THE FIBONACCI NUMBERS, SCHUR'S POLYNOMIALS AND THE ROGERS–RAMANUJAN IDENTITIES

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ABSTRACT: The Fibonacci numbers are a sequence of numbers named after Leonardo of Pisa, known as Fibonacci. The first Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... The n-th Fibonacci number $fib(n)$ can be interpreted as the number of ways summing 1's and 2's to $n-1$, with the convention that $fib(0)=0$.

I begin by reviewing some well-known formulas for Fibonacci numbers such as Binet's formula and Cassini's identity. Next, I will discuss a bit more esoteric results:

$$\sum_{r=0}^{L}(-1)^{L+r}\frac{2L}{L+r}\binom{L+r}{2r}fib(2r+1)=\begin{cases}0, & \text{if } L\equiv\pm1 \mod 5,\\ -1, & \text{if } L\equiv\pm2 \mod 5,\\ 2, & \text{if } L\equiv 0 \mod 5.\end{cases}$$

and

$$\sum_{a=0}^{L-1}\sum_{t=0}^{\frac{L-1-a}{2}}(-1)^{t+a}fib(a+2t)\binom{a+t}{a}\binom{L-t-a-1}{t}=\begin{cases}(-1)^{L+1}, & \text{if } L\equiv 2 \mod 5,\\ (-1)^{L}, & \text{if } L\equiv -2 \mod 5,\\ 0, & \text{otherwise.}\end{cases}$$

Remarkably, there are two straightforward q-analogues of the Fibonacci numbers known as Schur's polynomials. These polynomials lead to a natural q-generalizations of the above formulas. Moreover, they can be used to prove the celebrated Rogers–Ramanujan identities:

$$\sum_{n\geq0}\frac{q^{n^2}}{(q)_n}=\frac{1}{\prod_{j\geq0}(1-q^{5j+1})(1-q^{5j+4})},$$

$$\sum_{n\geq0}\frac{q^{n^2+n}}{(q)_n}=\frac{1}{\prod_{j\geq0}(1-q^{5j+2})(1-q^{5j+3})},$$

Here,

$$(q)_n=:\prod_{j=1}^{n}(1-q^j).$$