COMBINATORIAL INTERPRETATIONS OF RAMANUJAN'S PARTITION CONGRUENCES

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0. Introduction. This paper is mainly concerned with combinatorial aspects of the following congruences due to Ramanujan:

$$p(5n+4) \equiv 0 \pmod{5}, \tag{0.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{0.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{0.3}$$

In §1 we give a brief survey of Ramanujan's partition congruences. In §2 we state Dyson's [13] combinatorial interpretations of (0.1) and (0.2) in terms of the rank. See Dyson [14; II] for more background on the rank. Dyson conjectured the existence of what he called the *crank* which would explain (0.3) just as the rank explains (0.1) and (0.2). In §3 we give such a crank. It is in terms of a weighted count of what we call vector partitions. The results on vector partitions have been taken from [16], [17]. In §4 we show how these results are related to identities from Ramanujan's 'lost' notebook, the work of Atkin and Swinnerton-Dyer [6] and the mock theta conjectures mentioned by George Andrews in his talk.

Finally in §5 we prove some combinatorial results that are related to the crank and rank differences. The day after this conference ended the *true* crank (given in terms of ordinary partitions rather than vector partitions) was discovered by George Andrews and myself. This result is announced.

1. Ramanujan's partition congruences. The partition congruences modulo 5 and 7 namely (0.1) and (0.2) were proved by Ramanujan in [20]. In [21] he proved (0.3) by a different method. These three congruences are special cases of more general results. In 1919 Ramanujan conjectured that if $\alpha \geq 1$ and $\delta_{t,\alpha}$ is the reciprocal modulo t^α of 24 then

$$p(5^\alpha n+\delta_{5,\alpha}) \equiv 0 \pmod{5^\alpha}, \tag{1.1}$$

$$p(7^\alpha n+\delta_{7,\alpha}) \equiv 0 \pmod{7^\alpha}, \tag{1.2}$$

$$p(11^\alpha n+\delta_{11,\alpha}) \equiv 0 \pmod{11^\alpha}. \tag{1.3}$$

As noticed by Chowla [12] (1.2) fails for $\alpha = 3$ since $p(243)$ is divisible by 7^2 but not by 7^3. The correct version is

$$p(7^\alpha n+\delta_{7,\alpha}) \equiv 0 \pmod{7^{[(\alpha+2)/2]}}. \tag{1.4}$$

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(1.1) and (1.4) were proved by Watson [27] in 1938. (1.3) was finally proved by Atkin [7] in 1967. Atkin [9] has also simplified Watson's proofs of (1.1) and (1.4). Elementary proofs of (1.1) and (1.4) have been given by Hirschhorn and Hunt [18] and Garvan [15] respectively. Congruences analogous to (0.1) – (0.3) for other primes have also been found. For example Atkin [8], [10] has found

$$p(59^4 \cdot 13n + 111247) \equiv 0 \pmod{13}, \tag{1.5}$$

$$p(23^3 \cdot 17n + 2623) \equiv 0 \pmod{17}. \tag{1.6}$$

There are further congruence results to be found in some of Ramanujan's unpublished manuscripts. As noted by Rushforth [23] and Rankin [22] Hardy extracted [21] from an unpublished manuscript entitled "*Properties of* $p(n)$ *and* $\tau(n)$ *defined by the equations*

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots},$$
$$\sum_{n=1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24}."$$

Following Rankin we shall refer to this manuscript as MS. MS was sent to Hardy by Ramanujan a few months before his death. Apart from Rankin's and Rushforth's papers references to MS may be found in Birch [11] and Watson [25]. In MS Ramanujan indicates that the case $\alpha = 2$ of (1.3) can be proved in the same way as in the case $\alpha = 1$. The details are carried out by Rushforth in [23, §8]. MS also contains congruences for $p(n)$ in terms of generalizations of Ramanujan's τ-function modulo 13, 17 and 23. See Rushforth [23, §9]. The mod 13 case is closely related to work of Zuckerman [29]. Rankin [22] notes MS contains some congruences for $p(n)$ other than those mentioned in Rushforth [23]. For example, it contains the congruence

$$p\left(\frac{11 \cdot 13^\lambda + 1}{24}\right) + 2^{(5\lambda-3)/2} \equiv 0 \pmod{13}.$$

Results of this type were later discovered by Newman [19]. According to Rankin [22] there is another unpublished manuscript, referred to by Birch [11] as Fragment [VII] which is a sequel to MS. It contains amongst other things a sketch of a proof of (1.1) that is very similar to Watson's. Birch claims that Ramanujan's results are stronger than Watson's; Ramanujan states that if $\alpha \geq 1$ then there is a constant c_α such that

$$\sum_{n=0}^{\infty} p(5^\alpha n + \delta_{5,\alpha})q^n \equiv \begin{cases} c_\alpha 5^\alpha \{(1-q)(1-q^2)\cdots\}^{19} \pmod{5^{\alpha+1}}, & \text{if } \alpha \text{ odd}, \\ c_\alpha 5^\alpha \{(1-q)(1-q^2)\cdots\}^{23} \pmod{5^{\alpha+1}}, & \text{if } \alpha \text{ even}, \end{cases} \tag{1.7}$$

but this result follows almost immediately from equations (3.43) and (3.44) in Watson [27] .

