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CUBIC ANALOGUES OF THE JACOBIAN THETA FUNCTION $\theta(z,q)$

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Abstract. There are three modular forms $a(q)$, $b(q)$, $c(q)$ involved in the parametrization of the hypergeometric function ${}_2F_1(\frac{1}{3},\frac{2}{3};\frac{}{1};\cdot)$ analogous to the classical $\theta_2(q)$, $\theta_3(q)$, $\theta_4(q)$ and the hypergeometric function ${}_2F_1(\frac{1}{2},\frac{1}{2};\frac{}{1};\cdot)$. We give elliptic function generalizations of $a(q)$, $b(q)$, $c(q)$ analogous to the classical theta-function $\theta(z,q)$. A number of identities are proved. The proofs are self-contained, relying on nothing more than the Jacobi triple product identity.

§ 1 Introduction and statement of results

In a recent paper, Borwein, Borwein and Garvan [B-B-G] introduce three functions,

$$a(q) = \sum q^{m^2+mn+n^2},$$
$$b(q) = \sum \omega^{m-n} q^{m^2+mn+n^2} \quad (\omega^3 = 1,\ \omega \neq 1)$$

and (essentially)

$$c(q) = \sum q^{m^2+mn+n^2+m+n}.$$

These play a role for the hypergeometric function ${}_2F_1(\frac{1}{3},\frac{2}{3};\frac{}{1};.)$ analogous to that of the classical $\theta_2,\ \theta_3,\ \theta_4$ for the hypergeometric function ${}_2F_1(\frac{1}{2},\frac{1}{2};\frac{}{1};.)$ [B-B2].

They prove many results concerning $a(q)$, $b(q)$ and $c(q)$, culminating in a result of Ramanujan's ((1.9) below), which we cast as

$$(1.1)\qquad \left(1+9\left(\frac{\eta(q^9)}{\eta(q)}\right)^3\right)\left(1+9\left(\frac{\eta(q^{18})}{\eta(q^2)}\right)^3\right) = 1+9\left(\frac{\eta(q^3)\eta(q^6)}{\eta(q)\eta(q^2)}\right)^4$$

where $\eta(q) = q^{\frac{1}{24}}(q;q)_\infty$, and yielding on the way the classical result

$$(1.2)\qquad 1+27\left(\frac{\eta(q^3)}{\eta(q)}\right)^{12} = \left(1+9\left(\frac{\eta(q^9)}{\eta(q)}\right)^3\right)^3.$$

Throughout this paper we use the standard q-notation:

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \qquad |q|<1.$$

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Both (1.1) and (1.2) appear in Ramanujan's second notebook [R]. (1.1) is equivalent to the first equation on page 259, and (1.2) is Entry 1(iv) of Chapter 20 [Be, p.354]. The principal results of [B-B-G] are

$$a(q) = a(q^3) + 2qc(q^3), \tag{1.3}$$

$$b(q) = a(q^3) - qc(q^3), \tag{1.4}$$

$$a(q) = 1 + 6\sum_{n\ge1}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right), \tag{1.5}$$

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}, \tag{1.6}$$

$$c(q) = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} \tag{1.7}$$

$$a(q)^3 = b(q)^3 + qc(q)^3 \tag{1.8}$$

and

$$a(q)a(q^2) = b(q)b(q^2) + qc(q)c(q^2). \tag{1.9}$$

Other proofs of some of these and related results may be found in [Be].

Our object in writing this paper is to provide simple proofs of all these results. Indeed, we give generalisations ("z-analogs") of (1.3)-(1.9). We also give some further results obtained in the course of our investigations. In all of this we use nothing more than the triple product identity,

$$(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty(q^2;q^2)_\infty = \sum a^n q^{n^2} \tag{1.10}$$

[B-B1, (3.1.1) p.62], [W-W, p.469]. [Note that here, and throughout this paper, unless otherwise stated, it is understood that the summation index or indices range over all integer values.]

Thus, let

$$a'(q,z) = \sum q^{m^2+mn+n^2} z^n, \tag{1.11}$$

$$a(q,z) = \sum q^{m^2+mn+n^2} z^{m-n}, \tag{1.12}$$

$$b(q,z) = \sum \omega^{m-n} q^{m^2+mn+n^2} z^n, \tag{1.13}$$

(1.14). $$c(q,z) = \sum q^{m^2+mn+n^2+m+n} z^{m-n}$$

Then

$$a'(q,z) = z^2q^3a'(q,zq^3), \tag{1.15}$$

$$a(q,z) = z^2qa(q,zq), \tag{1.16}$$

$$b(q,z) = z^2q^3b(q,zq^3), \tag{1.17}$$

$$c(q,z) = z^2qc(q,zq), \tag{1.18}$$

$$a'(q,z) = a(q^3,z) + 2qc(q^3,z), \tag{1.19}$$

$$b(q,z) = a(q^3,z) - qc(q^3,z), \tag{1.20}$$

$$a(q,z) = \frac{1}{3}\left(1+z+z^{-1}\right)\left\{1 + 6\sum_{n\ge1}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\} \cdot \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2}\,\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \tag{1.21}$$

$$+\frac{1}{3}\left(2-z-z^{-1}\right)\frac{(q;q)_\infty^5}{(q^3;q^3)_\infty^3}(zq;q)_\infty^2(z^{-1}q;q)_\infty^2,$$

(1.22)
$$b(q,z)=(q;q)_\infty(q^3;q^3)_\infty\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty},$$

(1.23)
$$c(q,z)=\left(1+z+z^{-1}\right)(q;q)_\infty(q^3;q^3)_\infty\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty},$$

(1.24)
$$\frac{b(q,z)c(q,z)}{b(q^3,z^3)c(q^3,z)}=\frac{b(q)c(q)}{b(q^3)c(q^3)}=\frac{(q;q)_\infty^2}{(q^9;q^9)_\infty^2},$$

(1.25)
$$a(q,z)^3=b(q)^2b(q,z^3)+qc(q,z)^3$$

and

(1.26)
$$a(q,z)a(q^2,z^2)=b(q^2)b(q,z^3)+qc(q,z)c(q^2,z^2).$$

We show also that, although $a(q)$ is not a simple product, $a(q)$ can be written in several ways as the difference of two products. Thus

(1.27)
$$\begin{aligned}a(q)&=\frac{4}{3}\frac{(-q;q)_\infty^6(q;q)_\infty^3}{(-q^3;q^3)_\infty^2(q^3;q^3)_\infty}-\frac{1}{3}\frac{(q;q)_\infty^3(-q^3;q^3)_\infty}{(-q;q)_\infty^3(q^3;q^3)_\infty}\\&=\frac{4}{3}\frac{b(q^2)^2}{b(q)}-\frac{1}{3}\frac{b(q)^2}{b(q^2)},\end{aligned}$$

(1.28)
$$\begin{aligned}a(q)&=4\frac{(-q;q)_\infty^6(q;q)_\infty^3}{(-q^3;q^3)_\infty^2(q^3;q^3)_\infty}-3\frac{(-q;q)_\infty(q^3;q^3)_\infty^3}{(q;q)_\infty(-q^3;q^3)_\infty^3}\\&=4\frac{b(q^2)^2}{b(q)}-\frac{c(q)^2}{c(q^2)},\end{aligned}$$

(1.29)
$$\begin{aligned}a(q)&=\frac{3}{2}\frac{(-q;q)_\infty(q^3;q^3)_\infty^3}{(q;q)_\infty(-q^3;q^3)_\infty^3}-\frac{1}{2}\frac{(q;q)_\infty^3(-q^3;q^3)_\infty}{(-q;q)_\infty^3(q^3;q^3)_\infty}\\&=\frac{1}{2}\frac{c(q)^2}{c(q^2)}-\frac{1}{2}\frac{b(q)^2}{b(q^2)},\end{aligned}$$

and

(1.30)
$$\begin{aligned}a(q)&=3\frac{(q^6;q^6)_\infty^3(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty^3(q^2;q^2)_\infty}-2\frac{(q;q)_\infty^3(-q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(-q^6;q^6)_\infty}\\&=\frac{c(q^2)^2}{c(q^4)}-2\frac{b(q)b(q^4)}{b(q^2)}.\end{aligned}$$

