal{A}(B_k z).$$

For each $1 \le k \le \ell - 1$ choose $1 \le k^* \le \ell - 1$ such that $576kk^* \equiv -1 \pmod{\ell}$. Then

$$B_k S = C_k B_{k^*},$$

where

$$C_k = \begin{pmatrix} 24k & \frac{-1-576kk^*}{\ell} \\ \ell & -24k^* \end{pmatrix} \in \Gamma_0(\ell).$$

Then

$$\mathcal{A}(B_k S z) = \mathcal{A}(C_k B_{k^*} z) = z^{3/2}\left(\frac{-24k^*}{\ell}\right)e^{\pi i\ell/4}\mathcal{A}(B_{k^*} z),$$

by Theorem 2.1 since

$$\nu_\eta(C_k) = \left(\frac{-24k^*}{\ell}\right)e^{-\pi i\ell/4},$$

by [20, p.51]. Define

$$S_\ell = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}.$$

By Theorem 2.1,

$$\begin{aligned} \mathcal{A}\left(\frac{1}{\ell}S_\ell z\right) &= e^{\pi i/4}(z\ell^2)^{3/2}\mathcal{A}(\ell^2 z), \\ \mathcal{A}\left(\ell S_\ell z\right) &= e^{\pi i/4}z^{3/2}\mathcal{A}(z). \end{aligned}$$

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Hence, if we define

$$H_\ell(z) := U_\ell^*(\mathcal{A}) - \ell\chi_{12}(\ell)\,\mathcal{A}(\ell z), \tag{2.17}$$

then

$$\begin{aligned} &H_\ell(S_\ell z) \\ &= \ell z^{3/2}e^{\pi i/4}\left(\ell\,\mathcal{A}(\ell^2 z) + \frac{1}{\sqrt{\ell}}e^{\pi i(\ell-1)/4}\sum_{k=1}^{\ell-1}\left(\frac{-24k}{\ell}\right)\mathcal{A}\left(z + \tfrac{24k}{\ell}\right) - \chi_{12}(\ell)\mathcal{A}(z)\right). \end{aligned}$$

Replacing z by $24z$ gives

$$\begin{aligned} &H_\ell(S_\ell 24z) \\ &= \ell(24z)^{3/2}e^{\pi i/4}\left(\ell\,\mathcal{M}(\ell^2 z) + \frac{1}{\sqrt{\ell}}\chi_{12}(\ell)\epsilon_\ell^3\sum_{k=1}^{\ell-1}\left(\frac{-k}{\ell}\right)\mathcal{M}\left(z + \tfrac{k}{\ell}\right) - \chi_{12}(\ell)\mathcal{M}(z)\right), \end{aligned}$$

since

$$e^{\pi i(\ell-1)/4}\left(\frac{24}{\ell}\right) = \chi_{12}(\ell)\epsilon_\ell^3.$$

Here

$$\epsilon_\ell = \begin{cases} 1 & \text{if } \ell \equiv 1 \pmod 4, \\ i & \text{if } \ell \equiv 3 \pmod 4. \end{cases}$$

By [24, p.451] we have

$$\begin{aligned} H_\ell(S_\ell 24z) &= \ell(24z)^{3/2}e^{\pi i/4}\left(\mathcal{M}|T(\ell^2) - \chi_{12}(\ell)\mathcal{M}(z) - U_{\ell^2}(\mathcal{M})\right), \\ &= \ell(24z)^{3/2}e^{\pi i/4}\left((\mathcal{M}|T(\ell^2) - \chi_{12}(\ell)(1+\ell)\mathcal{M}(z)) - (U_{\ell^2}(\mathcal{M}) - \ell\chi_{12}(\ell)\mathcal{M}(z))\right), \end{aligned}$$

where $T(\ell^2)$ is the Hecke operator which acts on harmonic Maass forms of weight $\frac{3}{2}$, and was used by Ono [22]. When the form is meromorphic it corresponds to the usual Hecke operator as described by Shimura [24]. Ono [22] showed that function

$$\mathcal{M}_\ell(z) = \mathcal{M}|T(\ell^2) - \chi_{12}(\ell)(1+\ell)\mathcal{M}(z))$$

is a weakly holomorphic modular form. In fact, he showed that

$$\mathcal{F}_\ell(z) := \eta(z)^{\ell^2}\mathcal{M}_\ell(z/24) \tag{2.18}$$

is a weight $(l^2+3)/2$ entire modular form on $\mathrm{SL}_2(\mathbb{Z})$. See [22, Theorem 2.2]. We also note that the function

$$U_{\ell^2}(\mathcal{M}) - \ell\chi_{12}(\ell)\mathcal{M}(z) = U_\ell\left(U_\ell(\mathcal{M}) - \ell\chi_{12}(\ell)\mathcal{M}(\ell z)\right)$$

is holomorphic for $\Im(z) > 0$ by the remarks in Part (i). Thus we find that

(2.19)

$$G_\ell\left(\frac{-1}{\ell z}\right) = -(iz\ell)^{\ell+1}\frac{E(q^\ell)^{2\ell}}{E(q)}\sum_{n=-s_\ell}^{\infty}\left(\chi_{12}(\ell)\mathbf{a}(n)\left(\left(\frac{1-24n}{\ell}\right)-1\right)+\ell\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right)\right)q^{n+2s_\ell},$$

where $s_\ell = \frac{\ell^2-1}{24}$. It follows that $G_\ell(z)$ is holomorphic at the cusp 0. □

Since $G_\ell(z) \in M_{\ell+1}(\Gamma_0(\ell))$, the function $z^{-\ell-1}G_\ell\left(\frac{-1}{\ell z}\right) \in M_{\ell+1}(\Gamma_0(\ell))$ by [5, Lemma 1]. Thus if we define

$$\beta_\ell(z) := \sum_{n=-s_\ell}^{\infty}\left(\chi_{12}(\ell)\mathbf{a}(n)\left(\left(\frac{1-24n}{\ell}\right)-1\right)+\ell\mathbf{a}\left(\frac{n+s_\ell}{\ell^2}\right)\right)q^{n-\frac{1}{24}}, \tag{2.20}$$

then the proof of Part (iv) of Theorem 2.2 yields

Corollary 2.3. If $\ell \geq 5$ is prime, then

$$J_\ell(z) := \beta_\ell(z)\,\frac{\eta^{2\ell}(\ell z)}{\eta(z)} \in M_{\ell+1}(\ell). \tag{2.21}$$

We illustrate the case $\ell = 5$. For ℓ prime we define

$$\mathcal{E}_{2,\ell}(z) := \frac{1}{\ell-1}\left(\ell E_2(\ell z) - E_2(z)\right). \tag{2.22}$$

It is well-known that $\mathcal{E}_{2,\ell}(z) \in M_2(\Gamma_0(\ell))$. We also note that $\mathcal{E}_{2,\ell}(z)$ has integral coefficients when $\ell = 5$, 7 or 13. By [19, Theorem 3.8] $\dim M_6(\Gamma_0(5)) = 3$, and it can be shown that

$$\{\mathcal{E}_{2,5}(z)\frac{\eta(5z)^{10}}{\eta(z)^2}, \mathcal{E}_{2,5}(z)\eta(5z)^4\eta(z)^4, \mathcal{E}_{2,5}(z)\frac{\eta(z)^{10}}{\eta(5z)^2}\}$$

is a basis. We find that

$$G_5(z) = 5\,\mathcal{E}_{2,5}(z)\left(125\,\eta(5z)^4\eta(z)^4 - \frac{\eta(z)^{10}}{\eta(5z)^2}\right),$$

and

$$J_5(z) = 5\,\mathcal{E}_{2,5}(z)\left(\frac{\eta(5z)^{10}}{\eta(z)^2} - \eta(5z)^4\eta(z)^4\right).$$

Thus

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(5n-1)+5\,\mathbf{a}\left(\frac{n}{5}\right)\right)q^{n-\frac{5}{24}} = 5\,\frac{\mathcal{E}_{2,5}(z)}{\eta(5z)}\left(125\,\frac{\eta(5z)^6}{\eta(z)^6}-1\right), \tag{2.23}$$

and

(2.24)

$$\sum_{n=-1}^{\infty}\left(-\mathbf{a}(n)\left(\left(\frac{1-24n}{5}\right)-1\right)+5\,\mathbf{a}\left(\frac{n+1}{25}\right)\right)q^{n-\frac{1}{24}} = 5\,\frac{\mathcal{E}_{2,5}(z)}{\eta(z)}\left(1-\frac{\eta(z)^6}{\eta(5z)^6}\right).$$

3. The Congruences

In this section we derive explicit formulas for the generating functions of

$$\mathbf{a}(\ell^a n + d_{\ell,a}) - \chi_{12}(\ell)\,\ell\,\mathbf{a}(\ell^{a-2}n + d_{\ell,a-2}), \tag{3.1}$$

when $\ell = 5$, 7, and 13. As before, $d_{\ell,a}$ is the least positive integer such that $24d_{\ell,a} \equiv 1 \pmod{\ell^a}$. The presentation of the identities is analogous to those of the partition function as given by Hirschhorn and Hunt [18] and the author [15]. In each case we start by using Theorem 2.2 to find identities for $\alpha_\ell(z)$. This basically gives the initial case $a = 1$. Then we use Watson's [25] and Atkin's [8] method of modular equations to do the induction step and study the arithmetic properties of the coefficients in these identities. The main congruences (1.6)-(1.8) then follow in a straightforward way.

