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A Survey of Mock Theta Functions I

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ABSTRACT. In his last letter to Hardy, Ramanujan defined 17 functions $F(q)$, where $|q| < 1$. He called them mock θ-functions, observing that as q radially approaches any root of unity ζ, there is a θ-function $T_\zeta(q)$ with $F(q) - T_\zeta(q) = O(1)$.

1. Introduction

We begin with the partition generating function $P(q) = (q)_\infty^{-1}$, where as usual

$$(q)_0 = 1\,, \quad (q)_n = \prod_{m=1}^{n} (1-q^m) \quad \text{and} \quad (q)_\infty = \prod_{m=1}^{\infty} (1-q^m)\,, \quad |q|<1\,.$$

More generally, we put

$$(a\,;q^k)_0 = 1\,, \quad (a\,;q^k)_n = \prod_{m=0}^{n-1} (1-aq^{mk}) \quad \text{and} \quad (a\,;q^k)_\infty = \prod_{m=0}^{\infty} (1-aq^{mk})\,,$$

so that $(q)_n = (q\,;q)_n$ and $(q)_\infty = (q\,;q)_\infty$. We have

$$(a\,;q^k)_n = \frac{(a\,;q^k)_\infty}{(aq^{nk};q^k)_\infty}$$

for $n \geq 0$, and for other real n, we take this as the definition of $(a\,;q^k)_n$. $P(q)$ satisfies the Euler and Durfee identities

$$P(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}\,. \tag{1.1}$$

These express $P(q)$ in what S. Ramanujan, in his last letter to G.H. Hardy [**R1**, pp. 354–355; **R2**, pp. 127–131; **W1**, pp. 56–61], called *transformed Eulerian form.* Other examples are provided by the Rogers-Ramanujan identities

$$\left.\begin{aligned} G(q) &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-4})(1-q^{5m-1})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}\,, \\ H(q) &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-3})(1-q^{5m-2})} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n}\,. \end{aligned}\right\} \tag{1.2}$$

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In his letter, Ramanujan remarked that as q tends radially to exponential singularities at roots of unity, the functions $P(q)$ and $G(q)$ have asymptotic approximations involving "closed exponential factors". To express these approximations, he introduced a complex variable α with $\mathrm{Re}(\alpha) > 0$ and put $q = e^{-\alpha}$. Then, for example, if α is real and $\alpha \to 0^+$, we have

$$\left.\begin{aligned} P(q) &= \sqrt{\frac{\alpha}{2\pi}}\,\exp\!\left(\frac{\pi^2}{6\alpha} - \frac{\alpha}{24}\right) + o(1)\,, \\ G(q) &= \sqrt{\frac{2}{5-\sqrt{5}}}\,\exp\!\left(\frac{\pi^2}{15\alpha} - \frac{\alpha}{60}\right) + o(1)\,, \end{aligned}\right\} \tag{1.3}$$

with similar results near exponential singularities at other roots of unity. Ramanujan noted that for other q-series in Eulerian form, approximations analogous to (1.3) may or may not hold. He stated that if

$$F(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q)_n^2}\,,$$

and $q = e^{-\alpha}$ with $\alpha \to 0^+$, then for each positive integer m we have

$$F(q) = \sqrt{\frac{\alpha}{2\pi\sqrt{5}}}\,\exp\!\left(\frac{\pi^2}{5\alpha} + \frac{\alpha}{8\sqrt{5}} + c_2\alpha^2 + \cdots + c_m\alpha^m + O(\alpha^{m+1})\right) \tag{1.4}$$

with infinitely many $c_j \neq 0$. Ramanujan said that in this case "the exponential factor does not close", but an actual proof has not yet been found. An example of a q-series in Eulerian form having an approximation with an unclosed exponential factor is given by

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)+rn}}{(q)_n} = \prod_{\nu=0}^{\infty}(1 - q^{\nu+r})\,,$$

where $0 < r < 1$, $r \neq \frac{1}{2}$. (To obtain a function holomorphic for $|q| < 1$, take $r = \frac{a}{b}$ rational and replace q by q^b.) A proof is given in [**M2**].