Indeed we prove that

(1.31)
$$\begin{aligned}a(q,z)&=\frac{1}{3}\left(2+z+z^{-1}\right)\frac{(-q;q)_\infty^2(q;q)_\infty^3}{(-q^3;q^3)_\infty^2(q^3;q^3)_\infty}(-zq;q)_\infty^2(-z^{-1}q;q)_\infty^2\\&\quad+\frac{1}{3}\left(1-z-z^{-1}\right)\frac{(q;q)_\infty^3}{(-q;q)_\infty(q^6;q^6)_\infty}\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}\end{aligned}$$

(1.32)
$$\begin{aligned}a(q,z)&=\left(2+z+z^{-1}\right)\frac{(-q;q)_\infty^2(q;q)_\infty^3}{(-q^3;q^3)_\infty^2(q^3;q^3)_\infty}(-zq;q)_\infty^2(-z^{-1}q;q)_\infty^2\\&\quad-\left(1+z+z^{-1}\right)\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty^3}\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty},\end{aligned}$$

and

(1.33)

$$
\begin{aligned}
a(q,z) &= \frac{1}{2}\left(1+z+z^{-1}\right)\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)^3_\infty}\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\
&\quad+\frac{1}{2}\left(1-z-z^{-1}\right)\frac{(q;q)^3_\infty}{(-q;q)_\infty(q^6;q^6)_\infty}\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}.
\end{aligned}
$$

We also observe that

$$a(q) = a(q^4) + 6q(-q^2;q^2)^2_\infty(q^2;q^2)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty, \tag{1.34}$$

$$b(q) = b(q^4) - 3q\frac{(-q^6;q^6)^3_\infty(q^6;q^6)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}, \tag{1.35}$$

$$c(q) = qc(q^4) + 3\frac{(-q^2;q^2)^3_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty}{(-q^6;q^6)_\infty}. \tag{1.36}$$

There are two routes to the proof of most of these results: a direct manipulative approach in the style of some of Jacobi's work and a function theoretic approach. We choose the former but in the final section illustrate the latter to re-prove (1.26).

§ 2 Proofs of (1.1), (1.2)

We begin by showing how (1.1) and (1.2) follow from (1.3)-(1.9).

Thus

$$
\begin{aligned}
1+27\left(\frac{\eta(q^3)}{\eta(q)}\right)^{12} &= 1+q\left(\frac{c(q)}{b(q)}\right)^3 \quad \text{by (1.6) and (1.7)} \\
&= \frac{b(q)^3+qc(q)^3}{b(q)^3} \\
&= \frac{a(q)^3}{b(q)^3} \quad \text{by (1.8)} \\
&= \left(\frac{a(q)}{b(q)}\right)^3 \\
&= \left(1+\frac{a(q)-b(q)}{b(q)}\right)^3 \\
&= \left(1+\frac{3qc(q^3)}{b(q)}\right)^3 \quad \text{by (1.3) and (1.4)} \\
&= \left(1+9\left(\frac{\eta(q^9)}{\eta(q)}\right)^3\right)^3 \quad \text{by (1.6) and (1.7),}
\end{aligned} \tag{2.1}
$$

which is (1.2), and

(2.2)

$$
\begin{aligned}
1+9\left(\frac{\eta(q^3)\eta(q^6)}{\eta(q)\eta(q^2)}\right)^4 &= 1+q\frac{c(q)}{b(q)}\frac{c(q^2)}{b(q^2)} \text{ by (1.6) and (1.7)}\\
&= \frac{b(q)b(q^2)+qc(q)c(q^2)}{b(q)b(q^2)}\\
&= \frac{a(q)a(q^2)}{b(q)b(q^2)} \text{ by (1.9)}\\
&= \left(1+\frac{a(q)-b(q)}{b(q)}\right)\left(1+\frac{a(q^2)-b(q^2)}{b(q^2)}\right)\\
&= \left(1+3q\frac{c(q^3)}{b(q)}\right)\left(1+3q^2\frac{c(q^6)}{b(q^2)}\right) \text{ by (1.3) and (1.4)}\\
&= \left(1+9\left(\frac{\eta(q^9)}{\eta(q)}\right)^3\right)\left(1+9\left(\frac{\eta(q^{18})}{\eta(q^2)}\right)^3\right) \text{ by (1.6) and (1.7),}
\end{aligned}
$$

which is (1.1).

§ 3 Proofs of (1.15)-(1.18).

From (1.11),

$$
\begin{aligned}
z^2q^3a'(q,zq^3) &= \sum q^{m^2+mn+n^2+3n+3}z^{n+2}\\
&= \sum q^{(m-1)^2+(m-1)(n+2)+(n+2)^2}z^{n+2}\\
&= \sum q^{m^2+mn+n^2}z^n\\
&= a'(q,z),
\end{aligned}
\tag{3.1}
$$

which is (1.15).

From (1.12),

$$
\begin{aligned}
a(q,z) &= \sum q^{(m-n)^2+3(m-n)n+3n^2}z^{m-n}\\
&= \sum q^{3m^2+3mn+n^2}z^n,
\end{aligned}
\tag{3.2}
$$

so

$$
\begin{aligned}
z^2qa(q,zq) &= \sum q^{3m^2+3mn+n^2+n+1}z^{n+2}\\
&= \sum q^{3(m-1)^2+3(m-1)(n+2)+(n+2)^2}z^{n+2}\\
&= \sum q^{3m^2+3mn+n^2}z^n\\
&= a(q,z),
\end{aligned}
\tag{3.3}
$$

which is (1.16).

Similarly, from (1.14),

$$
c(q,z) = \sum q^{3m^2+3mn+n^2+2m+n}z^n,
\tag{3.4}
$$

so

$$
\begin{aligned}
z^2qc(q,zq) &= \sum q^{3m^2+3mn+n^2+2m+2n+1}z^{n+2}\\
&= \sum q^{3(m-1)^2+3(m-1)(n+2)+(n+2)^2+2(m-1)+(n+2)}z^{n+2}\\
&= \sum q^{3m^2+3mn+n^2+2m+n}z^n\\
&= c(q,z),
\end{aligned}
\tag{3.5}
$$

which is (1.18), and finally, from (1.13),

$$(3.6) \qquad \begin{aligned} z^2q^3b(q,zq^3) &= \sum \omega^{m-n}q^{m^2+mn+n^2+3n+3}z^{n+2} \\ &= \sum \omega^{(m-1)-(n+2)}q^{(m-1)^2+(m-1)(n+2)+(n+2)^2}z^{n+2} \\ &= \sum \omega^{m-n}q^{m^2+mn+n^2}z^n \\ &= b(q,z), \end{aligned}$$

which is (1.17).

§ 4 Proofs of (1.19), (1.20)

From (1.11) we have

(4.1)

$$a'(q,z) = \sum_{m-n\equiv 0 \bmod 3} q^{m^2+mn+n^2}z^n + \sum_{m-n\equiv 1 \bmod 3} q^{m^2+mn+n^2}z^n + \sum_{m-n\equiv -1 \bmod 3} q^{m^2+mn+n^2}z^n.$$

In the first sum, set $m+2n=3r,\ m-n=3s$,
in the second, set $m+2n-1=3r,\ m-n-1=3s$,
and in the third set $m+2n+1=-3s,\ m-n+1=-3r$,

and we find

$$(4.2) \qquad \begin{aligned} a'(q,z) &= \sum q^{(r+2s)^2+(r+2s)\,(r-s)+(r-s)^2}z^{r-s} \\ &\quad + \sum q^{(r+2s+1)^2+(r+2s+1)\,(r-s)+(r-s)^2}z^{r-s} \\ &\quad + \sum q^{(2r+s+1)^2-(2r+s+1)\,(r-s)+(r-s)^2}z^{r-s} \\ &= \sum q^{3r^2+3rs+3s^2}z^{r-s} + 2q\sum q^{3r^2+3rs+3s^2+3r+3s}z^{r-s} \\ &= a(q^3,z) + 2qc(q^3,z) \text{ by (1.12) and (1.14),} \end{aligned}$$

which is (1.19).

Similarly, from (1.13),

(4.3)

$$\begin{aligned} b(q,z) &= \sum_{m-n\equiv 0 \bmod 3} q^{m^2+mn+n^2}z^n + \omega \sum_{m-n\equiv 1 \bmod 3} q^{m^2+mn+n^2}z^n \\ &\qquad + \omega^2 \sum_{m-n\equiv -1 \bmod 3} q^{m^2+mn+n^2}z^n \\ &= a(q^3,z) + (\omega+\omega^2)qc(q^3,z) \\ &= a(q^3,z) - qc(q^3,z) \text{ by (1.12) and (1.14),} \end{aligned}$$

which is (1.20).