3.1. The SPT-function modulo powers of 5.

Theorem 3.1. *If $a \geq 1$ then*

$$\sum_{n=0}^{\infty} \left(\mathbf{a}(5^{2a-1}n - t_a) + 5\,\mathbf{a}(5^{2a-3}n - t_{a-1})\right) q^{n-\frac{5}{24}} = \frac{\mathcal{E}_{2,5}(z)}{\eta(5z)} \sum_{i\geq 0} x_{2a-1,i} Y^i, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \left(\mathbf{a}(5^{2a}n - t_a) + 5\,\mathbf{a}(5^{2a-2}n - t_{a-1})\right) q^{n-\frac{1}{24}} = \frac{\mathcal{E}_{2,5}(z)}{\eta(z)} \sum_{i\geq 0} x_{2a,i} Y^i, \tag{3.3}$$

where

$$t_a = \frac{1}{24}(5^{2a}-1), \qquad Y(z) = \frac{\eta(5z)^6}{\eta(z)^6},$$

$$\vec{x}_1 = (x_{1,0}, x_{1,1}, \cdots) = (-5, 5^4, 0, 0, 0, \cdots),$$

and for $a \geq 1$

$$\vec{x}_{a+1} = \begin{cases} \vec{x}_a A, & a \text{ odd}, \\ \vec{x}_a B, & a \text{ even}. \end{cases} \tag{3.4}$$

Here $A = (a_{i,j})_{i\geq 0, j\geq 0}$ and $B = (b_{i,j})_{i\geq 0, j\geq 0}$ are defined by

$$a_{i,j} = m_{6i,i+j}, \qquad b_{i,j} = m_{6i+1,i+j}, \tag{3.5}$$

where the matrix $M = (m_{i,j})_{i,j\geq 0}$ is defined as follows: The first five rows of M are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 5^3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 4\cdot 5^2 & 5^5 & 0 & 0 & 0 & \cdots \\ 0 & 9\cdot 5 & 9\cdot 5^4 & 5^7 & 0 & 0 & \cdots \\ 0 & 2\cdot 5 & 44\cdot 5^3 & 14\cdot 5^6 & 5^9 & 0 & \cdots \end{pmatrix}$$

and for $i \geq 5$, $m_{i,0} = 0$ and for $j \geq 1$,

$$m_{i,j} = 25\,m_{i-1,j-1} + 25\,m_{i-2,j-1} + 15\,m_{i-3,j-1} + 5\,m_{i-4,j-1} + m_{i-5,j-1}. \tag{3.6}$$

Lemma 3.2. *If n is a positive integer then there are integers c_m ($\lceil \frac{n}{5} \rceil \leq m \leq n$) such that*

$$U_5(\mathcal{E}_{2,5} Z^n) = \mathcal{E}_{2,5} \sum_{m=\lceil \frac{n}{5} \rceil}^{n} c_m Y^m,$$

where

$$Z(z) = \frac{\eta(25z)}{\eta(z)}, \qquad Y(z) = \frac{\eta(5z)^6}{\eta(z)^6}. \tag{3.7}$$

Proof. We need the following dimension formulas which follow from [13] and [19, Theorem 3.8]. For k even,

$$\dim M_k(\Gamma_0(5)) = 2\left\lfloor \frac{k}{4} \right\rfloor + 1,$$

$$\dim M_k(\Gamma_0(5), \left(\frac{\cdot}{5}\right)) = k - 2\left\lfloor \frac{k}{4} \right\rfloor.$$

Let n be a positive integer. Then

$$U_5(\mathcal{E}_{2,5}Z^n) = U_5\left(\mathcal{E}_{2,5}(z)\left(\frac{\eta(5z)^5}{\eta(z)}\right)^n\left(\frac{\eta(25z)}{\eta(5z)^5}\right)^n\right) = U_5\left(\mathcal{E}_{2,5}(z)\left(\frac{\eta(5z)^5}{\eta(z)}\right)^n\right)\left(\frac{\eta(5z)}{\eta(z)^5}\right)^n.$$

When n is even the function

$$\mathcal{E}_{2,5}(z)\left(\frac{\eta(5z)^5}{\eta(z)}\right)^n$$

belongs to the space $M_{2n+2}(\Gamma_0(5))$, which has as a basis

$$\left\{\mathcal{E}_{2,5}(z)\eta(z)^{5n-6m}\eta(5z)^{6m-n}\ ,\ 0 \le m \le n\right\}.$$

This follows from the dimension formula. We note that

$$\operatorname{ord}\left(\mathcal{E}_{2,5}(z)\eta(z)^{5n-6m}\eta(5z)^{6m-n}; i\infty\right) = m.$$

The operator U_5 preserves the space $M_{2n+2}(\Gamma_0(5))$. It follows that there are integers c_m ($\lceil \frac{n}{5} \rceil \le m \le n$) such that

$$U_5(\mathcal{E}_{2,5}Z^n) = \mathcal{E}_{2,5}(z)\sum_{m=\lceil \frac{n}{5} \rceil}^{n} c_m\,\eta(z)^{5n-6m}\eta(5z)^{6m-n}\left(\frac{\eta(5z)}{\eta(z)^5}\right)^n = \mathcal{E}_{2,5}(z)\sum_{m=\lceil \frac{n}{5} \rceil}^{n} c_m Y^m.$$

When n is odd the proof is similar except this time one needs to work in the space $M_{2n+2}(\Gamma_0(5), \left(\frac{\cdot}{5}\right))$. ☐

Corollary 3.3.

$$U_5(\mathcal{E}_{2,5}) = \mathcal{E}_{2,5} \tag{3.8}$$

$$U_5(\mathcal{E}_{2,5}Z) = 5^3\,\mathcal{E}_{2,5}Y \tag{3.9}$$

$$U_5(\mathcal{E}_{2,5}Z^2) = 5^2\,\mathcal{E}_{2,5}(4Y + 5^3Y^2) \tag{3.10}$$

$$U_5(\mathcal{E}_{2,5}Z^3) = 5\,\mathcal{E}_{2,5}(9Y + 9\cdot 5^3Y^2 + 5^6Y^3) \tag{3.11}$$

$$U_5(\mathcal{E}_{2,5}Z^4) = 5\,\mathcal{E}_{2,5}(2Y + 44\cdot 5^2Y^2 + 14\cdot 5^5Y^3 + 5^8Y^4). \tag{3.12}$$

Proof. Equation (3.8) is elementary. It also follows from the fact that $\dim M_2(\Gamma_0(5)) = 1$. Equations (3.9)–(3.12) follow from Lemma 3.2 and straightforward calculation. ☐

We need the 5th order modular equation that was used by Watson to prove Ramanujan's partition congruences for powers of 5.

$$Z^5 = \left(25Z^4 + 25Z^3 + 15Z^2 + 5Z + 1\right)Y(5z). \tag{3.13}$$

Lemma 3.4. *For $i \ge 0$*

$$U_5(\mathcal{E}_{2,5} Z^i) = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i}{5} \rceil}^{i} m_{i,j} Y^j,$$

where $Z = Z(z)$, $Y = Y(z)$ are defined in (3.7), and the $m_{i,j}$ are defined in Theorem 3.1.