At this point we need to clarify what Ramanujan meant by a θ-function. For this purpose, we recall the definition of the Jacobi triple product

$$j(x,q) = (x\,;q)_\infty (x^{-1}q\,;q)_\infty (q\,;q)_\infty\,, \tag{1.5}$$

and the identity

$$j(x,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} x^n\,.$$

Following Hickerson [**H1**], we define a *θ-product* to be an expression of the form

$$Cq^e x_1^{f_1}\cdots x_r^{f_r} L_1^{g_1}\cdots L_s^{g_s}\,,$$

where C is a complex number, e, f_i, g_i are integers, and each L_i has the form

$$j(Dq^h x_1^{k_1} \cdots x_r^{k_r}, \pm q^m)$$

for some complex number D (usually $D = \pm 1$) and integers h, k_i and $m \geq 1$. A *θ-function* is a finite sum of θ-products. Thus $(q)_\infty = j(q, q^3)$ is a θ-function, even though it lacks the factor $q^{\frac{1}{24}}$ needed to make it a modular form.

Every θ-function with an exponential singularity at a root of unity ζ has an asymptotic approximation near ζ like (1.3), where there may be several terms, each with $c_j = 0$ for all $j \geq 2$, and $o(1)$ may be $O(1)$. As an example of an approximation with more than one term, Ramanujan gave $(q)_\infty^{-120}$. A simpler example is provided by $(q)_\infty^{-48}$. By the functional equation of the Dedekind η-function (see for example [**Ap**, p. 48]), we have

$$q^{\frac{1}{24}}(q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{6}}(q_1^4; q_1^4)_\infty \,,$$

where $q = e^{-\alpha}$ and $q_1 = e^{-\frac{\pi^2}{\alpha}}$. Hence as $\alpha \to 0^+$,

$$\begin{aligned}
(q)_\infty^{-48} &= q^2[q^{\frac{1}{24}}(q)_\infty]^{-48} \\
&= q^2\left[\sqrt{\frac{2\pi}{\alpha}}\, q_1^{\frac{1}{6}}(q_1^4; q_1^4)_\infty\right]^{-48} \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-8}[1 - q_1^4 + O(q_1^8)]^{-48} \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-8}[1 + 48q_1^4 + O(q_1^8)] \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-8} + \frac{48\alpha^{24}}{(2\pi)^{24}} q^2 q_1^{-4} + o(1) \\
&= \frac{\alpha^{24}}{(2\pi)^{24}} \exp\left(\frac{8\pi^2}{\alpha} - 2\alpha\right) + \frac{48\alpha^{24}}{(2\pi)^{24}} \exp\left(\frac{4\pi^2}{\alpha} - 2\alpha\right) + o(1)\,.
\end{aligned}$$

A *mock θ-function* is a function $M(q)$, holomorphic for $|q| < 1$, such that

(i) $M(q)$ has infinitely many exponential singularities at roots of unity.

(ii) Under radial approach to every such singularity, $M(q)$ has an approximation consisting of a finite number of terms with closed exponential factors, and an error term $O(1)$.

(iii) There is no θ-function $T(q)$ which differs from $M(q)$ by a “trivial function”, i.e. a function bounded under radial approach to every root of unity.

If $L(q)$ satisfies (i), (ii), (iii) and has an expansion

$$L(q) = \sum_{n=0}^{\infty} a_n q^{\frac{n}{d}} \ ,$$

convergent for $|q| < 1$, where d is a positive integer, then $M(q) = L(q^d)$ is a mock θ-function. By abuse of language, we will sometimes refer to such an $L(q)$ as a mock θ-function. In this paper, we do not require $M(q)$ to be an Eulerian q-series.

In his letter, Ramanujan next introduced the function

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q\,;q)_n^2} \ , \tag{1.6}$$

and the θ-function

$$T(q) = \frac{(q)_\infty^3}{(q^2;q^2)_\infty^2} \ .$$

He stated that if ζ is a primitive ν^{th} root of unity, and $q \to \zeta$ radially, then

$$f(q) = \begin{cases} O(1), & \nu \text{ odd}\,, \\ -T(q) + O(1), & \nu \equiv 2 \,(\text{mod } 4)\,, \\ T(q) + O(1), & \nu \equiv 0 \,(\text{mod } 4)\,. \end{cases}$$

Thus $M(q) = f(q)$ satisfies the following property:

(iv) At each root of unity ζ, there is a θ-function $T_\zeta(q)$ such that $M(q) = T_\zeta(q) + O(1)$ as $q \to \zeta$ radially.