§ 5 Proofs of (1.21) and (1.23)

We start by showing that

(5.1)

$$\begin{aligned} &(-azq^{\frac12};q)_\infty(-az^{-1}q^{\frac12};q)_\infty(-aq^{\frac12};q)_\infty(-a^{-1}z^{-1}q^{\frac12};q)_\infty(-a^{-1}zq^{\frac12};q)_\infty(-a^{-1}q^{\frac12};q)_\infty(q;q)_\infty^3 \\ &= c_0(-a^3q^{\frac32};q^3)_\infty(-a^{-3}q^{\frac32};q^3)_\infty(q^3;q^3)_\infty \\ &+c_1\left(a(-a^3q^{\frac52};q^3)_\infty(-a^{-3}q^{\frac12};q^3)_\infty(q^3;q^3)_\infty + a^{-1}(-a^3q^{\frac12};q^3)_\infty(-a^{-3}q^{\frac52};q^3)_\infty(q^3;q^3)_\infty\right), \end{aligned}$$

where

(5.2)

$$c_0=\frac{1}{3}\left(1+z+z^{-1}\right)\left\{1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\}\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2}\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}$$

$$+\frac{1}{3}\left(2-z-z^{-1}\right)\frac{(q;q)_\infty^5}{(q^3;q^3)_\infty^3}(zq;q)_\infty^2(z^{-1}q;q)_\infty^2$$

and

(5.3)

$$c_1=\left(1+z+z^{-1}\right)q^{\frac{1}{2}}(q;q)_\infty(q^3;q^3)_\infty\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}.$$

Thus, let $f(a)$ denote the left side of (5.1). Then

$$\begin{aligned}f(aq)&=\frac{(1+a^{-1}z^{-1}q^{-\frac{1}{2}})\,(1+a^{-1}zq^{-\frac{1}{2}})\,(1+a^{-1}q^{-\frac{1}{2}})}{(1+azq^{\frac{1}{2}})\,(1+az^{-1}q^{\frac{1}{2}})\,(1+aq^{\frac{1}{2}})}f(a)\\&=a^{-3}q^{-\frac{3}{2}}f(a).\end{aligned}$$

If we let $f(a)=\sum c_na^n$, it follows that

$$q^nc_n=q^{-\frac{3}{2}}c_{n+3}$$

Also, $f\left(a^{-1}\right)=f(a)$, so

$$c_{-n}=c_n.$$

It is an easy induction to show that

$$\begin{aligned}c_{3n}&=q^{3n^2/2}c_0,\\c_{3n+1}&=q^{(3n^2+2n)/2}c_1\end{aligned}$$

and

$$c_{3n-1}=q^{(3n^2-2n)/2}c_{-1}=q^{(3n^2-2n)/2}c_1,$$

and so

$$f(a)=c_0\sum a^{3n}q^{3n^2/2}+c_1a\sum a^{3n}q^{(3n^2+2n)/2}+c_1a^{-1}\sum a^{3n}q^{(3n^2-2n)/2}$$

$$=c_0(-a^3q^{\frac{3}{2}};q^3)_\infty(-a^{-3}q^{\frac{3}{2}};q^3)_\infty(q^3;q^3)_\infty$$

$$+c_1\left(a(-a^3q^{\frac{5}{2}};q^3)_\infty(-a^{-3}q^{\frac{1}{2}};q^3)_\infty(q^3;q^3)_\infty+a^{-1}(-a^3q^{\frac{1}{2}};q^3)_\infty(-a^{-3}q^{\frac{5}{2}};q^3)_\infty(q^3;q^3)_\infty\right)$$

which is (5.1), and where we have used (1.10) to sum the three series.

We now determine c_0 and c_1. In (5.1), set $a^2q^{\frac{1}{2}}$ for a, then multiply by a^3, and we obtain

(5.4)

$$(a+a^{-1}z)\,(a+a^{-1}z^{-1})\,(a+a^{-1}).$$

$$.\,(-a^2zq;q)_\infty(-a^2z^{-1}q;q)_\infty(-a^2q;q)_\infty(-a^{-2}z^{-1}q;q)_\infty(-a^{-2}zq;q)_\infty(-a^{-2}q;q)_\infty(q;q)_\infty^3$$

$$=c_0(a^3+a^{-3})(-a^6q^3;q^3)_\infty(-a^{-6}q^3;q^3)_\infty(q^3;q^3)_\infty$$

$$+c_1q^{-\frac{1}{2}}\left(a(-a^6q^2;q^3)_\infty(-a^{-6}q;q^3)_\infty(q^3;q^3)_\infty+a^{-1}(-a^6q;q^3)_\infty(-a^{-6}q^2;q^3)_\infty(q^3;q^3)_\infty\right).$$

Set $a = e^{i\pi/6}$ in (5.4) and after a little simplification we obtain

$$c_1 = \left(1 + z + z^{-1}\right) q^{\frac{1}{2}} (q;q)_\infty (q^3;q^3)_\infty \frac{(z^3q^3;q^3)_\infty (z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty},$$

which is (5.3).

We can write (5.4) as

(5.5)

$$\left(a^3 + \left(1 + z + z^{-1}\right) a + \left(1 + z + z^{-1}\right) a^{-1} + a^{-3}\right).$$

$$. \ (-a^2zq;q)_\infty (-a^2z^{-1}q;q)_\infty (-a^2q;q)_\infty (-a^{-2}z^{-1}q;q)_\infty (-a^{-2}zq;q)_\infty (-a^{-2}q;q)_\infty (q;q)_\infty^3$$

$$= c_0 (a^3 + a^{-3})(-a^6q^3;q^3)_\infty (-a^{-6}q^3;q^3)_\infty (q^3;q^3)_\infty$$

$$+ c_1 q^{-\frac{1}{2}} \left(a(-a^6q^2;q^3)_\infty (-a^{-6}q;q^3)_\infty (q^3;q^3)_\infty + a^{-1}(-a^6q;q^3)_\infty (-a^{-6}q^2;q^3)_\infty (q^3;q^3)_\infty \right).$$

If we apply the operator $\theta_a = a\dfrac{d}{da}$ to (5.5) and substitute $a = i$, we obtain, after some simplification,

(5.6)

$$c_0 = \frac{1}{3} c_1 q^{-\frac{1}{2}} \frac{(q;q)_\infty}{(q^3;q^3)_\infty^3} \left\{ 1 + 6 \sum_{n \geq 1} \left(\frac{q^{3n-2}}{1 - q^{3n-2}} - \frac{q^{3n-1}}{1 - q^{3n-1}} \right) \right\}$$
$$+ \frac{1}{3} \left(2 - z - z^{-1}\right) \frac{(q;q)_\infty^5}{(q^3;q^3)_\infty^3} (zq;q)_\infty^2 (z^{-1}q;q)_\infty^2$$

which, by (5.3), is (5.2).

Now we turn to establishing (1.21).

By CT_a we will mean the operator which operates on the series $\sum c_n a^n$ to pick out the "Constant Term", that is,

$$CT_a(\sum c_n a^n) = c_0.$$

We have, from (1.12),

(5.7)

$$\begin{aligned}
a(q,z) &= \sum_{m+n+p=0} q^{(m^2+n^2+p^2)/2} z^{m-n} \\
&= CT_a \left\{ \sum a^m z^m q^{m^2/2} . \sum a^n z^{-n} q^{n^2/2} . \sum a^p q^{p^2/2} \right\} \\
&= CT_a \Big\{ (-azq^{\frac{1}{2}};q)_\infty (-a^{-1}z^{-1}q^{\frac{1}{2}};q)_\infty (-az^{-1}q^{\frac{1}{2}};q)_\infty \\
&\qquad . \ (-a^{-1}zq^{\frac{1}{2}};q)_\infty (-aq^{\frac{1}{2}};q)_\infty (-a^{-1}q^{\frac{1}{2}};q)_\infty (q;q)_\infty^3 \Big\} \\
&= CT_a(f(a)) \\
&= CT_a \left(\sum c_n a^n \right) \\
&= c_0
\end{aligned}$$

which, by (5.2), is (1.21).