Proof. The result holds for $0 \le i \le 4$ by Corollary 3.3. By (3.13) we have

$$\begin{aligned} U_5(\mathcal{E}_{2,5} Z^i) = \big(&25 U_5(\mathcal{E}_{2,5} Z^{i-1}) + 25 U_5(\mathcal{E}_{2,5} Z^{i-2}) + 15 U_5(\mathcal{E}_{2,5} Z^{i-3}) + 5 U_5(\mathcal{E}_{2,5} Z^{i-4}) \\ &+ U_5(\mathcal{E}_{2,5} Z^{i-5})\big)\, Y(z), \end{aligned}$$

for $i \ge 5$. The result follows by induction on i using the recurrence (3.6). ☐

Lemma 3.5. *For $i \ge 0$,*

$$U_5(\mathcal{E}_{2,5} Y^i) = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i}{5} \rceil}^{5i} a_{i,j} Y^j, \tag{3.14}$$

$$U_5(\mathcal{E}_{2,5} Z Y^i) = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i+1}{5} \rceil}^{5i+1} b_{i,j} Y^j, \tag{3.15}$$

where the $a_{i,j}$, $b_{i,j}$ are defined in (3.5).

Proof. Suppose $i \ge 0$. By Lemma 3.4

$$\begin{aligned} U_5(\mathcal{E}_{2,5} Y^i) &= U_5(\mathcal{E}_{2,5} Z^{6i} Y(5z)^{-i}) = Y^{-i} U_5(\mathcal{E}_{2,5} Z^{6i}) \\ &= Y^{-i} \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{6i}{5} \rceil}^{6i} m_{6i,j} Y^j \\ &= \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i}{5} \rceil}^{5i} m_{6i,i+j} Y^j = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i}{5} \rceil}^{5i} a_{i,j} Y^j, \end{aligned}$$

which is (3.14). Similarly

$$\begin{aligned} U_5(\mathcal{E}_{2,5} Z Y^i) &= U_5(\mathcal{E}_{2,5} Z^{6i+1} Y(5z)^{-i}) = Y^{-i} U_5(\mathcal{E}_{2,5} Z^{6i+1}) \\ &= Y^{-i} \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{6i+1}{5} \rceil}^{6i+1} m_{6i+1,j} Y^j \\ &= \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i+1}{5} \rceil}^{5i+1} m_{6i+1,i+j} Y^j = \mathcal{E}_{2,5}(z) \sum_{j=\lceil \frac{i+1}{5} \rceil}^{5i+1} b_{i,j} Y^j, \end{aligned}$$

which is (3.15). ☐

Proof of Theorem 3.1. We proceed by induction. The case $a = 1$ of (3.2) is (2.23). We now suppose $a \ge 1$ is fixed and (3.2) holds. Thus

$$E(q^5) \sum_{n=0}^{\infty} \big(\mathbf{a}(5^{2a-1} n - t_a) + 5\, \mathbf{a}(5^{2a-3} n - t_{a-1})\big)\, q^n = \mathcal{E}_{2,5}(z) \sum_{i \ge 0} x_{2a-1,i} Y^i.$$

We now apply the U_5 operator to both sides and use Lemma 3.5.

$$E(q)\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}n-t_a)+5\,\mathbf{a}(5^{2a-2}n-t_{a-1})\right)q^n=\sum_{i\geq 0}x_{2a-1,i}U_5(\mathcal{E}_{2,5}(z)Y^i)$$

$$=\mathcal{E}_{2,5}(z)\sum_{i\geq 0}x_{2a-1,i}\sum_{j\geq 0}a_{i,j}Y^j=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}\left(\sum_{i\geq 0}x_{2a-1,i}a_{i,j}\right)Y^j$$

$$=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}x_{2a,j}Y^j.$$

We obtain (3.3) by dividing both sides by $\eta(z)$.

Now again suppose a is fixed and (3.3) holds. Multiplying both sides by $\eta(25z)$ gives

$$E(q^{25})\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}n-t_a)+5\,\mathbf{a}(5^{2a-2}n-t_{a-1})\right)q^{n+1}=\mathcal{E}_{2,5}(z)\sum_{i\geq 0}x_{2a,i}ZY^i.$$

We apply the U_5 operator to both sides.

$$E(q^5)\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a}(5n-1)-t_a)+5\,\mathbf{a}(5^{2a-2}(5n-1)-t_{a-1})\right)q^n$$
$$=\sum_{i\geq 0}x_{2a,i}U_5(\mathcal{E}_{2,5}(z)ZY^i).$$

Using Lemma 3.5 and the fact that $t_{a+1}=5^{2a}+t_a$ we have

$$E(q^5)\sum_{n=0}^{\infty}\left(\mathbf{a}(5^{2a+1}n-t_{a+1})+5\,\mathbf{a}(5^{2a-1}n-t_a)\right)q^n=\mathcal{E}_{2,5}(z)\sum_{i\geq 0}x_{2a,i}\sum_{j\geq 0}b_{i,j}Y^j$$

$$=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}\left(\sum_{i\geq 0}x_{2a,i}b_{i,j}\right)Y^j=\mathcal{E}_{2,5}(z)\sum_{j\geq 0}x_{2a+1,j}Y^j.$$

We obtain (3.2) with a replaced by $a+1$ after dividing both sides by $\eta(5z)$. This completes the proof of the theorem. ☐

Throughout this section we will make repeated use of the following lemma which we leave as an exercise.

Lemma 3.6. *Suppose x, y, $n\in\mathbb{Z}$ and $n>0$. Then*

$$\left\lfloor\frac{x}{n}\right\rfloor+\left\lfloor\frac{y}{n}\right\rfloor\geq\left\lfloor\frac{x+y-n+1}{n}\right\rfloor. \tag{3.16}$$

For any prime ℓ we let $\pi(n)=\pi_\ell(n)$ denote the exact power of ℓ that divides n. Then we have

Lemma 3.7.

$$\pi_5(m_{i,j})\geq\lfloor\tfrac{1}{2}(5j-i+1)\rfloor,$$

where the matrix $M=(m_{i,j})_{i,j\geq 0}$ is defined in Theorem 3.1.

Proof. First we verify the result for $0\leq i\leq 4$. The result is easily proven for $i\geq 5$ using the recurrence (3.6). ☐

Corollary 3.8.

$$\pi_5(a_{i,j}) \ge \lfloor \tfrac{1}{2}(5j-i+1) \rfloor, \qquad \pi_5(b_{i,j}) \ge \lfloor \tfrac{1}{2}(5j-i) \rfloor,$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (3.5).

Lemma 3.9. *For $b \ge 2$, and $j \ge 1$,*

$$\pi_5(x_{2b-1,j}) \ge 5b-6+\max(0, \lfloor \tfrac{1}{2}(5j-7) \rfloor), \tag{3.17}$$

$$\pi_5(x_{2b,j}) \ge 5b-4+\lfloor \tfrac{1}{2}(5j-4) \rfloor. \tag{3.18}$$

Proof. A calculation gives

$$\begin{aligned}
\vec{x}_3 &= (x_{3,0}, x_{3,1}, x_{3,2}, \cdots) \\
&= (0, 669303124 \cdot 5^4, 3328977476 \cdot 5^{11}, 366098988268 \cdot 5^{14}, \\
&\quad 201318006648837 \cdot 5^{15}, 1618593700646527 \cdot 5^{18}, 6370852555263938 \cdot 5^{21}, \\
&\quad 2900024541422883 \cdot 5^{25}, 4237895677971369 \cdot 5^{28}, 21327793208615511 \cdot 5^{30}, \\
&\quad 15532659183030861 \cdot 5^{33}, 8481639849706179 \cdot 5^{36}, 3564573506915806 \cdot 5^{39}, \\
&\quad 1175454967692313 \cdot 5^{42}, 1542192101361916 \cdot 5^{44}, 325171329708596 \cdot 5^{47}, \\
&\quad 55431641829564 \cdot 5^{50}, 1532152033009 \cdot 5^{54}, 171561318777 \cdot 5^{57}, \\
&\quad 77490966671 \cdot 5^{59}, 5598792206 \cdot 5^{62}, 318906274 \cdot 5^{65}, \\
&\quad 2799863 \cdot 5^{69}, 91379 \cdot 5^{72}, 10439 \cdot 5^{74}, 149 \cdot 5^{77}, \\
&\quad 5^{80}, 0, \cdots),
\end{aligned}$$