This implies (ii). We call (iv) the *strong approximation property.* Andrews and Hickerson [**AH**] actually take (i), (iii) and (iv) as the definition of a mock θ-function. Such functions will be called *strong mock θ-functions.*

Ramanujan said "it is inconceivable that a single θ-function could be found to cut out the singularities of $f(q)$." Thus he indicated that $f(q)$ satisfies property (iii), and is therefore a strong mock θ-function. His assertion remains unproved. Henceforth when we speak of mock θ-functions, it is with the understanding that they have not yet been shown to possess property (iii).

¿From now on we will use the abbreviations θf and mf for θ-functions and mock θ-functions respectively. The notation mf$_\nu$ stands for an mf of order ν.

Ramanujan listed 17 mf's, to which he assigned orders 3, 5 and 7. (The order appears to be analogous to the level of a modular form.) Watson [**W1**] found three more mf$_3$'s, and in constructing transformation laws for them, Gordon and McIntosh [**GM2**] found two more. Still other mf's, to which orders 2, 6, 8 and 10 have been attributed, are discussed in [**M1**], [**AH**], [**M4**], [**GM1**] and [**C1**]–[**C4**].

2. The Watson-Selberg era

The above title was first used by Andrews [**A2**] in discussing the ground-breaking work on mf's done in the 1930's. This work dealt with the 17 functions of orders 3, 5 and 7 defined in Ramanujan's letter. They are the following:

order 3:

$$\left.\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q\,;q)_n^2}\,, \qquad & \phi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}\,, \\ \psi(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q\,;q^2)_n}\,, \qquad & \chi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q)_n}{(-q^3;q^3)_n}\,, \end{aligned}\right\} \tag{2.1}$$

order 5:

$$\left.\begin{aligned} f_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q\,;q)_n}\,, \qquad & f_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q\,;q)_n}\,, \\ F_0(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q\,;q^2)_n}\,, \qquad & F_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q\,;q^2)_{n+1}}\,, \\ \phi_0(q) &= \sum_{n=0}^{\infty} q^{n^2}(-q\,;q^2)_n\,, \qquad & \phi_1(q) &= \sum_{n=0}^{\infty} q^{(n+1)^2}(-q\,;q^2)_n\,, \\ \psi_0(q) &= \sum_{n=0}^{\infty} q^{\frac12(n+1)(n+2)}(-q\,;q)_n\,, \qquad & \psi_1(q) &= \sum_{n=0}^{\infty} q^{\frac12 n(n+1)}(-q\,;q)_n\,, \\ \chi_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n}}{(q^{n+1};q)_n}\,, \qquad & \chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n}}{(q^{n+1};q)_{n+1}}\,, \end{aligned}\right\} \tag{2.2}$$

order 7:

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}\,, \quad \mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1};q)_{n+1}}\,, \quad \mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1};q)_{n+1}}\,. \tag{2.3}$$

G.N. Watson wrote two papers [**W1**], [**W2**] dealing with (2.1), (2.2) repectively, while A. Selberg [**S1**], [**S2**] dealt with (2.3). In [**W1**], Watson showed that Ramanujan's functions (2.1), together with the 3 further ones

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q\,;q^2)_{n+1}^2}\,, \quad \upsilon(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q\,;q^2)_{n+1}}\,, \quad \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}(q\,;q^2)_{n+1}}{(q^3;q^6)_{n+1}}\,, \tag{2.4}$$

have the strong property (iv). This was accomplished by obtaining modular transformation laws for all but $\chi(q)$ and $\rho(q)$. The last two laws, found in [**GM2**], involve two more mf_3's

$$\xi(q) = 1 + 2\sum_{n=1}^{\infty} \frac{q^{6n(n-1)}}{(q\,;q^6)_n(q^5;q^6)_n}\,, \qquad \sigma(q) = \sum_{n=1}^{\infty} \frac{q^{3n(n-1)}}{(-q\,;q^3)_n(-q^2;q^3)_n}\,. \tag{2.5}$$

A more detailed account of transformation theory is given in §4 below.