In similar fashion, from (1.14),

$$
\begin{aligned}
(5.8)\qquad c(q,z) &= \sum_{m+n+p=0} q^{(m^2+m+n^2+n+p^2-p)/2} z^{m-n} \\
&= CT_a\left\{\sum a^m z^m q^{(m^2+m)/2}.\sum a^n z^{-n} q^{(n^2+n)/2}.\sum a^p q^{(p^2-p)/2}\right\} \\
&= CT_a\Big\{(-azq;q)_\infty(-a^{-1}z^{-1};q)_\infty(-az^{-1}q;q)_\infty(-a^{-1}z;q)_\infty \\
&\qquad\qquad .\ (-a;q)_\infty(-a^{-1}q;q)_\infty(q;q)_\infty^3\Big\} \\
&= CT_a(af(aq^{\frac{1}{2}})) \\
&= CT_a\left(a\sum q^{n/2}c_n a^n\right) \\
&= q^{-\frac{1}{2}}c_{-1} = q^{-\frac{1}{2}}c_1
\end{aligned}
$$

which, by (5.3), is (1.23).

§ 6 Proofs of (1.22), (1.24), (1.25)

We have, from (1.13),

(6.1)

$$
\begin{aligned}
b(q,z) &= \sum_{n\text{ even}} \omega^{m-n} q^{m^2+mn+n^2} z^n + \sum_{n\text{ odd}} \omega^{m-n} q^{m^2+mn+n^2} z^n \\
&= \sum \omega^{m-2k} q^{m^2+2km+4k^2} z^{2k} + \sum \omega^{m-(2k+1)} q^{m^2+(2k+1)m+4k^2+4k+1} z^{2k+1} \\
&= \sum \omega^{m+k} q^{(m+k)^2+3k^2} z^{2k} \\
&\qquad + \sum \omega^{m+k-1} q^{(m+k)^2+(m+k)+3k^2+3k+1} z^{2k+1} \\
&= \sum \omega^m q^{m^2} \sum q^{3k^2} z^{2k} \\
&\qquad + \omega^{-1} qz \sum \omega^m q^{m^2+m} \sum q^{3k^2+3k} z^{2k} \\
&= (-\omega q;q^2)_\infty(-\omega^{-1}q;q^2)_\infty(q^2;q^2)_\infty(-z^2q^3;q^6)_\infty(-z^{-2}q^3;q^6)_\infty(q^6;q^6)_\infty \\
&+ \omega^{-1}qz(-\omega q^2;q^2)_\infty(-\omega^{-1};q^2)_\infty(q^2;q^2)_\infty(-z^2q^6;q^6)_\infty(-z^{-2};q^6)_\infty(q^6;q^6)_\infty \\
&= \frac{(-q^3;q^6)_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty}{(-q;q^2)_\infty}(-z^2q^3;q^6)_\infty(-z^{-2}q^3;q^6)_\infty \\
&- q(z+z^{-1})\frac{(-q^6;q^6)_\infty(q^2;q^2)_\infty(q^6;q^6)_\infty}{(-q^2;q^2)_\infty}(-z^2q^6;q^6)_\infty(-z^{-2}q^6;q^6)_\infty.
\end{aligned}
$$

Now let
$$G(z) = \frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty}.$$
Then
$$G(q^3z) = z^{-2}q^{-3}G(z).$$

If we let $G(z) = \sum c_n z^n$, then $q^{3n}c_n = q^{-3}c_{n+2}$.

It follows that

$$G(z) = c_0\sum q^{3n^2}z^{2n} + c_1 z\sum q^{3n^2+3n}z^{2n},$$

or

(6.2)

$$\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty} = c_0(-z^2q^3;q^6)_\infty(-z^{-2}q^3;q^6)_\infty(q^6;q^6)_\infty + c_1(z+z^{-1})(-z^2q^6;q^6)_\infty(-z^{-2}q^6;q^6)_\infty(q^6;q^6)_\infty.$$

Putting $z=i$ in (6.2) gives, after a little manipulation,

$$c_0 = \frac{(-q^2;q^2)_\infty(-q^3;q^6)_\infty}{(q^3;q^3)_\infty}. \tag{6.3}$$

Putting $z = iq^{-\frac{3}{2}}$ in (6.2) gives

$$c_1 = -q\frac{(-q;q^2)_\infty(-q^6;q^6)_\infty}{(q^3;q^3)_\infty}. \tag{6.4}$$

Combining (6.2), (6.3) and (6.4) yields

$$\begin{aligned}(6.5)\qquad &\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty} = \\ &= \frac{(-q^2;q^2)_\infty(-q^3;q^6)_\infty(q^6;q^6)_\infty}{(q^3;q^3)_\infty}\,(-z^2q^3;q^6)_\infty(-z^{-2}q^3;q^6)_\infty \\ &- q(z+z^{-1})\frac{(-q;q^2)_\infty(-q^6;q^6)_\infty(q^6;q^6)_\infty}{(q^3;q^3)_\infty}\,(-z^2q^6;q^6)_\infty(-z^{-2}q^6;q^6)_\infty.\end{aligned}$$

From (6.1) and (6.5) it follows that

$$b(q,z) = (q;q)_\infty(q^3;q^3)_\infty\frac{(zq;q)_\infty(z^{-1}q;q)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty}, \tag{6.6}$$

which is (1.22).

If in (1.20) we put $q,\ \omega q,\ \omega^2 q$ for q in turn, multiply the results, use the fact that

$$b(q,z)b(\omega q,z)b(\omega^2 q,z) = b(q^3)^2 b(q^3,z^3) \tag{6.7}$$

which follows from (1.22) and (1.6), then put q for q^3, we obtain (1.25).

Finally, from (1.22) and (1.23), (1.6) and (1.7) we have

$$\begin{aligned}(6.8)\qquad b(q,z)c(q,z) &= (1+z+z^{-1})(q;q)_\infty^2(q^3;q^3)_\infty^2\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty} \\ &= \frac{1}{3}(1+z+z^{-1})b(q)c(q)\,\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty}\end{aligned}$$

and

$$\begin{aligned}(6.9)\qquad b(q,z^3)c(q,z) &= (1+z+z^{-1})(q;q)_\infty^2(q^3;q^3)_\infty^2\frac{(z^3q;q)_\infty(z^{-3}q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\ &= \frac{1}{3}(1+z+z^{-1})b(q)c(q)\frac{(z^3q;q)_\infty(z^{-3}q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}.\end{aligned}$$

From (6.8) and (6.9), (1.6) and (1.7), we have

$$
\text{(6.10)} \qquad \begin{aligned}
\frac{b(q,z)c(q,z)}{b(q^3,z^3)c(q^3,z)} &= \frac{b(q)c(q)}{b(q^3)c(q^3)} \\
&= \frac{3(q;q)_\infty^2(q^3;q^3)_\infty^2}{3(q^3;q^3)_\infty^2(q^9;q^9)_\infty^2} \\
&= \frac{(q;q)_\infty^2}{(q^9;q^9)_\infty^2},
\end{aligned}
$$

which is (1.24).

§ **7 Proof of (1.26)**

We start by giving, for the sake of completeness, the companion identities to (6.1) for $a(q,z)$ and $c(q,z)$.

From (3.2), we have

(7.1)

$$
\begin{aligned}
a(q,z) &= \sum_{n \text{ even}} q^{3m^2+3mn+n^2} z^n + \sum_{n \text{ odd}} q^{3m^2+3mn+n^2} z^n \\
&= \sum q^{3m^2+6mn+4n^2} z^{2n} + \sum q^{3m^2+6mn+4n^2+3m+4n+1} z^{2n+1} \\
&= \sum q^{3(m+n)^2+n^2} z^{2n} + \sum q^{3(m+n)^2+3(m+n)+n^2+n+1} z^{2n+1} \\
&= \sum q^{3m^2} \sum q^{n^2} z^{2n} + \sum q^{3m^2+3m} \sum q^{n^2+n+1} z^{2n+1} \\
&= (-q^3;q^6)_\infty^2 (q^6;q^6)_\infty (q^2;q^2)_\infty (-z^2q;q^2)_\infty (-z^{-2}q;q^2)_\infty \\
&\quad + 2q(z+z^{-1})(-q^6;q^6)_\infty^2 (q^6;q^6)_\infty (q^2;q^2)_\infty (-z^2q^2;q^2)_\infty (-z^{-2}q^2;q^2)_\infty .
\end{aligned}
$$

(Note that when $z = 1$, this gives $a(q) = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3)$ [B-B-G, Lemma (2.1) (i) (a)]).