$$\begin{aligned}
\pi_5(\vec{x}_3) = (&\infty, 4, 11, 14, 15, 18, 21, 25, 28, 30, 33, 36, 39, 42, 44, 47, 50, 54, 57, 59, 62, \\
&65, 69, 72, 74, 77, 80, \infty, \infty, \cdots),
\end{aligned}$$

and (3.17) holds for $b = 2$. Now suppose $b \ge 2$ is fixed and (3.17) holds. By (3.4)

$$x_{2b,j} = \sum_{i \ge 1} x_{2b-1,i} a_{i,j}.$$

Then using Corollary 3.8

$$\pi_5(x_{2b,1}) \ge \min(\{5b-4\} \cup \{5b-6+\lfloor \tfrac{1}{2}(5i-7) \rfloor + \lfloor (\tfrac{1}{2}(6-i) \rfloor \,:\, 2 \le i \le 5\}) = 5b-4,$$

and (3.18) holds for $j = 1$. Suppose $j \ge 2$. Then

$$\begin{aligned}
\pi_5(x_{2b,j}) &\ge \min_{1 \le i \le 5j} (\pi_5(x_{2b-1,i}) + \pi_5(a_{i,j})) \\
&\ge \min_{2 \le i \le 5j} (\pi_5(x_{2b-1,1}) + \pi_5(a_{1,j}), (\pi_5(x_{2b-1,i}) + \pi_5(a_{i,j})) \\
&\ge \min(\{5b-6+\lfloor \tfrac{1}{2}(5j) \rfloor\} \cup \{5b-6+\lfloor \tfrac{1}{2}(5i-7) \rfloor) + \lfloor \tfrac{1}{2}(5j-i+1) \rfloor \,:\, 2 \le i \le 5j\}).
\end{aligned}$$

Now

$$5b-6+\lfloor \tfrac{1}{2}(5j) \rfloor = 5b-4+\lfloor \tfrac{1}{2}(5j-4) \rfloor.$$

If $2 \le i \le 5j$, then using Lemma 3.6 we have

$$\begin{aligned}
5b-6+\lfloor \tfrac{1}{2}(5i-7) \rfloor) + \lfloor \tfrac{1}{2}(5j-i+1) \rfloor &\ge 5b-6+\lfloor \tfrac{1}{2}(5j+4i-7) \rfloor \\
&\ge 5b-6+\lfloor \tfrac{1}{2}(5j+1) \rfloor = 5b-4+\lfloor \tfrac{1}{2}(5j-3) \rfloor
\end{aligned}$$

and (3.18) holds. Now suppose $b \ge 2$ is fixed and (3.18) holds. By (3.4)

$$x_{2b+1,j} = \sum_{i \ge 1} x_{2b,i} b_{i,j}.$$

We observe that $\pi_5(b_{1,1}) = \pi_5(500) = 3$. Then using Corollary 3.8

$$\pi_5(x_{2b+1,1}) \geq \min(\{5b-1\} \cup \{5b-4+\lfloor\tfrac{1}{2}(5i-4)+\lfloor(\tfrac{1}{2}(5-i)\rfloor \,:\, 2 \leq i \leq 4\}) = 5b-1,$$

and (3.17) holds for $j = 1$ with b replaced by $b+1$. Suppose $j \geq 2$. Then

$$\begin{aligned}
&\pi_5(x_{2b+1,j}) \geq \min_{1 \leq i \leq 5j-1} (\pi_5(x_{2b,i}) + \pi_5(b_{i,j})) \\
&\geq \min_{2 \leq i \leq 5j-1} (\pi_5(x_{2b,1}) + \pi_5(b_{1,j}), (\pi_5(x_{2b,i}) + \pi_5(b_{i,j})) \\
&\geq \min(\{5b-4+\lfloor\tfrac{1}{2}(5j-1)\rfloor\} \cup \{5b-4+\lfloor\tfrac{1}{2}(5i-4)\rfloor) + \lfloor\tfrac{1}{2}(5j-i)\rfloor \,:\, 2 \leq i \leq 5j-1\}).
\end{aligned}$$

Now

$$5b-4+\lfloor\tfrac{1}{2}(5j-1)\rfloor = 5b-1+\lfloor\tfrac{1}{2}(5j-7)\rfloor.$$

If $2 \leq i \leq 5j-1$, then again using Lemma 3.6 we have

$$\begin{aligned}
5b-4+\lfloor\tfrac{1}{2}(5i-4)\rfloor) + \lfloor\tfrac{1}{2}(5j-i)\rfloor &\geq 5b-4+\lfloor\tfrac{1}{2}(5j+4i-5)\rfloor \\
&\geq 5b-4+\lfloor\tfrac{1}{2}(5j+3)\rfloor = 5b-1+\lfloor\tfrac{1}{2}(5j-3)\rfloor
\end{aligned}$$

and (3.17) holds with b replaced by $b+1$. Lemma 3.9 follows by induction. □

Corollary 3.10. For $b \geq 2$,

$$\mathbf{a}(5^{2b-1}n+\delta_{2b+1}) + 5\,\mathbf{a}(5^{2b-3}n+\delta_{2b-3}) \equiv 0 \pmod{5^{5b-6}}, \tag{3.19}$$

$$\mathbf{a}(5^{2b}n+\delta_{2b}) + 5\,\mathbf{a}(5^{2b-2}n+\delta_{2b-2}) \equiv 0 \pmod{5^{5b-4}}. \tag{3.20}$$

For $a \geq 1$,

$$\mathrm{spt}(5^{a+2}n+\delta_{a+2}) + 5\,\mathrm{spt}(5^{a}n+\delta_{a}) \equiv 0 \pmod{5^{2a+1}}, \tag{3.21}$$

$$\mathrm{spt}(5^{a}n+\delta_{a}) \equiv 0 \pmod{5^{\lfloor\frac{a+1}{2}\rfloor}}. \tag{3.22}$$

Proof. The congruences (3.19)–(3.20) follow from Theorem 3.1 and Lemma 3.9. Let

$$\mathrm{dp}(n) = (24n-1)p(n).$$

Then

$$\mathrm{dp}(5^{a}n+\delta_{a}) \equiv 0 \pmod{5^{2a}}, \tag{3.23}$$

by (1.12). The congruence (3.21) follows from (3.19)–(3.20), and (3.23). Andrews' congruence (1.2) implies that (3.22) holds for $a = 1$, 2. The general result follows by induction using (3.21). □

We note that when $a = 0$ there is a stronger congruence than (3.21). We prove that

$$\mathrm{spt}(25n-1) + 5\,\mathrm{spt}(n) \equiv 0 \pmod{25}. \tag{3.24}$$

We have calculated

$$\begin{aligned}
\vec{x}_2 &= (x_{2,0}, x_{2,1}, x_{2,2}, \cdots) \\
&= (-5^1, 63\cdot 5^6, 104\cdot 5^9, 189\cdot 5^{11}, 24\cdot 5^{14}, 5^{17}, 0, \cdots).
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(25n-1)+5\,\mathbf{a}(n)\right)q^{n-\frac{1}{24}} \tag{3.25}$$
$$=5\frac{\mathcal{E}_{2,5}(z)}{\eta(z)}\Bigg(-1+63\cdot 5^5\frac{\eta^6(5z)}{\eta^6(z)}+104\cdot 5^8\frac{\eta^{12}(5z)}{\eta^{12}(z)}+189\cdot 5^{10}\frac{\eta^{18}(5z)}{\eta^{18}(z)}$$
$$+24\cdot 5^{13}\frac{\eta^{24}(5z)}{\eta^{24}(z)}+5^{16}\frac{\eta^{30}(5z)}{\eta^{30}(z)}\Bigg),$$

and

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(25n-1)+5\,\mathbf{a}(n)\right)q^{n-\frac{1}{24}}\equiv 20\frac{E_2(z)}{\eta(z)}\pmod{25}.$$

But from (2.6) we see that

$$\sum_{n=0}^{\infty}\left(dp(25n-1)+5\,dp(n)\right)q^{n-\frac{1}{24}}\equiv 20\frac{E_2(z)}{\eta(z)}\pmod{25},$$

and

$$12\sum_{n=0}^{\infty}\left(\mathrm{spt}(25n-1)+5\,\mathrm{spt}(n)\right)q^{n-\frac{1}{24}}$$
$$=\sum_{n=0}^{\infty}\left(\mathbf{a}(25n-1)+5\,\mathbf{a}(n)\right)q^{n-\frac{1}{24}}-\sum_{n=0}^{\infty}\left(dp(25n-1)+5\,dp(n)\right)q^{n-\frac{1}{24}}$$
$$\equiv 0\pmod{25},$$

which gives (3.24).