Watson also proved a number of linear relations connecting the functions (2.1), such as

$$4\chi(q) - f(q) = 3\theta_4^2(0, q^3)(q)_\infty^{-1}, \tag{2.6}$$

where

$$\theta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz}.$$

In [**W2**], Watson went on to consider the ten fifth order functions (2.2). He was unable to obtain transformation laws for these, so prodeeded differently. He first proved a number of linear relations stated by Ramanujan, such as

$$2\psi_0(-q^2) - f_0(q) = \theta_4(0, q)G(q).$$

He also found and proved similar relations not stated by Ramanujan. Each relation has three terms, one of which is a θf, while the other two are of the form $q^r\mu(\pm q^s)$, with functions $\mu(q)$ appearing in (2.2). Next Watson determined, directly from their definitions, which of the functions (2.2) are bounded under radial approach to certain roots of unity. Using the linear relations he then obtained strong approximations for all the functions (2.2) with singularities at these and other roots of unity.

In [**S**], Selberg proved that Ramanujan's functions (2.3) are strong mf's. The transformation theory was not yet available, and in contrast to orders 3 and 5, there are no linear relations between the $\mathcal{F}_i$. Thus a new approach was required. To deal with $\mathcal{F}_0$, for example, Selberg obtained an identity of the form

$$\mathcal{F}_0(q) = \mathcal{A}(q) + \mathcal{B}(q)\phi(q) + \mathcal{C}(q), \tag{2.7}$$

where $\mathcal{A}(q)$ and $\mathcal{B}(q)$ are θf's, and $\phi(q)$ is the third order function listed in (2.1). He then proved that $\mathcal{C}(q)$ is bounded under radial approach to every root of unity ζ. Since $\phi(q)$ can be strongly approximated at ζ, equation (2.7) provides the required approximation to $\mathcal{F}_0(q)$ there. Similar identities for $\mathcal{F}_1(q)$ and $\mathcal{F}_2(q)$ show that they are also strong mf's.

3. The Andrews-Hickerson era

The next major advances were made starting in the 1950's. As noted above, Watson's paper [**W1**] showed that the mf_3's (2.1), (2.4) could be strongly approximated by θf's at every root of unity. This raised the possibility of applying the circle method to obtain convergent or asymptotic series expansions for the Taylor coefficients of these mf's. Such an expansion for the partition function $p(n)$, whose generating function

$$\sum_{n=0}^{\infty} p(n)q^n = (q)_\infty^{-1}$$

is a θf, had been found earlier by Hardy and Ramanujan [**R1**]. Subsequently H. Rademacher [**Ra**] had improved their result by obtaining a convergent series expansion of $p(n)$. Work on mf$_3$'s was begun by L. Dragonette [**D**], who selected the function

$$f(q) = \sum_{n=0}^{\infty} a(n) q^n$$

of (2.1) for detailed study. Watson's paper [**W1**] gave only the transformation laws for $f(q)$ under the generators $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ of the modular group Γ (where $q = e^{\pi i \tau}$), and Dragonette first needed to determine laws under all the transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ of Γ. After doing so, she used Cauchy's formula

$$a(n) = \frac{1}{2\pi i} \int_C \frac{f(q)}{q^{n+1}}\, dq\,,$$

taking for C the circle $|q| = e^{-\frac{\pi}{n}}$. The next step was to divide C into Farey arcs of order $N = \lfloor n^{\frac{1}{2}} \rfloor$. With the aid of the transformation laws, in each arc $f(q)$ was replaced by another mf$_3$ plus an "error term" (a Mordell integral), which was then estimated. Evaluation of the resulting integrals over the arcs with centers $e^{-\frac{\pi}{n} + i\frac{\pi h}{k}}$ (k fixed) lead to an exponential sum $\lambda(k) = \lambda(k, n)$ involving some unevaluated roots of unity $\epsilon_{h,k}$. The final result was the series

$$a(n) = \sum_{k=1}^{\lfloor n^{\frac{1}{2}} \rfloor} \frac{\lambda(k) \exp\left(\pi (n - \frac{1}{24})^{\frac{1}{2}} / k\sqrt{6}\right)}{k^{\frac{1}{2}} (n - \frac{1}{24})^{\frac{1}{2}}} + O(n^{\frac{1}{2}} \log n)\,. \tag{3.1}$$