Similarly, from (3.4),

(7.2)

$$
\begin{aligned}
c(q,z) &= (-q;q^6)_\infty (-q^5;q^6)_\infty (q^6;q^6)_\infty (q^2;q^2)_\infty (-z^2q;q^2)_\infty (-z^{-2}q;q^2)_\infty \\
&\quad + (z+z^{-1})(-q^2;q^6)_\infty (-q^4;q^6)_\infty (q^6;q^6)_\infty (q^2;q^2)_\infty (-z^2q^2;q^2)_\infty (-z^{-2}q^2;q^2)_\infty .
\end{aligned}
$$

We now turn to the proof of (1.26).

Let $f(z) = a(q,z)a(q^2,z^2)$.

From (1.16) it follows that

$$
\text{(7.3)} \qquad z^6q^3 f(q,qz) = f(q,z)
$$

and from (1.12),

$$
\text{(7.4)} \qquad f(q,z^{-1}) = f(q,z).
$$

Hence, if we set $f(q,z) = \sum f_n(q) z^n$,

$$
\text{(7.5)} \qquad a(q,z)a(q^2,z^2) = \sum f_n(q) z^n
$$

$$= f_0(q)(-z^6q^3;q^6)_\infty(-z^{-6}q^3;q^6)_\infty(q^6;q^6)_\infty$$
$$+ f_1(q)\Big(z(-z^6q^4;q^6)_\infty(-z^{-6}q^2;q^6)_\infty(q^6;q^6)_\infty + z^{-1}(-z^6q^2;q^6)_\infty(-z^{-6}q^4;q^6)_\infty(q^6;q^6)_\infty\Big)$$
$$+ f_2(q)\Big(z^2(-z^6q^5;q^6)_\infty(-z^{-6}q;q^6)_\infty(q^6;q^6)_\infty + z^{-2}(-z^6q;q^6)_\infty(-z^{-6}q^5;q^6)_\infty(q^6;q^6)_\infty\Big)$$
$$+ f_3(q)(z^3+z^{-3})(-z^6q^6;q^6)_\infty(-z^{-6}q^6;q^6)_\infty(q^6;q^6)_\infty.$$

On the other hand, from (3.2),

$$a(q,z)a(q^2,z^2) = \sum q^{3m^2+3mn+n^2}z^n.\sum q^{6r^2+6rs+2s^2}z^{2s},$$

so

$$\begin{aligned} f_0(q) &= \sum_{n+2s=0} q^{3m^2+3mn+n^2+6r^2+6rs+2s^2} \\ &= \sum q^{3m^2-6ms+6r^2+6rs+6s^2} \\ &= \sum q^{3(m-s)^2+3(s+r)^2+3r^2} \\ &= \sum q^{3a^2+3b^2+3c^2} \\ &= (-q^3;q^6)^6_\infty(q^6;q^6)^3_\infty. \end{aligned} \tag{7.6}$$

Similarly

$$f_1(q) = 2q(-q^2;q^2)^2_\infty(q^6;q^6)^3_\infty, \tag{7.7}$$
$$f_2(q) = q(-q;q^2)^2_\infty(q^6;q^6)^3_\infty \tag{7.8}$$

and

$$f_3(q) = 8q^3(-q^6;q^6)^6_\infty(q^6;q^6)^3_\infty. \tag{7.9}$$

Combining (7.5), (7.6), (7.7), (7.8) and (7.9), we obtain

(7.10) $\quad a(q,z)a(q^2,z^2) =$

$$= (-q^3;q^6)^6_\infty(q^6;q^6)^4_\infty(-z^6q^3;q^6)_\infty(-z^{-6}q^3;q^6)_\infty$$
$$+ 2q(-q^2;q^2)^2_\infty(q^6;q^6)^4_\infty\Big(z(-z^6q^4;q^6)_\infty(-z^{-6}q^2;q^6)_\infty + z^{-1}(-z^6q^2;q^6)_\infty(-z^{-6}q^4;q^6)_\infty\Big)$$
$$+ q(-q;q^2)^2_\infty(q^6;q^6)^4_\infty\Big(z^2(-z^6q^5;q^6)_\infty(-z^{-6}q;q^6)_\infty + z^{-2}(-z^6q;q^6)_\infty(-z^{-6}q^5;q^6)_\infty\Big)$$
$$+ 8q^3(-q^6;q^6)^6_\infty(q^6;q^6)^4_\infty(z^3+z^{-3})(-z^6q^6;q^6)_\infty(-z^{-6}q^6;q^6)_\infty.$$

In precisely similar fashion, we can show from (1.18), (1.14) and (3.4) that

(7.11) $\quad c(q,z)c(q^2,z^2) =$

$$= (-q;q^6)^3_\infty(-q^5;q^6)^3_\infty(q^6;q^6)^4_\infty(-z^6q^3;q^6)_\infty(-z^{-6}q^3;q^6)_\infty$$
$$+ 2(-q^2;q^2)^2_\infty(q^6;q^6)^4_\infty\Big(z(-z^6q^4;q^6)_\infty(-z^{-6}q^2;q^6)_\infty + z^{-1}(-z^6q^2;q^6)_\infty(-z^{-6}q^4;q^6)_\infty\Big)$$
$$+ (-q;q^2)^2_\infty(q^6;q^6)^4_\infty\Big(z^2(-z^6q^5;q^6)_\infty(-z^{-6}q;q^6)_\infty + z^{-2}(-z^6q;q^6)_\infty(-z^{-6}q^5;q^6)_\infty\Big)$$
$$+ (-q^2;q^6)^3_\infty(-q^4;q^6)^3_\infty(q^6;q^6)^4_\infty(z^3+z^{-3})(-z^6q^6;q^6)_\infty(-z^{-6}q^6;q^6)_\infty\ .$$

It follows from (7.10) and (7.11) that

(7.12) $a(q,z)a(q^2,z^2) - qc(q,z)c(q^2,z^2) =$

$$= C_0(q)(-z^6q^3;q^6)_\infty(-z^{-6}q^3;q^6)_\infty(q^6;q^6)_\infty$$

$$- qC_3(q)(z^3+z^{-3})(-z^6q^6;q^6)_\infty(-z^{-6}q^6;q^6)_\infty(q^6;q^6)_\infty$$

where

$$C_0(q) = (-q^3;q^6)^6_\infty(q^6;q^6)^3_\infty - q(-q;q^6)^3_\infty(-q^5;q^6)^3_\infty(q^6;q^6)^3_\infty \tag{7.13}$$

and

$$C_3(q) = (-q^2;q^6)^3_\infty(-q^4;q^6)^3_\infty(q^6;q^6)^3_\infty - 8q^2(-q^6;q^6)^6_\infty(q^6;q^6)^3_\infty. \tag{7.14}$$

Now

(7.15)
$$\begin{aligned}
C_0(q) &= \sum q^{3a^2+3b^2+3c^2} - q\sum q^{3a^2+2a+3b^2+2b+3c^2+2c}\\
&= \sum_{a\equiv b\equiv c\equiv 0 \bmod 3} q^{\frac13(a^2+b^2+c^2)} - \sum_{a\equiv b\equiv c\equiv 1 \bmod 3} q^{\frac13(a^2+b^2+c^2)}\\
&= \sum_{a\equiv b\equiv c \bmod 3} \omega^a q^{\frac13(a^2+b^2+c^2)}\\
&= \sum \omega^a q^{\frac13(a^2+(a+3k)^2+(a+3l)^2)}\\
&= \sum \omega^a q^{a^2+2ka+3k^2+2la+3l^2}\\
&= \sum \omega^a q^{(a+k+l)^2+2k^2-2kl+2l^2}\\
&= \sum \omega^{a+k+l} q^{(a+k+l)^2}.\sum \omega^{-k-l} q^{2k^2-2kl+2l^2}\\
&= \sum \omega^a q^{a^2}\sum \omega^{k-l} q^{2k^2+2kl+2l^2}\\
&= \frac{(-q^3;q^6)_\infty(q^2;q^2)_\infty}{(-q;q^2)_\infty} b(q^2)
\end{aligned}$$

and

(7.16)

$$\begin{aligned}
C_3(q) &= \sum q^{3a^2+a+3b^2+b+3c^2+c} - q^2\sum q^{3a^2+3a+3b^2+3b+3c^2+3c}\\
&= \sum_{a\equiv b\equiv c\equiv 0 \bmod 3} q^{\frac13(a^2+b^2+c^2+a+b+c)} - \sum_{a\equiv b\equiv c\equiv 1 \bmod 3} q^{\frac13(a^2+b^2+c^2+a+b+c)}\\
&= -\sum_{a\equiv b\equiv c \bmod 3} \omega^{a-1} q^{\frac13(a^2+b^2+c^2+a+b+c)}\\
&= -\sum \omega^{a-1} q^{\frac13(a^2+(a+3k)^2+(a+3l)^2+a+(a+3k)+(a+3l))}\\
&= -\sum \omega^{a-1} q^{a^2+2ak+3k^2+2al+3l^2+a+k+l}\\
&= -\sum \omega^{a-1} q^{(a+k+l)^2+(a+k+l)+2k^2-2kl+2l^2}\\
&= -\omega^{-1}\sum \omega^a q^{a^2+a}\sum \omega^{-k-l} q^{2k^2-2kl+2l^2}\\
&= \frac{(-q^6;q^6)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} b(q^2).
\end{aligned}$$