3.2. **The SPT-function modulo powers of 7.**

Theorem 3.11. *If $a\ge 1$ then*

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(7^{2a-1}n-u_a)+7\,\mathbf{a}(7^{2a-3}n-u_{a-1})\right)q^{n-\frac{7}{24}}=\frac{\mathcal{E}_{2,7}(z)}{\eta(7z)}\sum_{i\ge 0}x_{2a-1,i}Y^i, \tag{3.26}$$

$$\sum_{n=0}^{\infty}\left(\mathbf{a}(7^{2a}n-u_a)+7\,\mathbf{a}(7^{2a-2}n-u_{a-1})\right)q^{n-\frac{1}{24}}=\frac{\mathcal{E}_{2,7}(z)}{\eta(z)}\sum_{i\ge 0}x_{2a,i}Y^i, \tag{3.27}$$

where

$$u_a=\frac{1}{24}(7^{2a}-1),\qquad Y(z)=\frac{\eta(7z)^4}{\eta(z)^4},$$

$$\vec{x}_1=(x_{1,0},x_{1,1},\cdots)=(-7,3\cdot 7^3,7^5,0,0,\cdots),$$

and for $a\ge 1$

$$\vec{x}_{a+1}=\begin{cases}\vec{x}_aA, & a\ \text{odd},\\ \vec{x}_aB, & a\ \text{even}.\end{cases}$$

Here $A=(a_{i,j})_{i\ge 0,j\ge 0}$ and $B=(b_{i,j})_{i\ge 0,j\ge 0}$ are defined by

$$a_{i,j}=m_{4i,i+j},\qquad b_{i,j}=m_{4i+1,i+j}, \tag{3.28}$$

where the matrix $M = (m_{i,j})_{i,j\ge 0}$ is defined as follows: The first seven rows of M are defined so that

$$U_7(\mathcal{E}_{2,7}Z^i) = \sum_{j=\lceil \frac{2i}{7}\rceil}^{2i} m_{i,j}Y^j \qquad (0 \le i \le 6),$$

where

$$Z(z) = \frac{\eta(49z)}{\eta(z)}.$$

and for $i \ge 7$, $m_{i,0} = 0$, $m_{i,1} = 0$, and for $j \ge 2$,

(3.29)

$$\begin{aligned} m_{i,j} &= 49\, m_{i-1,j-1} + 35\, m_{i-2,j-1} + 7\, m_{i-3,j-1} + 343\, m_{i-1,j-2} + 343\, m_{i-2,j-2} \\ &\quad + 147\, m_{i-3,j-2} + 49\, m_{i-4,j-2} + 21\, m_{i-5,j-2} + 7\, m_{i-6,j-2} + m_{i-7,j-2}. \end{aligned}$$

The proof of the following lemma is analogous to that of Lemma 3.2.

Lemma 3.12. *If n is a positive integer then there are integers c_m ($\lceil \frac{2n}{7}\rceil \le m \le 2n$) such that*

$$U_7(\mathcal{E}_{2,7}Z^n) = \mathcal{E}_{2,7} \sum_{m=\lceil \frac{2n}{7}\rceil}^{2n} c_m Y^m,$$

where

$$Z(z) = Z_7(z) = \frac{\eta(49z)}{\eta(z)}, \qquad Y(z) = \frac{\eta(7z)^4}{\eta(z)^4}. \tag{3.30}$$

Corollary 3.13.

(3.31)

$$U_7(\mathcal{E}_{2,7}) = \mathcal{E}_{2,7}$$

(3.32)

$$U_7(\mathcal{E}_{2,7}Z) = 7^2\mathcal{E}_{2,7}(3\,Y + 7^2\,Y^2)$$

(3.33)

$$U_7(\mathcal{E}_{2,7}Z^2) = 7\mathcal{E}_{2,7}(10\,Y + 27\cdot 7^2\,Y^2 + 10\cdot 7^4\,Y^3 + 7^6\,Y^4)$$

(3.34)

$$U_7(\mathcal{E}_{2,7}Z^3) = 7\mathcal{E}_{2,7}(Y + 190\cdot 7\,Y^2 + 255\cdot 7^3\,Y^3 + 104\cdot 7^5\,Y^4 + 17\cdot 7^7\,Y^5 + 7^9\,Y^6)$$

(3.35)

$$\begin{aligned} U_7(\mathcal{E}_{2,7}Z^4) &= 7^2\mathcal{E}_{2,7}(82\,Y^2 + 352\cdot 7^2\,Y^3 + 2535\cdot 7^3\,Y^4 + 1088\cdot 7^5\,Y^5 + 230\cdot 7^7\,Y^6 \\ &\quad + 24\cdot 7^9\,Y^7 + 7^{11}\,Y^8) \end{aligned}$$

(3.36)

$$\begin{aligned} U_7(\mathcal{E}_{2,7}Z^5) &= 7\mathcal{E}_{2,7}(114\,Y^2 + 253\cdot 7^3\,Y^3 + 4169\cdot 7^4\,Y^4 + 3699\cdot 7^6\,Y^5 + 11495\cdot 7^7\,Y^6 \\ &\quad + 2852\cdot 7^9\,Y^7 + 405\cdot 7^{11}\,Y^8 + 31\cdot 7^{13}\,Y^9 + 7^{15}\,Y^{10}) \end{aligned}$$

(3.37)

$$\begin{aligned} U_7(\mathcal{E}_{2,7}Z^6) &= 7\mathcal{E}_{2,7}(9\,Y^2 + 736\cdot 7^2\,Y^3 + 27970\cdot 7^3\,Y^4 + 6808\cdot 7^6\,Y^5 + 38475\cdot 7^7\,Y^6 \\ &\quad + 17490\cdot 7^9\,Y^7 + 33930\cdot 7^{10}\,Y^8 + 5890\cdot 7^{12}\,Y^9 + 629\cdot 7^{14}\,Y^{10} \\ &\quad + 38\cdot 7^{16}\,Y^{11} + 7^{18}\,Y^{12}) \end{aligned}$$

We need the 7th order modular equation that was used by Watson to prove Ramanujan's partition congruences for powers of 7.

$$Z^7 = (1 + 7\,Z + 21\,Z^2 + 49\,Z^3 + 147\,Z^4 + 343\,Z^5 + 343\,Z^6)\,Y(7z)^2 + (7\,Z^4 + 35\,Z^5 + 49\,Z^6)\,Y(7z). \tag{3.38}$$

Lemma 3.14. *For $i \ge 0$*

$$U_7(\mathcal{E}_{2,7} Z^i) = \mathcal{E}_{2,7}(z) \sum_{j=\lceil \frac{2i}{7} \rceil}^{2i} m_{i,j} Y^j,$$

where $Z = Z(z)$, $Y = Y(z)$ are defined in (3.30), and the $m_{i,j}$ are defined in Theorem 3.11.

Lemma 3.15. *For $i \ge 0$,*

$$U_7(\mathcal{E}_{2,7} Y^i) = \mathcal{E}_{2,7}(z) \sum_{j=\lceil \frac{i}{7} \rceil}^{7i} a_{i,j} Y^j, \tag{3.39}$$

$$U_7(\mathcal{E}_{2,7} Z Y^i) = \mathcal{E}_{2,7}(z) \sum_{j=\lceil \frac{i+2}{7} \rceil}^{7i+2} b_{i,j} Y^j \tag{3.40}$$

where the $a_{i,j}$, $b_{i,j}$ are defined in (3.28).

Let $\pi_7(n)$ denote the exact power of 7 dividing n. Then we have

Lemma 3.16.

$$\pi_7(m_{i,j}) \ge \lfloor \tfrac{1}{4}(7j - 2i + 3) \rfloor,$$

where the matrix $M = (m_{i,j})_{i,j\ge 0}$ is defined in Theorem 3.11.