2. Dyson's rank. In 1944 F.J. Dyson [13] discovered empirically some remarkable combinatorial interpretations of (0.1) and (0.2). Dyson defined the *rank* of a partition as the largest part minus the number of parts. For example, the partition $4+4+3+2+1+1+1$ has rank $4-7=-3$. Let $N(m,n)$ denote the number of partitions of n with rank m and let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Dyson conjectured that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5} \qquad 0 \le k \le 4, \tag{2.1}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{7} \qquad 0 \le k \le 6. \tag{2.2}$$

(2.1) and (2.2) were later proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [6] in 1953. These are the combinatorial interpretations of (0.1) and (0.2). Atkin and Swinnerton-Dyer's proof is analytic relying heavily on the properties of modular functions. No combinatorial proof is known. All that is known combinatorially about the rank is that

$$N(m,n) = N(-m,n), \tag{2.3}$$

which follows from the fact that the operation of conjugation reverses the sign of the rank. A trivial consequence is that

$$N(m,t,n) = N(t-m,t,n). \tag{2.4}$$

More than (2.1) and (2.2) is true. Dyson also conjectured

$$N(1,5,5n+1) = N(2,5,5n+1), \tag{2.5}$$
$$N(0,5,5n+2) = N(2,5,5n+2), \tag{2.6}$$
$$N(2,7,7n) = N(3,7,7n), \tag{2.7}$$
$$N(1,7,7n+1) = N(2,7,7n+1) = N(3,7,7n+1), \tag{2.8}$$
$$N(0,7,7n+2) = N(3,7,7n+2), \tag{2.9}$$
$$N(0,7,7n+3) = N(2,7,7n+3), \quad N(1,7,7n+3) = N(3,7,7n+3), \tag{2.10}$$
$$N(0,7,7n+4) = N(1,7,7n+4) = N(3,7,7n+4), \tag{2.11}$$
$$N(0,7,7n+6) + N(1,7,7n+6) = N(2,7,7n+6) + N(3,7,7n+6). \tag{2.12}$$

These were proved by Atkin and Swinnerton-Dyer. In fact they calculated the generating functions for $N(a,t,tn+k) - N(b,t,tn+k)$ for $t=5,7$ and all possible values for a, b and k. Before giving their result for $t = 5$ we need some notation. We define

$$r_a(d) = r_a(d,t) = \sum_{n=0}^{\infty} N(a, t, tn + d) q^n \tag{2.13}$$

and

$$r_{a,b}(d) = r_{a,b}(d,t) = r_a(d) - r_b(d). \tag{2.14}$$

Theorem 1 (Atkin and Swinnerton-Dyer [6]**).** *For $t = 5$,*

$$r_{1,2}(0) = q \prod_{n=1}^{\infty} (1 - q^{5n})^{-1} \sum_{m=-\infty}^{\infty} (-1)^m \frac{q^{15m(m+1)/2}}{1 - q^{5m+1}}, \tag{2.15}$$

$$r_{0,2}(0) + 2r_{1,2}(0) = \prod_{n=1}^{\infty} \frac{(1 - q^{5n-3})(1 - q^{5n-2})(1 - q^{5n})}{(1 - q^{5n-4})^2(1 - q^{5n-1})^2} - 1, \tag{2.16}$$

$$r_{0,2}(1) = \prod_{n=1}^{\infty} \frac{(1 - q^{5n})}{(1 - q^{5n-4})(1 - q^{5n-1})}, \tag{2.17}$$

$$r_{1,2}(1) = r_{0,2}(2) = 0, \tag{2.18}$$

$$r_{1,2}(2) = \prod_{n=1}^{\infty} \frac{(1 - q^{5n})}{(1 - q^{5n-3})(1 - q^{5n-2})}, \tag{2.19}$$

$$r_{0,2}(3) = -q \prod_{n=1}^{\infty} (1 - q^{5n})^{-1} \sum_{m=-\infty}^{\infty} (-1)^m \frac{q^{15m(m+1)/2}}{1 - q^{5m+2}}, \tag{2.20}$$

$$r_{0,1}(3) + r_{0,2}(3) = \prod_{n=1}^{\infty} \frac{(1 - q^{5n-4})(1 - q^{5n-1})(1 - q^{5n})}{(1 - q^{5n-3})^2(1 - q^{5n-2})^2}, \tag{2.21}$$

$$r_{0,2}(4) = r_{1,2}(4) = 0. \tag{2.22}$$

We shall show later that Theorem 1 is embodied in an identity from Ramanujan's 'lost' notebook. We note that (2.1), (2.5) and (2.6) follow directly from (2.18) and (2.22). Dyson also conjectured some identities for the generating functions for the rank, namely

$$\sum_{n=0}^{\infty} N(m,n) q^n = \sum_{n=1}^{\infty} (-1)^{n-1} q^{\frac{1}{2}n(3n-1)+|m|n} (1 - q^n) \prod_{k=1}^{\infty} (1 - q^k)^{-1}, \tag{2.23}$$

$$\sum_{n=0}^{\infty} N(m,t,n) q^n = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^n q^{\frac{1}{2}n(3n+1)} \frac{(q^{mn} + q^{n(t-m)})}{(1 - q^{tn})} \prod_{k=1}^{\infty} (1 - q^k)^{-1}, \tag{2.24}$$

which were also proved by Atkin and Swinnerton-Dyer.