It follows from (7.12), (7.15), (7.16) and (6.1) that

$$(7.17)\qquad a(q,z)a(q^2,z^2)-qc(q,z)c(q^2,z^2)=$$

$$\begin{aligned}
&=\frac{(-q^3;q^6)_\infty(q^2;q^2)_\infty}{(-q;q^2)_\infty}b(q^2)(-z^6q^3;q^6)_\infty(-z^{-6}q^3;q^6)_\infty(q^6;q^6)_\infty\\
&-q\frac{(-q^6;q^6)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}b(q^2)(z^3+z^{-3})(-z^6q^6;q^6)_\infty(-z^{-6}q^6;q^6)_\infty(q^6;q^6)_\infty.\\
&=b(q^2)b(q,z^3),
\end{aligned}$$

which is (1.26).

§ 8 Proofs of (1.27) – (1.33)

We begin by writing (5.5) as follows

$$(8.1)\qquad \left(a^3+\left(1+z+z^{-1}\right)a+\left(1+z+z^{-1}\right)a^{-1}+a^{-3}\right).$$

$$.\,(-a^2zq;q)_\infty(-a^2z^{-1}q;q)_\infty(-a^2q;q)_\infty(-a^{-2}z^{-1}q;q)_\infty(-a^{-2}zq;q)_\infty(-a^{-2}q;q)_\infty(q;q)^3_\infty$$

$$\begin{aligned}
&=a(q,z)\,(a^3+a^{-3})(-a^6q^3;q^3)_\infty(-a^{-6}q^3;q^3)_\infty(q^3;q^3)_\infty\\
&+c(q,z)\left(a(-a^6q^2;q^3)_\infty(-a^{-6}q;q^3)_\infty(q^3;q^3)_\infty+a^{-1}(-a^6q;q^3)_\infty(-a^{-6}q^2;q^3)_\infty(q^3;q^3)_\infty\right).
\end{aligned}$$

Here we have used (5.7) and (5.8). If we set $a=1$ and divide by 2, we obtain

$$(8.2)\qquad \left(2+z+z^{-1}\right)(-q;q)^2_\infty(q;q)^3_\infty(-zq;q)^2_\infty(-z^{-1}q;q)^2_\infty$$

$$=a(q,z)(-q^3;q^3)^2_\infty(q^3;q^3)_\infty+c(q,z)(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty,$$

if we set $a=e^{i\pi/3}$ we obtain

(8.3)

$$\left(1-z-z^{-1}\right)\frac{(-q^3;q^3)_\infty(q;q)^3_\infty}{(-q;q)_\infty}\;\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}$$

$$=2a(q,z)(-q^3;q^3)^2_\infty(q^3;q^3)_\infty-c(q,z)(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty,$$

while if we set $a=e^{i\pi/6}$ and divide by $\sqrt{3}$, we obtain

(8.4)

$$\left(1+z+z^{-1}\right)(q^3;q^3)_\infty(q;q)^2_\infty\;\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}=c(q,z)(q;q)_\infty$$

(which is (1.23)).

From (8.2) and (8.3) we deduce

$$(8.5)\qquad a(q,z)(-q^3;q^3)^2_\infty(q^3;q^3)_\infty$$

$$\begin{aligned}
&=\frac{1}{3}\Big\{\left(2+z+z^{-1}\right)(-q;q)^2_\infty(q;q)^3_\infty(-zq;q)^2_\infty(-z^{-1}q;q)^2_\infty\\
&+\left(1-z-z^{-1}\right)\frac{(-q^3;q^3)_\infty(q;q)^3_\infty}{(-q;q)_\infty}\;\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}\Big\}
\end{aligned}$$

which is (1.31), and

$$
\begin{aligned}
(8.6)\qquad & c(q,z)(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty \\
&= \frac{1}{3}\Big\{2\left(2+z+z^{-1}\right)(-q;q)^2_\infty(q;q)^3_\infty(-zq;q)^2_\infty(-z^{-1}q;q)^2_\infty \\
&- \left(1-z-z^{-1}\right)\frac{(-q^3;q^3)_\infty(q;q)^3_\infty}{(-q;q)_\infty}\,\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}\Big\}.
\end{aligned}
$$

Comparing (8.4) and (8.6), we find

(8.7)

$$
\begin{aligned}
&\left(1+z+z^{-1}\right)(q^3;q^3)^2_\infty(q;q)_\infty(-q;q^3)_\infty(-q^2;q^3)_\infty\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\
&= \frac{1}{3}\Big\{2\left(2+z+z^{-1}\right)(-q;q)^2_\infty(q;q)^3_\infty(-zq;q)^2_\infty(-z^{-1}q;q)^2_\infty \\
&- \left(1-z-z^{-1}\right)\frac{(-q^3;q^3)_\infty(q;q)^3_\infty}{(-q;q)_\infty}\,\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}\Big\}.
\end{aligned}
$$

From (8.5) and (8.7) we deduce

$$
\begin{aligned}
(8.8)\qquad & a(q,z)(-q^3;q^3)^2_\infty(q^3;q^3)_\infty \\
&+ \left(1+z+z^{-1}\right)(q^3;q^3)^2_\infty(q;q)_\infty(-q;q^3)_\infty(-q^2;q^3)_\infty\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\
&= \left(2+z+z^{-1}\right)(-q;q)^2_\infty(q;q)^3_\infty(-zq;q)^2_\infty(-z^{-1}q;q)^2_\infty
\end{aligned}
$$

which is (1.32), and

$$
\begin{aligned}
(8.9)\qquad & 2a(q,z)(-q^3;q^3)^2_\infty(q^3;q^3)_\infty \\
&- \left(1+z+z^{-1}\right)(q^3;q^3)^2_\infty(q;q)_\infty(-q;q^3)_\infty(-q^2;q^3)_\infty\,\frac{(z^3q^3;q^3)_\infty(z^{-3}q^3;q^3)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} \\
&= \left(1-z-z^{-1}\right)\frac{(-q^3;q^3)_\infty(q;q)^3_\infty}{(-q;q)_\infty}\,\frac{(-z^3q^3;q^3)_\infty(-z^{-3}q^3;q^3)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}
\end{aligned}
$$

which is (1.33).

If we put $z=1$ in (1.31), (1.32) and (1.33), we obtain (1.27), (1.28) and (1.29).

All that remains is to establish (1.30).

We have, from (1.12) and (1.21)

$$
\begin{aligned}
(8.10)\qquad & \sum q^{m^2+mn+n^2}z^{m-n} \\
&= \frac{1}{3}\left\{1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\}.\frac{1}{(q^3;q^3)^2_\infty} \\
&\quad .\ e^{i\pi/3}(-e^{i\pi/3}zq;q)_\infty(-e^{-i\pi/3}z^{-1};q)_\infty(q;q)_\infty \\
&\quad .\ (-e^{-i\pi/3}z;q)_\infty(-e^{i\pi/3}z^{-1}q;q)_\infty(q;q)_\infty \\
&+\frac{1}{3}\frac{(q;q)^3_\infty}{(q^3;q^3)^3_\infty}\ (zq;q)_\infty(z^{-1};q)_\infty(q;q)_\infty(z;q)_\infty(z^{-1}q;q)_\infty(q;q)_\infty \\
&= \frac{1}{3}\left\{1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\}.\ \frac{1}{(q^3;q^3)^2_\infty} \\
&\quad .\ e^{i\pi/3}\ \sum(e^{i\pi/3})^m z^m q^{(m^2+m)/2}\sum(e^{-i\pi/3})^n z^n q^{(n^2-n)/2} \\
&+ \frac{1}{3}\frac{(q;q)^3_\infty}{(q^3;q^3)^3_\infty}\sum(-1)^m z^m q^{(m^2+m)/2}\sum(-1)^n z^n q^{(n^2-n)/2}.
\end{aligned}
$$

Applying CT_z to this, we find after some simplification that

$$(8.11)\qquad (-q^3;q^6)_\infty^2(q^6;q^6)_\infty$$
$$= \frac{1}{3}\left\{1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)\right\}\frac{(-q^6;q^6)_\infty(q^2;q^2)_\infty}{(q^3;q^3)_\infty^2(-q^2;q^2)_\infty}$$
$$+\frac{2}{3}\frac{(q;q)_\infty^3(-q^2;q^2)_\infty^2(q^2;q^2)_\infty}{(q^3;q^3)_\infty^3},$$

or,

$$(8.12)\qquad 1+6\sum_{n\geq 1}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)$$
$$=3\frac{(-q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q^2;q^2)_\infty(-q^6;q^6)_\infty}\left\{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty-\frac{2}{3}\frac{(q;q)_\infty^3(-q^2;q^2)_\infty^2(q^2;q^2)_\infty}{(q^3;q^3)_\infty^3}\right\},$$

which by (1.5) simplifies to (1.30).