Corollary 3.17.

$$\pi_7(a_{i,j}) \ge \lfloor \tfrac{1}{4}(7j - i + 3) \rfloor, \qquad \pi_7(b_{i,j}) \ge \lfloor \tfrac{1}{4}(7j - i + 1) \rfloor,$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (3.28).

Lemma 3.18. *For $b \ge 2$, and $j \ge 1$,*

$$\pi_7(x_{2b-1,j}) \ge 3b - 3 + \lfloor \tfrac{1}{4}(7j - 4) \rfloor. \tag{3.41}$$

$$\pi_7(x_{2b,j}) \ge 3b - 1 + \lfloor \tfrac{1}{4}(7j - 6) \rfloor. \tag{3.42}$$

Corollary 3.19. For $b \ge 2$,

$$\mathbf{a}(7^{2b-1}n + \lambda_{2b+1}) + 7 \cdot \mathbf{a}(7^{2b-3}n + \lambda_{2b-3}) \equiv 0 \pmod{7^{3b-3}}, \tag{3.43}$$

$$\mathbf{a}(7^{2b}n + \lambda_{2b}) + 7 \cdot \mathbf{a}(7^{2b-2}n + \lambda_{2b-2}) \equiv 0 \pmod{7^{3b-1}}. \tag{3.44}$$

For $a \ge 1$,

$$\mathrm{spt}(7^{a+2}n + \lambda_{a+2}) + 7 \cdot \mathrm{spt}(7^a n + \lambda_a) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3a+4) \rfloor}}, \tag{3.45}$$

$$\mathrm{spt}(7^a n + \lambda_a) \equiv 0 \pmod{7^{\lfloor \frac{a+1}{2} \rfloor}}. \tag{3.46}$$

We note that (3.45) also holds for $a = 0$ taking $\lambda_0 = 1$. The proof of the congruence

$$\mathrm{spt}(49n - 2) + 7 \cdot \mathrm{spt}(n) \equiv 0 \pmod{49}. \tag{3.47}$$

is analogous to the proof of (3.24).

3.3. The SPT-function modulo powers of 13.

Theorem 3.20. *If $a \geq 1$ then*

(3.48)
$$\sum_{n=0}^{\infty} \left(\mathbf{a}(13^{2a-1}n - v_a) - 13\,\mathbf{a}(13^{2a-3}n - v_{a-1})\right) q^{n-\frac{13}{24}} = \frac{\mathcal{E}_{2,13}(z)}{\eta(13z)} \sum_{i\geq 0} x_{2a-1,i} Y^i,$$

(3.49)
$$\sum_{n=0}^{\infty} \left(\mathbf{a}(13^{2a}n - v_a) - 13\,\mathbf{a}(13^{2a-2}n - v_{a-1})\right) q^{n-\frac{1}{24}} = \frac{\mathcal{E}_{2,13}(z)}{\eta(z)} \sum_{i\geq 0} x_{2a,i} Y^i,$$

where

$$v_a = \frac{1}{24}(13^{2a} - 1), \qquad Y(z) = \frac{\eta(13z)^2}{\eta(z)^2},$$

$$\begin{aligned} \vec{x}_1 &= (x_{1,0}, x_{1,1}, \cdots) \\ &= (13, 11\cdot 13^2, 108\cdot 13^3, 190\cdot 13^4, 140\cdot 13^5, 54\cdot 13^6, 11\cdot 13^7, 13^8, 0, 0, 0, \cdots), \end{aligned}$$

and for $a \geq 1$

$$\vec{x}_{a+1} = \begin{cases} \vec{x}_a A, & a \text{ odd}, \\ \vec{x}_a B, & a \text{ even}. \end{cases}$$

Here $A = (a_{i,j})_{i\geq 0, j\geq 0}$ and $B = (b_{i,j})_{i\geq 0, j\geq 0}$ are defined by

$$a_{i,j} = m_{2i,i+j}, \qquad b_{i,j} = m_{2i+1,i+j}, \tag{3.50}$$

where the matrix $M = (m_{i,j})_{i\geq -12, j\geq -6}$ is defined as follows: The first 13 rows of M are

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 13^6 & 0 & 0 & \cdots \\
0 & 82\cdot 13 & 456\cdot 13^2 & 360\cdot 13^3 & 126\cdot 13^4 & 18\cdot 13^5 & 13^6 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 13^5 & 0 & 0 & \cdots \\
0 & 0 & 18\cdot 13 & -36\cdot 13^2 & -40\cdot 13^3 & -14\cdot 13^4 & -13^5 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 13^4 & 0 & 0 & \cdots \\
0 & 0 & 0 & -14\cdot 13 & -12\cdot 13^2 & 0 & 13^4 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 13^3 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 4\cdot 13 & 6\cdot 13^2 & 13^3 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 13^2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -13^2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 13 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -13 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots
\end{pmatrix}$$

and for $m_{k,\ell} = 0$ for $k \geq 1$ and $-6 \leq \ell \leq 0$; and for $i \geq 1$ and $j \geq 1$,

$$m_{i,j} = \sum_{r=1}^{13} \sum_{s=\lfloor \frac{1}{2}(r+2)\rfloor}^{7} \psi_{r,s} m_{i-r,j-s}, \tag{3.51}$$

where $\Psi = (\psi_{r,s})_{1\le r\le 13, 1\le s\le 7}$ *is the matrix*

(3.52)
$$\Psi = \begin{pmatrix}
11\cdot 13 & 36\cdot 13^2 & 38\cdot 13^3 & 20\cdot 13^4 & 6\cdot 13^5 & 13^6 & 13^6\\
0 & -204\cdot 13 & -346\cdot 13^2 & -222\cdot 13^3 & -74\cdot 13^4 & -13^6 & -13^6\\
0 & 36\cdot 13 & 126\cdot 13^2 & 102\cdot 13^3 & 38\cdot 13^4 & 7\cdot 13^5 & 7\cdot 13^5\\
0 & 0 & -346\cdot 13 & -422\cdot 13^2 & -184\cdot 13^3 & -37\cdot 13^4 & -3\cdot 13^5\\
0 & 0 & 38\cdot 13 & 102\cdot 13^2 & 56\cdot 13^3 & 13^5 & 15\cdot 13^4\\
0 & 0 & 0 & -222\cdot 13 & -184\cdot 13^2 & -51\cdot 13^3 & -5\cdot 13^4\\
0 & 0 & 0 & 20\cdot 13 & 38\cdot 13^2 & 13^4 & 19\cdot 13^3\\
0 & 0 & 0 & 0 & -74\cdot 13 & -37\cdot 13^2 & -5\cdot 13^3\\
0 & 0 & 0 & 0 & 6\cdot 13 & 7\cdot 13^2 & 15\cdot 13^2\\
0 & 0 & 0 & 0 & 0 & -13^2 & -3\cdot 13^2\\
0 & 0 & 0 & 0 & 0 & 13 & 7\cdot 13\\
0 & 0 & 0 & 0 & 0 & 0 & -13\\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

The proof of the following lemma is analogous to that of Lemma 3.2.

Lemma 3.21. *If* n *is a positive integer then there are integers* c_m $(\lceil \frac{7n}{13}\rceil \le m \le 7n)$ *such that*

$$U_{13}(\mathcal{E}_{2,13}Z^n) = \mathcal{E}_{2,13}\sum_{m=\lceil \frac{7n}{13}\rceil}^{7n} c_m Y^m,$$

where

$$Z(z) = Z_{13}(z) = \frac{\eta(169z)}{\eta(z)}, \qquad Y(z) = \frac{\eta(13z)^2}{\eta(z)^2}. \tag{3.53}$$

We need a version for Lemma 3.21 when n is negative.