Finally Dyson conjectures that there exists some analog of the rank that will explain (0.3):

> I hold in fact:
>
> That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and denote by $M(m, q, n)$ the number of partitions of n whose crank is congruent to m modulo q;
>
> that $M(m, q, n) = M(q - m, q, n)$;
>
> that
>
> $$\begin{aligned} M(0, 11, 11n+6) &= M(1, 11, 11n+6) = M(2, 11, 11n+6) \\ &= M(3, 11, 11n+6) = M(4, 11, 11n+6); \end{aligned}$$
>
> that numerous other relations exist analogous to (12)–(19), and in particular
>
> $$\begin{aligned} M(1, 11, 11n+1) &= M(2, 11, 11n+1) \\ &= M(3, 11, 11n+1) = M(4, 11, 11n+1); \end{aligned}$$
>
> that $M(m, 11, n)$ has a generating function not completely different in form from (24);
>
> that the values of the differences such as $M(0, 11, n) - M(4, 11, n)$ are always extremely small compared with $p(n)$.
>
> Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

The equations (12)–(19) and (24) referred to in the quotation above correspond to (2.5)–(2.12) and (2.24).

3. The crank for vector partitions. In this section we give a combinatorial interpretation of (0.3) as well as new interpretations of (0.1) and (0.2). Our main result does not actually divide the partitions of $11n + 6$ into 11 equal classes but rather it gives a combinatorial interpretation of $\frac{p(11n+6)}{11}$ in terms of the crank of vector partitions.

To describe our main result we need some more notation. For a partition, π, let $\sharp(\pi)$ be the number of parts of π and $\sigma(\pi)$ be the sum of the parts of π (or the number π is partitioning) with the convention $\sharp(\phi) = \sigma(\phi) = 0$ for the empty partition, ϕ, of 0. Let,

$$\begin{aligned} V = \{\, (\pi_1, \pi_2, \pi_3) \mid & \pi_1 \text{ is a partition into distinct parts} \\ & \pi_2, \pi_3 \text{ are unrestricted partitions}\}. \end{aligned}$$

We shall call the elements of V *vector partitions*. For $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ in V we define the sum of parts, s, a weight, ω, and a crank, r, by

$$s(\vec{\pi}) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3), \tag{3.1}$$

$$\omega(\vec{\pi}) = (-1)^{\sharp(\pi_1)}, \tag{3.2}$$

$$r(\vec{\pi}) = \sharp(\pi_2) - \sharp(\pi_3). \tag{3.3}$$

We say $\vec{\pi}$ is a vector partition of n if $s(\vec{\pi}) = n$. For example, if $\vec{\pi} = (5+3+2, 2+2+1, 2+1+1)$ then $s(\vec{\pi}) = 19$, $\omega(\vec{\pi}) = -1$, $r(\vec{\pi}) = 0$ and $\vec{\pi}$ is a vector partition of 19. The number of vector partitions of n (counted according to the weight ω) with crank m is denoted by $N_V(m,n)$, so that

$$N_V(m,n) = \sum_{\substack{\vec{\pi} \in V \\ s(\vec{\pi})=n \\ r(\vec{\pi})=m}} \omega(\vec{\pi}). \tag{3.4}$$

The number of vector partitions of n (counted according to the weight ω) with crank congruent to k modulo t is denoted by $N_V(k,t,n)$, so that

$$N_V(k,t,n) = \sum_{m=-\infty}^{\infty} N_V(mt+k,n) = \sum_{\substack{\vec{\pi} \in V \\ s(\vec{\pi})=n \\ r(\vec{\pi}) \equiv k \pmod{t}}} \omega(\vec{\pi}). \tag{3.5}$$

By considering the transformation that interchanges π_2 and π_3 we have

$$N_V(m,n) = N_V(-m,n) \tag{3.6}$$

so that

$$N_V(t-m,t,n) = N_V(m,t,n). \tag{3.7}$$

We have the following generating function for $N_V(m,n)$:

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}. \tag{3.8}$$

By putting $z = 1$ in (3.8) we find

$$\sum_{m=-\infty}^{\infty} N_V(m,n) = p(n). \tag{3.9}$$

Our main result is

Theorem 2.