§ 9 Proofs of (1.34) – (1.36)

We have

(9.1)

$$\begin{aligned} a(q) &= \sum q^{m^2+mn+n^2} \\ &= \sum_{\substack{m\equiv 0 \bmod 2\\ n\equiv 0 \bmod 2}} q^{m^2+mn+n^2} + 2\sum_{\substack{m\equiv 1 \bmod 2\\ n\equiv 0 \bmod 2}} q^{m^2+mn+n^2} + \sum_{\substack{m\equiv 1 \bmod 2\\ n\equiv 1 \bmod 2}} q^{m^2+mn+n^2}. \end{aligned}$$

The first sum is

$$(9.2)\qquad \begin{aligned} \sum_{\substack{m\equiv 0 \bmod 2\\ n\equiv 0 \bmod 2}} q^{m^2+mn+n^2} &= \sum q^{(2k)^2+(2k)\,(2l)+(2l)^2} \\ &= \sum q^{4k^2+4kl+4l^2} \\ &= a(q^4); \end{aligned}$$

the second sum is

$$(9.3)\qquad \begin{aligned} \sum_{\substack{m\equiv 1 \bmod 2\\ n\equiv 0 \bmod 2}} q^{m^2+mn+n^2} &= \sum q^{(2k+1)^2+(2k+1)(2l)+(2l)^2} \\ &= \sum q^{4k^2+4kl+4l^2+4k+2l+1} \end{aligned}$$

$$\begin{aligned} &= q\sum q^{(k-l)^2+3(k+l)^2+(k-l)+3(k+l)} \\ &= q\sum_{u\equiv v \bmod 2} q^{u^2+u+3v^2+3v} \\ &= q\left\{\sum_{u\equiv v\equiv 0 \bmod 2} q^{u^2+u+3v^2+3v}+\sum_{u\equiv v\equiv 1 \bmod 2} q^{u^2+u+3v^2+3v}\right\} \\ &= q\left\{\sum q^{(2r)^2+(2r)+3(2s)^2+3(2s)}+\sum q^{(2r-1)^2+(2r-1)+3(2s-1)^2+3(2s-1)}\right\} \\ &= q\left\{\sum q^{4r^2+2r+12s^2+6s}+\sum q^{4r^2-2r+12s^2-6s}\right\} \\ &= 2q(-q^2;q^8)_\infty(-q^6;q^8)_\infty(q^8;q^8)_\infty(-q^6;q^{24})_\infty(-q^{18};q^{24})_\infty(q^{24};q^{24})_\infty \\ &= 2q(-q^2;q^4)_\infty(-q^4;q^4)_\infty(q^4;q^4)_\infty(-q^6;q^{12})_\infty(-q^{12};q^{12})_\infty(q^{12};q^{12})_\infty \\ &= 2q(-q^2;q^2)_\infty^2(q^2;q^2)_\infty(-q^6;q^6)_\infty^2(q^6;q^6)_\infty; \end{aligned}$$

the third sum is

$$(9.4)\qquad \sum_{\substack{m\equiv 1 \bmod 2\\ n\equiv 1 \bmod 2}} q^{m^2+mn+n^2} = \sum q^{(2k+1)^2+(2k+1)\,(2l+1)+(2l+1)^2}$$
$$= \sum q^{4k^2+4kl+4l^2+6k+6l+3}$$
$$= \sum q^{(k-l)^2+3(k+l+1)^2}$$
$$= \sum_{u\not\equiv v \bmod 2} q^{u^2+3v^2}.$$

Now, points (u,v) with $u \not\equiv v$ mod 2 are given either by $u+v=4r+1,\ u-3v=4s+1$ or $u+v=4r-1,\ u-3v=4s-1$ (as is easily seen from a graph of Z^2).

So the third sum is

$$(9.5)\qquad \sum_{u\not\equiv v \bmod 2} q^{u^2+3v^2} = \sum q^{(3r+s+1)^2+3(r-s)^2} + \sum q^{(3r+s-1)^2+3(r-s)^2}$$
$$= \sum q^{12r^2+6r+4s^2+2s+1} + \sum q^{12r^2-6r+4s^2-2s+1}$$
$$= 2q(-q^2;q^2)_\infty^2(q^2;q^2)_\infty(-q^6;q^6)_\infty^2(q^6;q^6)_\infty.$$

Thus we have

$$(9.6)\qquad a(q) = a(q^4) + 6q(-q^2;q^2)_\infty^2(q^2;q^2)_\infty(-q^6;q^6)_\infty^2(q^6;q^6)_\infty$$

which is (1.34).

From (1.3), (1.4) and (1.34) we have

$$(9.7)\qquad b(q) = \frac{3}{2}a(q^3) - \frac{1}{2}a(q)$$

$$= \frac{3}{2}\left[a(q^{12}) + 6q^3(-q^6;q^6)_\infty^2(q^6;q^6)_\infty(-q^{18};q^{18})_\infty^2(q^{18};q^{18})_\infty\right]$$
$$- \frac{1}{2}\left[a(q^4) + 6q(-q^2;q^2)_\infty^2(q^2;q^2)_\infty(-q^6;q^6)_\infty^2(q^6;q^6)_\infty\right]$$
$$= b(q^4) - 3q(-q^6;q^6)_\infty^2(q^6;q^6)_\infty\left\{(-q^2;q^2)_\infty^2(q^2;q^2)_\infty - 3q^2(-q^{18};q^{18})_\infty^2(q^{18};q^{18})_\infty\right\}.$$

Now the expression in braces is

(9.8)

$$\sum q^{4n^2-2n} - 3q^2\sum q^{36n^2-18n}$$

$$= \sum q^{4(3n)^2-2(3n)} + \sum q^{4(3n+1)^2-2(3n+1)}$$
$$+ \sum q^{4(3n-1)^2-2(3n-1)} - 3q^2\sum q^{36n^2+18n}$$
$$= \sum q^{36n^2-6n} + q^6\sum q^{36n^2-30n} - 2q^2\sum q^{36n^2-18n}$$
$$= \sum 2\cos\left((2n+1)\frac{\pi}{3}\right) q^{4n^2-2n}$$
$$= \sum(\alpha^{2n+1}+\alpha^{-(2n+1)})q^{4n^2-2n} \quad \text{where} \quad \alpha = e^{i\pi/3}$$
$$= \left(\alpha+\frac{1}{\alpha}\right)(-\alpha^2q^2;q^2)_\infty(-\alpha^{-2}q^2;q^2)_\infty(q^2;q^2)_\infty$$
$$= (-q^6;q^6)_\infty(q^2;q^2)_\infty/(-q^2;q^2)_\infty.$$

Combining (9.7) and (9.8), we have

$$(9.9)\qquad b(q) = b(q^4) - 3q(-q^6;q^6)_\infty^3(q^6;q^6)_\infty(q^2;q^2)_\infty/(-q^2;q^2)_\infty$$

which is (1.35).