In [**A**], Andrews made a substantial improvement in evaluating both the "error terms" and the $\epsilon_{h,k}$. This enabled him to express $\lambda(k)$ in terms of the exponential sum $A_k(n)$ appearing in the Hardy-Ramanujan series for $p(n)$ [**R1**, pp. 284–285]. The improved result is that for every $\epsilon > 0$, the term $O(n^{\frac{1}{2}} \log n)$ in (3.1) can be replaced by $O(n^{\epsilon})$, and that

$$\lambda(k) = \begin{cases} \dfrac{1}{2}(-1)^{\frac{1}{2}(k+1)} A_{2k}(n), & k \text{ odd}\,, \\ \dfrac{1}{2}(-1)^{\frac{1}{2}k} A_{2k}\Big(n - \dfrac{1}{2}k\Big), & k \text{ even.} \end{cases}$$

Andrews conjectured that if $\exp x$ is replaced by $2 \sinh x$ in (3.1), and the resulting series is extended to infinity, it converges to $a(n)$. We will return to this in part II.

In 1976 Andrews discovered, in the mathematics library of Trinity College, Cambridge, a notebook written by Ramanujan towards the end of his life. This important work has come to be known as the Lost Notebook [**R2**]. In it, Ramanujan defined further mf's, and stated linear relations between them. We will discuss this in §5 below, dealing with mf's of even order.

The Lost Notebook also lists ten identities satisfied by the mf_5's (2.2). These have come to be known as the Mock Theta Conjectures. They can be uniformly stated with the aid of the function

$$g_3(x,q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(x\,;q)_{n+1}(x^{-1}q\,;q)_{n+1}}\,. \tag{3.2}$$

The conjectures are:

$$\left.\begin{aligned}
f_0(q) &= -2q^2 g_3(q^2,q^{10}) + \theta_4(0,q^5)G(q)\,,\\
F_0(q) - 1 &= q g_3(q,q^5) - q\psi(q^5)H(q^2)\,,\\
\phi_0(-q) &= -q g_3(q,q^5) + \frac{(q^5;q^5)_\infty G(q^2)H(q)}{H(q^2)}\,,\\
\psi_0(q) &= q^2 g_3(q^2,q^{10}) + q(q\,;q^{10})_\infty(q^9;q^{10})_\infty(q^{10};q^{10})_\infty H(q)\,,\\
\chi_0(q) - 2 &= 3q g_3(q,q^5) - \frac{(q^5;q^5)_\infty G(q)^2}{H(q)}\,,
\end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned}
f_1(q) &= -2q^3 g_3(q^4,q^{10}) + \theta_4(0,q^5)H(q)\,,\\
F_1(q) &= q g_3(q^2,q^5) + \psi(q^5)G(q^2)\,,\\
\phi_1(-q) &= q^2 g_3(q^2,q^5) - \frac{q(q^5;q^5)_\infty G(q)H(q^2)}{G(q^2)}\,,\\
\psi_1(q) &= q^3 g_3(q^4,q^{10}) + (q^3;q^{10})_\infty(q^7;q^{10})_\infty(q^{10};q^{10})_\infty G(q)\,,\\
\chi_1(q) &= 3q g_3(q^2,q^5) + \frac{(q^5;q^5)_\infty H(q)^2}{G(q)}\,,
\end{aligned}\right\} \tag{3.4}$$

where

$$\theta_4(0,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q)_\infty}{(-q\,;q)_\infty} = \frac{(q)_\infty^2}{(q^2;q^2)_\infty}\,,$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)} = \frac{(q^2;q^2)_\infty}{(q\,;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q)_\infty}\,,$$

and $G(q)$ and $H(q)$ are the functions defined in (1.2). These identities were proved by Hickerson [**H1**]. In this survey we will nonetheless refer to them, and other similar identities which have since been proved, as mock theta "conjectures".