$$N_V(k,5,5n+4)=\frac{p(5n+4)}{5}\qquad 0\le k\le 4, \tag{3.10}$$

$$N_V(k,7,7n+5)=\frac{p(7n+5)}{7}\qquad 0\le k\le 6, \tag{3.11}$$

$$N_V(k,11,11n+6)=\frac{p(11n+6)}{11}\qquad 0\le k\le 10. \tag{3.12}$$

We illustrate (3.10) with an example. The 41 vector partitions of 4 are given in the table below. From the this table we have

$$\begin{aligned} N_V(0,5,4) &= \omega(\vec{\pi}_6)+\omega(\vec{\pi}_9)+\omega(\vec{\pi}_{12})+\omega(\vec{\pi}_{13})+\omega(\vec{\pi}_{24}) \\ &\qquad +\omega(\vec{\pi}_{26})+\omega(\vec{\pi}_{33})+\omega(\vec{\pi}_{40})+\omega(\vec{\pi}_{41}) \\ &= 1+1+1+1-1-1-1-1+1 \\ &= 1. \end{aligned}$$

Similarly

$$N_V(0,5,4)=N_V(1,5,4)=\cdots=N_V(4,5,4)=1=\frac{p(4)}{5},$$

which is (3.10) with $n=0$.

	Weight	Crank		Weight	Crank
$\vec{\pi}_1=(\phi,\phi,4)$	$+1$	-1	$\vec{\pi}_{22}=(1,\phi,2+1)$	-1	-2
$\vec{\pi}_2=(\phi,\phi,3+1)$	$+1$	-2	$\vec{\pi}_{23}=(1,\phi,1+1+1)$	-1	-3
$\vec{\pi}_3=(\phi,\phi,2+2)$	$+1$	-2	$\vec{\pi}_{24}=(1,1,2)$	-1	0
$\vec{\pi}_4=(\phi,\phi,2+1+1)$	$+1$	-3	$\vec{\pi}_{25}=(1,1,1+1)$	-1	-1
$\vec{\pi}_5=(\phi,\phi,1+1+1+1)$	$+1$	-4	$\vec{\pi}_{26}=(1,2,1)$	-1	0
$\vec{\pi}_6=(\phi,1,3)$	$+1$	0	$\vec{\pi}_{27}=(1,1+1,1)$	-1	1
$\vec{\pi}_7=(\phi,1,2+1)$	$+1$	-1	$\vec{\pi}_{28}=(1,3,\phi)$	-1	1
$\vec{\pi}_8=(\phi,1,1+1+1)$	$+1$	-2	$\vec{\pi}_{29}=(1,2+1,\phi)$	-1	2
$\vec{\pi}_9=(\phi,2,2)$	$+1$	0	$\vec{\pi}_{30}=(1,1+1+1,\phi)$	-1	3
$\vec{\pi}_{10}=(\phi,2,1+1)$	$+1$	-1	$\vec{\pi}_{31}=(2,\phi,2)$	-1	-1
$\vec{\pi}_{11}=(\phi,1+1,2)$	$+1$	1	$\vec{\pi}_{32}=(2,\phi,1+1)$	-1	-2
$\vec{\pi}_{12}=(\phi,1+1,1+1)$	$+1$	0	$\vec{\pi}_{33}=(2,1,1)$	-1	0
$\vec{\pi}_{13}=(\phi,3,1)$	$+1$	0	$\vec{\pi}_{34}=(2,2,\phi)$	-1	1
$\vec{\pi}_{14}=(\phi,2+1,1)$	$+1$	1	$\vec{\pi}_{35}=(2,1+1,\phi)$	-1	2
$\vec{\pi}_{15}=(\phi,1+1+1,1)$	$+1$	2	$\vec{\pi}_{36}=(3,\phi,1)$	-1	-1
$\vec{\pi}_{16}=(\phi,4,\phi)$	$+1$	1	$\vec{\pi}_{37}=(2+1,\phi,1)$	$+1$	-1
$\vec{\pi}_{17}=(\phi,3+1,\phi)$	$+1$	2	$\vec{\pi}_{38}=(3,1,\phi)$	-1	1
$\vec{\pi}_{18}=(\phi,2+2,\phi)$	$+1$	2	$\vec{\pi}_{39}=(2+1,1,\phi)$	$+1$	1
$\vec{\pi}_{19}=(\phi,2+1+1,\phi)$	$+1$	3	$\vec{\pi}_{40}=(4,\phi,\phi)$	-1	0
$\vec{\pi}_{20}=(\phi,1+1+1+1,\phi)$	$+1$	4	$\vec{\pi}_{41}=(3+1,\phi,\phi)$	$+1$	0
$\vec{\pi}_{21}=(1,\phi,3)$	-1	-1			

4. A page from Ramanujan's 'lost' notebook. For an introduction to the 'lost' notebook see Andrews [2]. We give this page below correcting typos and adding equation numbers:

(4.1)
$$F(q) = \frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1-2q\cos\frac{2n\pi}{5}+q^2)(1-2q^2\cos\frac{2n\pi}{5}+q^4)\cdots},$$

(4.2)
$$f(q) = 1 + \frac{q}{(1-2q\cos\frac{2n\pi}{5}+q^2)} + \frac{q^4}{(1-2q\cos\frac{2n\pi}{5}+q^2)(1-2q^2\cos\frac{2n\pi}{5}+q^4)} + \cdots, \; n = 1 \text{ or } 2,$$