Lemma 3.22. *If* n *is a nonnegative integer then there are integers* c_m $(-6n \le m \le n - \lceil \frac{6n}{13}\rceil)$ *such that*

$$U_{13}(\mathcal{E}_{2,13}Z^{-n}) = \mathcal{E}_{2,13}\sum_{m=-6n}^{n-\lceil \frac{6n}{13}\rceil} c_m Y^{-m}.$$

Proof. The proof is analogous to Lemma 3.21. The main difference is that we write

$$U_{13}(\mathcal{E}_{2,13}Z^{-n}) = U_{13}\left(\mathcal{E}_{2,13}(z)\left(\eta(z)\eta^{11}(13z)\right)^n\right)\left(\eta^{11}(z)\eta(13z)\right)^{-n},$$

and use the fact that $\mathcal{E}_{2,13}(z)\left(\eta(z)\eta^{11}(13z)\right)^n \in M_{2+6n}(\Gamma_0(13), \left(\frac{\cdot}{13}\right)^n)$. $\square$

Corollary 3.23.

$$\begin{aligned}
U_{13}(\mathcal{E}_{2,13}) &= \mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-1}) &= -13\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-2}) &= 13\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-3}) &= -13^2\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-4}) &= 13^2\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-5}) &= 13\,\mathcal{E}_{2,13}(4\,Y^{-2} + 6\cdot 13\,Y^{-1} + 13^2)\\
U_{13}(\mathcal{E}_{2,13}Z^{-6}) &= 13^3\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-7}) &= 13\,\mathcal{E}_{2,13}(-14\,Y^{-3} - 12\cdot 13\,Y^{-2} + 13^3)\\
U_{13}(\mathcal{E}_{2,13}Z^{-8}) &= 13^4\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-9}) &= 13\,\mathcal{E}_{2,13}(18\,Y^{-4} - 36\cdot 13\,Y^{-3} - 40\cdot 13^2\,Y^{-2} - 14\cdot 13^3\,Y^{-1} - 13^4)\\
U_{13}(\mathcal{E}_{2,13}Z^{-10}) &= 13^5\,\mathcal{E}_{2,13}\\
U_{13}(\mathcal{E}_{2,13}Z^{-11}) &= 13\,\mathcal{E}_{2,13}(82\,Y^{-5} + 456\cdot 13\,Y^{-4} + 360\cdot 13^2\,Y^{-3} + 126\cdot 13^3\,Y^{-2}\\
&\quad + 18\cdot 13^4\,Y^{-1} + 13^5)\\
U_{13}(\mathcal{E}_{2,13}Z^{-12}) &= 13^6\,\mathcal{E}_{2,13}
\end{aligned}$$

We need the 13th order modular equation that was used by Atkin and O'Brien [6] to study properties of $p(n)$ modulo powers of 13. Lehner [21] derived this equation earlier.

$$Z^{13}(z) = \sum_{r=1}^{13} \sum_{s=\lfloor \frac{1}{2}(r+2)\rfloor}^{7} \psi_{r,s} Y^s(13z)\, Z^{13-r}(z), \tag{3.54}$$

where the matrix $\Psi = (\psi_{i,j})$ is given in (3.52), and $Y(z)$, $Z(z)$ are given in (3.53). The modular equation and the matrix Ψ are given explicitly in Appendix C in [6]

Lemma 3.24. *For $i \ge 0$*

$$U_{13}(\mathcal{E}_{2,13}Z^i) = \mathcal{E}_{2,13}(z) \sum_{j=\lceil \frac{7i}{13}\rceil}^{7i} m_{i,j} Y^j,$$

where $Z = Z(z)$, $Y = Y(z)$ are defined in (3.53), and the $m_{i,j}$ are defined in Theorem 3.20.

Lemma 3.25. *For $i \ge 0$,*

$$U_{13}(\mathcal{E}_{2,13}Y^i) = \mathcal{E}_{2,13}(z) \sum_{j=\lceil \frac{i}{13}\rceil}^{13i} a_{i,j} Y^j, \tag{3.55}$$

$$U_{13}(\mathcal{E}_{2,13}ZY^i) = \mathcal{E}_{2,13}(z) \sum_{j=\lceil \frac{i+7}{13}\rceil}^{13i+7} b_{i,j} Y^j \tag{3.56}$$

where the $a_{i,j}$, $b_{i,j}$ are defined in (3.50).

Let $\pi_{13}(n)$ denote the exact power of 13 dividing n. Then we have

Lemma 3.26. *For i, $j \geq 0$,*

$$\pi_{13}(m_{i,j}) \geq \lfloor \tfrac{1}{14}(13j - 7i + 13) \rfloor, \tag{3.57}$$

where the matrix $M = (m_{i,j})$ is defined in Theorem 3.20.

Proof. As noted in [6] we observe that

$$\pi_{13}(\psi_{r,s}) \geq \lfloor \tfrac{1}{14}(13s - 7r + 13) \rfloor, \tag{3.58}$$

for all $1 \leq r \leq 13$ and $1 \leq s \leq 7$. We verify the result for $0 \leq i \leq 12$ by direct computation using the recurrence (3.51). We use (3.58), the recurrence (3.51) and Lemma 3.6 to prove the general result by induction. □

Corollary 3.27.

$$\pi_{13}(a_{i,j}) \geq \lfloor \tfrac{1}{14}(13j - i + 13) \rfloor, \qquad \pi_{13}(b_{i,j}) \geq \lfloor \tfrac{1}{14}(13j - i + 6) \rfloor,$$

where the $a_{i,j}$, $b_{i,j}$ are defined by (3.50).

We provide more complete details for the proof of the following lemma since congruences for the spt-function modulo 13 are stronger than those for the partition function.

Lemma 3.28.

$$\pi_{13}(x_{2,0}) = 1, \tag{3.59}$$

$$\pi_{13}(x_{2,j}) \geq 3 + \lfloor \tfrac{1}{14}(13j) \rfloor \qquad \textit{for } j \geq 1 \tag{3.60}$$

$$\pi_{13}(x_{2b-1,j}) \geq 2b - 2 + \lfloor \tfrac{1}{14}(13j - 10) \rfloor \qquad \textit{for } b \geq 2, \textit{ and } j \geq 1 \tag{3.61}$$

$$\pi_{13}(x_{2b,j}) \geq 2b - 1 + \lfloor \tfrac{1}{14}(13j) \rfloor \qquad \textit{for } b \geq 2, \textit{ and } j \geq 1. \tag{3.62}$$

Proof. We have calculated $\vec{x}_2$ and verified (3.59)–(3.60). We note that $x_{2,j} = 0$ for $j > 91$. Now,

$$x_{3,j} = \sum_{i \geq 0} x_{2,i} b_{i,j},$$

and we note that $x_{3,0} = 0$. We have

$$\pi_{13}(x_{2,0} b_{0,j}) = 1 + \pi_{13}(b_{0,j}) \geq 2 + \lfloor \tfrac{1}{14}(13j - 8) \rfloor$$

by Corollary 3.27. For $i \geq 1$

$$\begin{aligned} \pi_{13}(x_{2,i} b_{i,j}) &= \pi_{13}(x_{2,i}) + \pi_{13}(b_{i,j}) \geq 3 + \lfloor \tfrac{1}{14}(13i) \rfloor + \lfloor \tfrac{1}{14}(13j - i + 6) \rfloor \\ &\geq 3 + \lfloor \tfrac{1}{14}(13j + 12i - 7) \rfloor \geq 2 + \lfloor \tfrac{1}{14}(13j - 9) \rfloor, \end{aligned}$$

again by Corollary 3.27. It follows that

$$\pi_{13}(x_{3,j}) \geq 2 + \lfloor \tfrac{1}{14}(13j - 9) \rfloor,$$

and (3.61) holds for $b = 2$. Now supposed $b \geq 2$ is fixed and that (3.61) holds. We have

$$x_{2b,j} = \sum_{i \geq 1} x_{2b-1,i} a_{i,j}.$$

Now

$$\pi_{13}(x_{2b-1,1} a_{1,j}) = \pi_{13}(x_{2b-1,1}) + \pi_{13}(a_{1,j}) \geq 2b - 2 + \pi_{13}(a_{1,j}) \geq 2b - 1 + \lfloor \tfrac{1}{14}(13j) \rfloor,$$

by a direct calculation noting that $a_{1,j} = 0$ for $j > 13$. For $i \ge 2$

$$\begin{aligned}\pi_{13}(x_{2b-1,i}a_{i,j}) &= \pi_{13}(x_{2b-1,i}) + \pi_{13}(a_{i,j}) \ge 2b-2+\lfloor \tfrac{1}{14}(13i-10)\rfloor + \lfloor \tfrac{1}{14}(13j-i+13)\rfloor \\ &\ge 2b-2+\lfloor \tfrac{1}{14}(13j+12i-10)\rfloor \ge 2b-1+\lfloor \tfrac{1}{14}(13j)\rfloor,\end{aligned}$$