Also, from (1.3), (1.4) and (1.34),

$$qc(q^3) = \frac{1}{2}a(q) - \frac{1}{2}a(q^3) \tag{9.10}$$

$$\begin{aligned}
&= \frac{1}{2}\left[a(q^4) + 6q(-q^2;q^2)_\infty^2(q^2;q^2)_\infty(-q^6;q^6)_\infty^2(q^6;q^6)_\infty\right] \\
&- \frac{1}{2}\left[a(q^{12}) + 6q^3(-q^6;q^6)_\infty^2(q^6;q^6)_\infty(-q^{18};q^{18})_\infty^2(q^{18};q^{18})_\infty\right] \\
&= q^4c(q^{12}) + 3q(-q^6;q^6)_\infty^2(q^6;q^6)_\infty\left\{(-q^2;q^2)_\infty^2(q^2;q^2)_\infty - q^2(-q^{18};q^{18})_\infty^2(q^{18};q^{18})_\infty\right\}.
\end{aligned}$$

The expression in braces is

(9.11)

$$\sum q^{4n^2-2n} - q^2\sum q^{36n^2-18n}$$

$$\begin{aligned}
&= \sum q^{4(3n)^2-2(3n)} + \sum q^{4(3n+1)^2-2(3n+1)} \\
&\qquad + \sum q^{4(3n-1)^2-2(3n-1)} - q^2\sum q^{36n^2+18n} \\
&= \sum q^{36n^2-6n} + q^6\sum q^{36n^2-30n} \\
&= \sum_{n \text{ even}} q^{9n^2-3n} + \sum_{n \text{ odd}} q^{9n^2-3n} \\
&= \sum q^{9n^2-3n} \\
&= (-q^6;q^{18})_\infty(-q^{12};q^{18})_\infty(q^{18};q^{18})_\infty \\
&= (-q^6;q^6)_\infty(q^{18};q^{18})_\infty/(-q^{18};q^{18})_\infty.
\end{aligned}$$

Combining (9.10) and (9.11), we have

(9.12)

$$qc(q^3) = q^4c(q^{12}) + 3q(-q^6;q^6)_\infty^3(q^6;q^6)_\infty(q^{18};q^{18})_\infty/(-q^{18};q^{18})_\infty.$$

If we divide by q, then replace q^3 by q, we find

(9.13)

$$c(q) = qc(q^4) + 3(-q^2;q^2)_\infty^3(q^2;q^2)_\infty(q^6;q^6)_\infty/(-q^6;q^6)_\infty,$$

which is (1.36).

§ 10 A Function Theoretic Proof of (1.26) and a Quintic Identity.

We now illustrate a function theoretic approach to proving our z-analogs. We need the following result due to Atkin and Swinnerton-Dyer [A-SD, Lemma 2].

Lemma 1. *Let $0 < |q| < 1$ be fixed and suppose $f(z)$ is an analytic function of z, except for a finite number of poles, in every region $0 < z_1 \leqq |z| \leqq z_2$. If*

$$f(zq) = Az^kf(z)$$

for some integer k (positive, zero or negative) and constant A, then either $f(z)$ has just k more poles than zeros in the region

$$|q| < |z| \leqq 1,$$

or $f(z)$ vanishes identically.

Let $0 < q < 1$ be fixed and consider

$$F(z) = a(q,z)a(q^2,z^2) - b(q^2)b(q,z^3) - qc(q,z)c(q^2,z^2).$$

From (1.16)-(1.18) we have

$$F(zq) = \frac{1}{z^6 q^3} F(z).$$

Our goal is to prove

$$F(z) \equiv 0.$$

In view of Lemma 1 it suffices to show that $F(z)$ vanishes for at least seven distinct values of z in the region $q < |z| \leqq 1$. In fact, we shall do this for eight distinct values. From (1.23) $c(q, \omega) = c(q, \omega^2) = 0$. Since $a(q, \omega) = b(q)$ it follows that $F(z) = 0$ for $z = \omega,\ \omega^2$. From (1.22) $b(q, q) = b(q, q^2) = 0$. Combining this with (1.20) implies $F(z) = 0$ for $z = \omega^k q^{(\frac{m}{3})}$ for $k = 0,\ 1,\ 2,\ m = 1,\ 2$. This gives a total of eight values of z. Thus the identity holds for all q with $z \neq 0$ and $0 < q < 1$. The general result holds for all $|q| < 1$ by analytic continuation. This completes the proof. It is interesting to note that we did not need the value at $z = 1$.

Ramanujan [R, p.259] has a quintic analog of (1.9),

(10.1)
$$a(q)a(q^5) = b(q)b(q^5) + q^2 c(q)c(q^5) + 3q\sqrt{b(q)b(q^5)c(q)c(q^5)}.$$

We note that

(10.2)
$$q\sqrt{b(q)b(q^5)c(q)c(q^5)} = 3\eta(q)\eta(q^3)\eta(q^5)\eta(q^{15}).$$

Below in (10.7) we give a z-analog of this identity. For $k \in \mathbb{Z}$ we define

(10.3)
$$\phi_k(q, z) = \sum q^{6m^2+km} z^{12m+k}.$$

Let

(10.4)
$$\alpha(q) = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15}),$$

(10.5)
$$\beta(q) = \frac{\eta(q^4)^3\eta(q^5)\eta(q^6)\eta(q^{15})}{\eta(q^2)\eta(q^{12})},$$

(10.6)
$$\gamma(q) = \frac{\eta(q^2)^3\eta(q^5)\eta(q^6)^3\eta(q^{15})}{\eta(q)\eta(q^3)\eta(q^4)\eta(q^{12})}.$$

Then

(10.7)
$$\begin{aligned}
&2a(q,z)a(q^5,z^5) - a(q,\omega z)a(q^5,\omega^2 z^5) - a(q,\omega^2 z)a(q^5,\omega z^5)\\
&- 2q^2 c(q,z)c(q^5,z^5) + q^2 c(q,\omega z)c(q^5,\omega^2 z^5) + q^2 c(q,\omega^2 z)c(q^5,\omega z^5)\\
&= 3(\alpha(q)+\beta(q))\frac{(\phi_1(q,z)+\phi_{-1}(q,z))}{\phi_1(q,1)}\\
&+ \frac{3}{2}(\alpha(q)+\gamma(q))\frac{(\phi_2(q,z)+\phi_{-2}(q,z))}{\phi_2(q,1)}\\
&+ \frac{3}{2}(\alpha(q)-\gamma(q))\frac{(\phi_4(q,z)+\phi_{-4}(q,z))}{\phi_4(q,1)}\\
&+ 3(\alpha(q)-\beta(q))\frac{(\phi_5(q,z)+\phi_{-5}(q,z))}{\phi_5(q,1)}
\end{aligned}$$

Since $a(q, \omega) = a(q, \omega^2) = b(q),\ c(q, \omega) = c(q, \omega^2) = 0$ and the right side of (10.7) simplifies to $18\alpha(q)$ when $z = 1$, this is a z-analog of (10.1) in view of (10.2).

We briefly indicate how (10.7) may be proved. Each term on the left side (and on the right side for that matter) satisfies the functional equation

(10.8)

$$F(zq) = \frac{1}{z^{12}q^6} F(z).$$

Thus to apply the method above would require verifying the identity for at least 13 values of z. In principle, this would require verifying at least 13 q-series identities. We can reduce the size of the problem as follows. The functional equation (10.8) and an analysis analogous to (7.5) reveals that the left side of (10.7) can be written as

$$\begin{aligned} &\alpha_1(q)(\phi_1(q,z) + \phi_{-1}(q,z)) \\ +&\alpha_2(q)(\phi_2(q,z) + \phi_{-2}(q,z)) \\ +&\alpha_4(q)(\phi_4(q,z) + \phi_{-4}(q,z)) \\ +&\alpha_5(q)(\phi_5(q,z) + \phi_{-5}(q,z)), \end{aligned}$$

for some functions $\alpha_i(q)$.

We may proceed as in (7.6) but instead of being able to write each $\alpha_i(q)$ as a product of three θ- functions it is possible to write each as the sum of two such products. Thus the problem is reduced to verifying four q-series identities.

The left side of each such identity involves θ- functions and the right side involves η-functions with all functions involved being modular forms on some congruence subgroup. As in [B-B-G] verifying such identities is a computable task. It would be desirable to find a more transparent proof. Finding a z-analog for $\alpha(q)$ may prove useful.

We describe another application of our z-analogs. At the bottom of page 257 of Ramanujan's second notebook [R] there is an identity which gives the Fourier series of the inverse function of a function which is a cubic analog of the incomplete elliptic integral of the first kind [W-W, p.494]. One of us, in joint work with Bruce Berndt and S. Bhargava, has been able to prove this identity [Be-Bh-G]. The proof depends crucially on identities for $b(q,z)$ and other z-analogs.

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