(4.3)
$$F(q^{\frac{1}{5}}) = A(q) - 4q^{\frac{1}{5}}\cos^2\frac{2n\pi}{5}B(q) + 2q^{\frac{2}{5}}\cos\frac{4n\pi}{5}C(q) - 2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}D(q),$$

(4.4)
$$f(q^{\frac{1}{5}}) = \left\{A(q) - 4\sin^2\frac{n\pi}{5}\phi(q)\right\} + q^{\frac{1}{5}}B(q) + 2q^{\frac{2}{5}}\cos\frac{2n\pi}{5}C(q) - 2q^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(q) + 4\sin^2\frac{2n\pi}{5}\frac{\psi(q)}{q}\right\},$$

(4.5)
$$A(q) = \frac{1-q^2-q^3+q^9+\cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots},$$

(4.6)
$$B(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q)(1-q^4)(1-q^6)\cdots},$$

(4.7)
$$C(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^2)(1-q^3)(1-q^7)\cdots},$$

(4.8)
$$D(q) = \frac{1-q-q^4+q^7+\cdots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\cdots},$$

(4.9)
$$\phi(q) = -1 + \left\{\frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})} + \cdots\right\},$$

(4.10)
$$\psi(q) = -1 + \left\{\frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^7)} + \frac{q^{20}}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})} + \cdots\right\},$$

(4.11)
$$\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots = 3\phi(q) + 1 - A(q),$$

(4.12)
$$\frac{q}{1-q}+\frac{q^2}{(1-q^2)(1-q^3)}+\frac{q^3}{(1-q^3)(1-q^4)(1-q^5)}+\cdots=3\psi(q)+qD(q),$$

(4.13)
$$\frac{q^2}{1-q}+\frac{q^8}{(1-q)(1-q^3)}+\frac{q^{18}}{(1-q)(1-q^3)(1-q^5)}+\cdots$$
$$=\phi(q)-q\frac{1+q^5+q^{15}+\cdots}{(1-q^4)(1-q^6)(1-q^{14})\cdots},$$

(4.14)
$$\frac{q}{1-q}+\frac{q^5}{(1-q)(1-q^3)}+\frac{q^{13}}{(1-q)(1-q^3)(1-q^5)}+\cdots$$
$$=\psi(q)+q\frac{1+q^5+q^{15}+\cdots}{(1-q^2)(1-q^8)(1-q^{12})\cdots}.$$

In (4.1)–(4.4) we may assume without loss of generality that $n=1$. If we let $\zeta=exp(\frac{2\pi i}{5})$ then we may write the definitions of $F(q)$ and $f(q)$ as

$$F(q)=\frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty}, \tag{4.15}$$

$$f(q)=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-\zeta q)(1-\zeta^{-1}q)\cdots(1-\zeta q^n)(1-\zeta^{-1}q^n)} \tag{4.16}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\zeta q)_n(\zeta^{-1}q)_n},$$

where

$$(a)_0=1,\qquad (a)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})\qquad \text{for } n\geq 1$$

and
$$(a)_\infty=\lim_{n\to\infty}(a)_n=\prod_{n=1}^{\infty}(1-aq^{n-1}).$$

After replacing q by q^5 we see that (4.3) and (4.4) are identities for $F(q)$ and $f(q)$ that split each function (as a power series in q) according to the residue of the exponent modulo 5. This splitting is in terms of the functions $A(q),\ldots,D(q),\phi(q),\psi(q)$ which are defined in (4.5)–(4.10). In Theorem 3 we show that (4.3) implies (3.10) which is the combinatorial interpretation of the partition congruence mod 5 in terms of the crank. In Theorem 4 we show not only does (4.4) imply (2.1), which is Dyson's combinatorial interpretation of the partition congruence mod 5 in terms of the rank, but that (4.4) is actually equivalent to Theorem 1 (due to Atkin and Swinnerton-Dyer). (4.11)–(4.14) are mock theta function identities. In Theorem 5 we show that (4.11) and (4.12) are equivalent to Andrews and Garvan's [4] mock theta conjectures.

We note that the numerators in the definitions of $A(q)$ and $D(q)$ are theta series in q and hence may be written as infinite products using Jacobi's triple product identity:

$$\prod_{n=1}^{\infty}(1-zq^n)(1-z^{-1}q^{n-1})(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{\frac12 n(n+1)}, \tag{4.17}$$

where $z \neq 0$ and $|q| < 1$. In fact we have

$$A(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n})}{(1-q^{5n-4})^2(1-q^{5n-1})^2}, \tag{4.18}$$

$$B(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{4.19}$$

$$C(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{5n-3})(1-q^{5n-2})}, \tag{4.20}$$

$$D(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n})}{(1-q^{5n-3})^2(1-q^{5n-2})^2}. \tag{4.21}$$

Theorem 3. *(4.3) $\Longrightarrow$ (3.10).*

Proof. We first write $F(q)$ in terms of $N_V(k,5,n)$. Substituting $z = \zeta$ into both sides of (3.8) we have

$$F(q) = \frac{(q)_\infty}{(\zeta q)_\infty (\zeta^{-1} q)_\infty} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) \zeta^m q^n$$

$$= \sum_{k=0}^{4} \zeta^k \sum_{n=0}^{\infty} \Big(\sum_{\substack{m \equiv k \\ \text{mod } 5}} N_V(m,n) \Big) q^n$$

$$= \sum_{k=0}^{4} \zeta^k \sum_{n=0}^{\infty} N_V(k,5,n) q^n \qquad \text{(by (3.5))}.$$