again by Corollary 3.27. It follows that

$$\pi_{13}(x_{2b,j}) \ge 2b-1+\lfloor \tfrac{1}{14}(13j)\rfloor,$$

and (3.62) holds. For $i \ge 1$

Again suppose $b \ge 2$ is fixed, and that (3.62) holds. We have

$$x_{2b+1,j} = \sum_{i\ge 1} x_{2b,i}b_{i,j}.$$

For $i \ge 1$

$$\begin{aligned}\pi_{13}(x_{2b,i}b_{i,j}) &= \pi_{13}(x_{2b,i}) + \pi_{13}(b_{i,j}) \ge 2b-1+\lfloor \tfrac{1}{14}(13i)\rfloor + \lfloor \tfrac{1}{14}(13j-i+6)\rfloor \\ &\ge 2b-1+\lfloor \tfrac{1}{14}(13j+12i-8)\rfloor \ge 2b+\lfloor \tfrac{1}{14}(13j-10)\rfloor,\end{aligned}$$

again by Corollary 3.27. It follows that

$$\pi_{13}(x_{2b+1,j}) \ge 2b+\lfloor \tfrac{1}{14}(13j-10)\rfloor,$$

and (3.61) holds with b replaced by $b+1$. Lemma 3.28 follows by induction. $\square$

Corollary 3.29. For $c \ge 3$,

$$\mathbf{a}(13^c n + \gamma_c) - 13\cdot \mathbf{a}(13^{c-2}n+\gamma_{c-2}) \equiv 0 \pmod{13^{c-1}}. \tag{3.63}$$

For $a \ge 1$,

$$\mathrm{spt}(13^{a+2}n+\gamma_{a+2}) - 13\cdot \mathrm{spt}(13^a n+\gamma_a) \equiv 0 \pmod{13^{a+1}}, \tag{3.64}$$

$$\mathrm{spt}(13^a n+\gamma_a) \equiv 0 \pmod{13^{\lfloor \frac{a+1}{2}\rfloor}}. \tag{3.65}$$

We note that (3.63) holds when $c = 2$ by taking $\gamma_0 = 1$. Also when $a = 0$ the congruence (3.64) has a stronger form. The proof of the congruence

$$\mathrm{spt}(169n-7) - 13\cdot \mathrm{spt}(n) \equiv 0 \pmod{169}. \tag{3.66}$$

is analogous to the proof of (3.24).

4. The spt-function modulo ℓ

In this section we improve on results in [16] and [10] for the spt-function and the second moment rank function modulo ℓ. We let

$$J_\ell(z) = \sum_{n=s_\ell}^{\infty} j_\ell(n)q^n,$$

where $J_\ell(z)$ is defined in (2.21), and define

$$K_\ell(z) := G_\ell(z) + (-1)^{\frac12(\ell-1)}\,\ell \sum_{n=\lceil \frac{s_\ell}{\ell}\rceil}^{\infty} j_\ell(\ell n)q^n, \tag{4.1}$$

where $G_\ell(z)$ is defined in (2.12). Then we have

Theorem 4.1. *If $\ell \ge 5$ is prime, then $K_\ell(z)$ is an entire modular form of weight $(\ell+1)$ on the full modular group $SL_2(\mathbb{Z})$.*

Proof. Suppose $\ell \geq 5$ is prime. We utilize Serre's [23, pp.223–224] results on the trace of a modular form on $\Gamma_0(\ell)$. By Theorem 2.2 we know that $G_\ell(z)$ is an entire modular form of weight $(\ell+1)$ on $\Gamma_0(\ell)$. By [23, Lemma 7],

$$\mathrm{Tr}(G_\ell) = G_\ell + \ell^{1-\frac{1}{2}(\ell+1)}\, G_\ell \mid W \mid U \tag{4.2}$$

is an entire modular form of weight $(\ell+1)$ on $\mathrm{SL}_2(\mathbb{Z})$. See [23, pp.223–224] for definition of W, U and the notation used. From (2.19) we find that

$$G_\ell \mid W = (-1)^{\frac{1}{2}(\ell-1)}\, \ell^{\frac{1}{2}(\ell+1)}\, J_\ell. \tag{4.3}$$

From (4.1), (4.2) and (4.3) we see that

$$K_\ell = \mathrm{Tr}(G_\ell)$$

is an entire modular form of weight $(\ell+1)$ on $\mathrm{SL}_2(\mathbb{Z})$. ∎

We observed special cases of the following Corollary in [16, Theorem 6.1].

Corollary 4.2. Suppose $\ell \geq 5$ is prime. Then

$$\sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} \mathrm{spt}(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \eta(z)^{r_\ell}\, L_\ell(z) \pmod{\ell} \tag{4.4}$$

for some integral entire modular form $L_\ell(z)$ on the full modular group of weight $\ell + 1 - 12\lceil \frac{\ell}{24} \rceil$, and where r_ℓ and s_ℓ are defined in (2.8).

Proof. Suppose $\ell \geq 5$ is prime. Since

$$(24n-1)\, p(n) \equiv 0 \pmod{\ell},$$

for $24n \equiv 1 \pmod{\ell}$, and using Theorem 4.1 we have

$$\frac{\eta(z)^{2\ell}}{\eta(\ell z)} \sum_{n=0}^{\infty} \mathbf{a}(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv P_\ell(z) \pmod{\ell},$$

for some integral $P_\ell(z) \in M_{\ell+1}(\Gamma(1))$. We note that

$$\mathrm{spt}(\ell n - s_\ell) \neq 0$$

implies that $\ell n - s_\ell \geq 1$ and $n \geq \lceil \frac{\ell}{24} \rceil$. It follows that

$$\frac{\eta(z)^{2\ell}}{\eta(\ell z)} \sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} \mathrm{spt}(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \Delta(z)^{c}\, L_\ell(z) \pmod{\ell},$$

where $\Delta(z)$ is Ramanujan's function

$$\Delta(z) := \eta(z)^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \tag{4.5}$$

$c = \lceil \frac{\ell}{24} \rceil$ and $L_\ell(z)$ is some integral modular form in $M_{\ell+1-12c}(\Gamma(1))$. Thus

$$\sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} \mathrm{spt}(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \eta(z)^{24c-\ell}\, L_\ell(z) \pmod{\ell},$$

and the result follows since

$$r_\ell = 24c - \ell.$$

∎

We conclude the paper by improving a result in [10] for the second rank moment function. From (1.1)

$$N_2(n) = 2n\,p(n) - 2\,\mathrm{spt}(n). \tag{4.6}$$

We note that the analog of Corollary 4.2 holds for the partition function $p(n)$ except the weight is 2 less. See either [16, Theorem 3.4] or [1, Theorem3]. This together with Corollary 4.2 and (4.6) implies

Corollary 4.3. Suppose $\ell \geq 5$ is prime. Then

$$\sum_{n=\lceil \frac{\ell}{24} \rceil}^{\infty} N_2(\ell n - s_\ell) q^{n-\frac{\ell}{24}} \equiv \eta(z)^{r_\ell} \left(Q_\ell(z) + L_\ell(z)\right) \pmod{\ell} \tag{4.7}$$

for some integral entire modular forms $Q_\ell(z)$ and $L_\ell(z)$ on the full modular group of weights k and $k+2$ respectively where $k = \ell - 1 - 12\lceil \frac{\ell}{24} \rceil$.

We illustrate Theorem 4.1 and Corollaries 4.2 and 4.3 in the case $\ell = 17$. We find that

$$K_{17}(z) = G_{17}(z) + 17 \sum_{n=1}^{\infty} j_{17}(17n) q^n = -17\,E_6(z)^3 - 26148\,\Delta(z)\,E_6(z),$$

$$\sum_{n=0}^{\infty} \mathrm{spt}(17n+5) q^{n+\frac{7}{24}} \equiv 14\,\eta(z)^7\,E_6(z) \pmod{17},$$

and

$$\sum_{n=0}^{\infty} N_2(17n+5) q^{n+\frac{7}{24}} \equiv \eta(z)^7 \left(2\,E_4(z) + 6\,E_6(z)\right) \pmod{17}.$$

Here $E_4(z)$ and $E_6(z)$ are the usual Eisenstein series

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{4.8}$$

where $\sigma_k(n) = \sum_{d|n} q^k$.

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