If we assume (4.3) then we find that

$$\sum_{k=0}^{4} N_V(k,5,5n+4)\zeta^k = \text{Coeff of } q^{5n+4} \text{ in } F(q) = 0. \tag{4.22}$$

The lefthand side of (4.22) is a polynomial in ζ over $\mathbb{Z}$. It follows that

$$N_V(0,5,5n+4) = N_V(1,5,5n+4) = \cdots = N_V(4,5,5n+4).$$

From (3.9) we have

$$p(5n+4) = \sum_{k=0}^{4} N_V(k,5,5n+4) = 5N_V(0,5,5n+4)$$

and (3.10) follows. □

(4.3) is easy to prove once we observe that

$$F(q) = \frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty} = \frac{(q)_\infty^2(\zeta^2 q)_\infty(\zeta^{-2}q)_\infty}{(q^5;q^5)_\infty} \tag{4.23}$$
$$= \frac{(q)_\infty\{(\zeta^2)_\infty(\zeta^{-2}q)_\infty(q)_\infty\}}{(q^5;q^5)_\infty(1-\zeta^2)}.$$

Here $(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1$. Now we can split $(\zeta^2)_\infty(\zeta^{-2}q)_\infty(q)_\infty$ utilizing Jacobi's triple product identity (4.17), split $(q)_\infty$ using Euler's result

$$(q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n(3n-1)} \qquad |q|<1, \tag{4.24}$$

and (4.3) follows with not much work. (3.11) can be proved in a similar way. The proof of (3.12) is analogous but depends on Winquist's identity [28, §1]. We refer the reader to [16], [17] for details.

We now turn to (4.4) and Atkin and Swinnerton-Dyer's Theorem. We first observe that the infinite products that occur in Theorem 1 are exactly $A(q)$, $B(q)$,$C(q)$ and $D(q)$. The remaining functions turn out to be $\phi(q)$ and $\psi(q)$. Utilizing Watson's [24] q-analog of Whipple's theorem or (2.23) it can be shown that

$$-1+\sum_{n=0}^{\infty}\frac{q^{n^2}}{(z)_{n+1}(z^{-1}q)_n} = \frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{\frac{3}{2}n(n+1)}}{1-zq^n}, \tag{4.25}$$

where $|q|<1$, $|q|<|z|<|q|^{-1}$ and $z\neq 1$. See [17, Lemma(7.9)]. It follows that

$$\phi(q) = -1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \tag{4.26}$$
$$= q\prod_{n=1}^{\infty}(1-q^{5n})^{-1}\sum_{m=-\infty}^{\infty}(-1)^m\frac{q^{\frac{15}{2}m(m+1)}}{1-q^{5m+1}}$$

and

$$\frac{\psi(q)}{q} = \frac{1}{q}\left\{-1+\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}\right\} \tag{4.27}$$
$$= q\prod_{n=1}^{\infty}(1-q^{5n})^{-1}\sum_{m=-\infty}^{\infty}(-1)^m\frac{q^{\frac{15}{2}m(m+1)}}{1-q^{5m+2}}.$$

Theorem 4. *Theorem 1 $\Longleftrightarrow$ (4.4).*

Proof. Utilizing the concept of the Durfee square (see Andrews [3]) it can be shown that

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N(m,n)z^m q^n = \sum_{n=1}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}. \tag{4.28}$$

Here following Atkin and Swinnerton-Dyer we agree that $N(0,0)=0$. Analogous to the proof of Theorem 3 we find that

$$\begin{aligned} f(q) &= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(\zeta q)_n(\zeta^{-1}q)_n}. \\ &= 1+\sum_{k=0}^{4}\zeta^k\big(\sum_{n=0}^{\infty}N(k,5,n)q^n\big). \end{aligned} \tag{4.29}$$

After replacing q by q^5 we can write (4.4) as

$$\begin{aligned} f(q) = \big\{A(q^5)+(\zeta+\zeta^{-1}-2)\phi(q^5)\big\} + qB(q^5) + (\zeta+\zeta^{-1})q^2C(q^5) \\ -(\zeta+\zeta^{-1})q^3\left\{D(q^5)-(\zeta^2+\zeta^{-2}-2)\frac{\psi(q^5)}{q^5}\right\}. \end{aligned} \tag{4.30}$$

We now sketch how (4.4) implies Theorem 1. By picking out the coefficient of q^{5n} in both sides of (4.30) we find that

$$1+\sum_{k=0}^{4}\zeta^k N(k,5,5n) = a_n + (\zeta+\zeta^{-1}-2)\phi_n \tag{4.31}$$

where a_n, ϕ_n are the coefficients of q^n in $A(q)$, $\phi(q)$ respectively. It follows that

$$1+N(0,5,5n)-a_n+2\phi_n = N(1,5,5n)-\phi_n = N(2,5,5n) \tag{4.32}$$

since all coefficients are rational integers. From (4.18) and (4.26) we have (2.15) and (2.16). The rest of Theorem 1 follows in the same way. ☐

Corollary.

$$r_{1,2}(0) = \phi(q), \tag{4.33}$$

$$r_{2,0}(3) = \frac{\psi(q)}{q}. \tag{4.34}$$

We now turn to the mock theta conjectures (Andrews and Garvan [4, §1]). The remainder of this section has been taken from [4]. There are ten (unproved) mock theta function identities [4, $(3.1)_R$–$(3.10)_R$]. In fact (4.11)–(4.14) are respectively $(3.1)_R$, $(3.6)_R$, $(3.2)_R$ and $(3.7)_R$. In [4] the ten identities were reduced to two combinatorial conjectures:

First Mock Theta Conjecture.

$$N(1,5,5n) = N(0,5,5n) + \rho_0(n), \tag{4.35}$$

where $\rho_0(n)$ is the number of partitions of n with unique smallest part and all other parts $\leq$ the double of the smallest part.

Second Mock theta conjecture.

$$2N(2,5,5n+3) = N(1,5,5n+3) + N(0,5,5n+3) + \rho_1(n) + 1, \tag{4.36}$$

where $\rho_1(n)$ is the number of partitions of n with unique smallest part and all other parts $\leq$ one plus the double of the smallest part.

We need the following result of Watson [26, (A_0)]:

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1})_{n+1}}. \tag{4.37}$$

Theorem 5 ([4, Theorem 2]**).** *(4.11) $\Longleftrightarrow$(4.35) and (4.12) $\Longleftrightarrow$ (4.36).*

Proof. Let $M_1(q) = \chi_0(q) - 2 - 3\phi(q) + A(q)$. Then (4.11) is equivalent to $M_1(q) = 0$ by (4.37). Now, by (2.17) and (4.33), we have

$$\begin{aligned} M_1(q) &= \chi_0(q) - 2 - 3\phi(q) + A(q) \\ &= \chi_0(q) - 1 + r_{0,2}(0) + 2r_{1,2}(0) - 3r_{1,2}(0) \\ &= \chi_0(q) - 1 - r_{1,0}(0). \end{aligned}$$

But by (4.37) we have

$$\chi_0(q) - 1 = \sum_{n=0}^{\infty} \rho_0(n) q^n$$

if we assume $\rho_0(0) = 0$. Hence (4.11) is equivalent to (4.35). Similarly we find (4.12) is equivalent to (4.36) by considering (2.21) and (4.34). □

5. The search for a better crank. Our crank is in terms of a weighted count of certain restricted triples of partitions. It would be nice if we could interpret $N_V(m,n)$ solely in terms of ordinary partitions. This may be possible (see **Announcement** in this section) since it turns out that all of the coefficients $N_V(m,n)$ except for one are nonnegative. All that is needed to prove this is the q-binomial theorem (Andrews [1, p.17]):

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(at)_\infty}{(t)_\infty}, \tag{5.1}$$

where $|q| < 1$ and $|t| < 1$.

Theorem 6. *For $m \geq 0$,*

$$\sum_{n=0}^{\infty} N_V(m,n) q^n = (1-q) \sum_{n=0}^{\infty} \frac{q^{n^2+nm+2n+m}}{(q)_{m+n}(q)_n}. \tag{5.2}$$

Proof.

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} \tag{5.3}$$

$$\begin{aligned}
&= \frac{(1-z)}{(z^{-1}q)_\infty} \sum_{n=0}^{\infty} \frac{(z^{-1}q)_n z^n}{(q)_n} && \text{(by (5.1) with } a = z^{-1}q \text{ and } t = z) \\
&= (1-z) \sum_{m=0}^{\infty} \frac{z^m}{(q)_m (z^{-1}q^{m+1})_\infty} \\
&= (1-z) \sum_{m=0}^{\infty} \frac{z^m}{(q)_m} \sum_{n=0}^{\infty} \frac{(z^{-1}q^{m+1})^n}{(q)_n} && \text{(by (5.1) with } a = 0 \text{ and } t = z^{-1}q^{m+1}) \\
&= (1-z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^{m-n} q^{nm+n}}{(q)_m (q)_n} \\
&= (1-z) \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} \frac{z^m q^{n(m+n+1)}}{(q)_{m+n} (q)_n} \\
&= \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} \frac{z^m q^{n(m+n+1)}}{(q)_{m+n} (q)_n} - \sum_{n=0}^{\infty} \sum_{m=-n+1}^{\infty} \frac{z^m q^{n(m+n)}}{(q)_{m+n-1} (q)_n}.
\end{aligned}$$

Picking out the coefficient of z^0 in (5.3) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N_V(0,m) q^n &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n^2} - \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q)_{n-1}(q)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n^2} - \sum_{n=1}^{\infty} \frac{q^{n^2}\{q^n + (1-q^n)\}}{(q)_{n-1}(q)_n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}(1-(1-q^n))}{(q)_n^2} - \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q)_{n-1}^2} \\
&= 1 - q + \sum_{n=1}^{\infty} \frac{q^{n(n+2)}}{(q)_n^2} - \sum_{n=2}^{\infty} \frac{q^{n^2}}{(q)_{n-1}^2} \\
&= (1-q) \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q)_n^2},
\end{aligned}$$

which is (5.2) when $m = 0$. The general case follows in a similar fashion. □

Corollary.

$$N_V(m,n) \geq 0 \qquad \textit{for } (m,n) \neq (0,1). \tag{5.4}$$

Theorem 6 yields combinatorial results involving the rank that are similar to but not as deep as the mock theta conjectures.

Theorem 7.

$$N(0,5,5n+1) = \beta_1(n) + N(2,5,5n+1), \tag{5.5}$$
$$N(1,5,5n+2) = \beta_2(n) + N(2,5,5n+2), \tag{5.6}$$

where for $i = 1,\ 2$ $\beta_i(n)$ *denotes the number of partitions of* n *into* i*'s and parts congruent to* 0 *or* $-i$ *modulo* 5 *with the largest part* $\equiv 0 \pmod 5 \leq 5$ *times the number of* i*'s* $\leq$ *the smallest part* $\equiv -i \pmod 5$.

Proof. From (5.3) we have

$$\frac{(q)_\infty}{(z)_\infty(z^{-1}q)_\infty} = \sum_{m=0}^{\infty} \frac{z^m}{(q)_m(z^{-1}q^{m+1})_\infty}. \tag{5.7}$$

Now the generating function for $\beta_1(n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_1(n)q^n &= \sum_{n=0}^{\infty}\frac{q^n}{(q^5;q^5)_n(q^{5n+4};q^5)_\infty}\\ &= \frac{(q^5;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} \qquad \text{(by (5.7) with } q \text{ replaced by } q^5 \text{ and } z \text{ by } q)\\ &= r_{0,2}(1) \qquad \text{(by (2.17))}\end{aligned}$$

and (5.5) follows. Similarly we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_2(n)q^n &= \sum_{n=0}^{\infty}\frac{q^{2n}}{(q^5;q^5)_n(q^{5n+3};q^5)_\infty}\\ &= \frac{(q^5;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = r_{1,2}(2)\end{aligned}$$

and (5.6) follows. □

Example (1) $N(0,5,36) = 3597$, $N(2,5,36) = 3595$, $\beta_1(7) = 2$ with the relevant partitions being $5+1+1$ and $1+1+1+1+1+1+1$.

(2) $N(1,5,37) = 4328$, $N(2,5,37) = 4327$, $\beta_2(7) = 1$ with the relevant partition being $5+2$.

Announcement. The day after this conference ended the *true* crank was discovered by George Andrews and myself. This result will appear in a joint paper with George. May I introduce the crank:

Definition. *For a partition π let $\lambda(\pi)$ denote the largest part of π, $\mu(\pi)$ denote the number of ones in π and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is given by*

$$c(\pi) = \begin{cases} \lambda(\pi), & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0. \end{cases}$$

Following Dyson we let $M(m,n)$ denote the number of partitions of n with crank m and let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t. It turns out that

Theorem 8 (Andrews and Garvan [5]**).** *For $n > 1$,*

$$M(m,n) = N_V(m,n). \tag{5.8}$$

Theorem 8 together with Theorem 2 yield the following new combinatorial interpretations of the congruences (0.1)–(0.3).

Corollary.

$$M(k,5,5n+4) = \frac{p(5n+4)}{5} \qquad 0 \le k \le 4, \tag{5.9}$$

$$M(k,7,7n+5) = \frac{p(7n+5)}{7} \qquad 0 \le k \le 6, \tag{5.10}$$

$$M(k,11,11n+6) = \frac{p(11n+6)}{11} \qquad 0 \le k \le 10. \tag{5.11}$$

The appearance of the functions $A(q)$, $B(q)$, $C(q)$ and $D(q)$ in both (4.3) and (4.4) leads to some rather *curious* relations between the rank and the crank:

Theorem 9. *For $n > 0$,*

$$N(0,5,5n) + 2N(1,5,5n) + M(1,5,5n) = M(0,5,5n) + 3N(2,5,5n), \tag{5.12}$$

$$N(0,5,5n+1) + M(2,5,5n+1) = M(1,5,5n+1) + N(2,5,5n+1), \tag{5.13}$$

$$N(1,5,5n+2) + M(1,5,5n+2) = N(2,5,5n+2) + M(2,5,5n+2), \tag{5.14}$$

$$\begin{aligned} &2N(0,5,5n+3) + M(1,5,5n+3) \\ &\quad = M(0,5,5n+3) + N(1,5,5n+3) + N(2,5,5n+3), \end{aligned} \tag{5.15}$$

$$N(k,5,5n+4) = M(k',5,5n+4) = \frac{p(5n+4)}{5} \qquad 0 \le k, k' \le 4. \tag{5.16}$$

I offer a prize of $25 (Australian) for a combinatorial proof of any one of the relations given above.

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