The first step in proving (3.3) and (3.4) was taken by Andrews and F. Garvan [**AG**], who showed that the identities (3.3) are all equivalent, as are the identities (3.4). This reduced the problem to proving the identities for $f_0(q)$ and $f_1(q)$.

[Insert, constant terms etc.]

Analogous to the mock theta "conjectures" are the following identities for mf$_7$'s stated and proved by Hickerson [**H2**]:

$$\mathcal{F}_0(q) - 2 = 2qg_3(q,q^7) - \frac{j(q^3,q^7)^2}{(q)_\infty},$$

$$\mathcal{F}_1(q) = 2q^2g_3(q^2,q^7) + \frac{qj(q,q^7)^2}{(q)_\infty},$$

$$\mathcal{F}_2(q) = 2q^2g_3(q^3,q^7) + \frac{j(q^2,q^7)^2}{(q)_\infty}.$$

For each of the mf$_3$'s $\kappa(q)$ in (2.1), (2.4), (2.5), either $\kappa(q)$ or $\kappa(-q)$ has the form $Aq^cg_3(q^a,q^b)+T(q)$, where a, b and c are nonnegative integers and $T(q)$ is a θf [**GM3**]. Indeed:

$$f(-q) = -4qg_3(q,q^4) + \frac{(q^2;q^2)_\infty^7}{(q)_\infty^3(q^4;q^4)_\infty^3},$$

$$\phi(q) = -2qg_3(q,q^4) + \frac{(q^2;q^2)_\infty^7}{(q)_\infty^3(q^4;q^4)_\infty^3},$$

$$\psi(q) = qg_3(q,q^4),$$

$$\chi(-q) = -qg_3(q,q^4) + \frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty(q^{12};q^{12})_\infty^2},$$

$$\omega(q) = g_3(q,q^2),$$

$$\upsilon(q) = -qg_3(q^2,q^4) + \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2},$$

$$\rho(q) = -\frac{g_3(q,q^2)}{2} + \frac{3(q^6;q^6)_\infty^4}{2(q^2;q^2)_\infty(q^3;q^3)_\infty^2}\,,$$

$$\xi(q) = 1 + 2qg_3(q,q^6)$$

$$= q^2 g_3(q^3,q^6) + \frac{(q^2;q^2)_\infty^4}{(q)_\infty^2(q^6;q^6)_\infty}\,,$$

$$\sigma(-q) = q^2 g_3(q^3,q^{12}) + \frac{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty^3}{(q)_\infty(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}\,.$$

These identities can be viewed as third order mock theta "conjectures".

4. Transformation theory

In discussing the approximation of mf's near roots of unity, we have adhered to the notation $q = e^{-\alpha}$, employed by Ramanujan and his early successors. This function maps the right half-plane $\mathrm{Re}(\alpha) > 0$ onto the punctured disc $0 < |q| < 1$. In the classical theory of θf's, as expounded for example in [**TM**] and [**WW**], it is customary to write instead $q = e^{\pi i\tau}$, where $\mathrm{Im}(\tau) > 0$. Thus $\alpha = -\pi i\tau$. The variable τ is then subjected to the transformations

$$\tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d}\,,$$

where a, b, c, d are integers with $ad - bc = 1$. These transformations form the modular group Γ; it is generated by

$$T\tau = \tau + 1, \qquad S\tau = -1/\tau = \tau_1\,.$$

These generators map $q = e^{\pi i\tau}$ to $-q$ and to $q_1 = e^{\pi i\tau_1} = e^{-\pi i/\tau}$, respectively. Equivalently, we have $q_1 = e^{-\beta}$, where $\alpha\beta = \pi^2$.

When a function $W(q)$ is being considered as a function of τ, it is customary to denote it by $W(\tau)$. The transformation laws for an mf $M(\tau)$ express $M(A\tau)$ (where $A \in \Gamma$) in terms of another mf $M^*(\tau)$ and a Mordell integral, to be defined below. Since Watson's fundamental paper [**W1**], it has become standard to write two laws for each $M(q)$, one expressing $M(q)$ in terms of $M^*(cq_1^r)$ (where $c = \pm 1$ and r is a rational number) and a Mordell integral, and the other doing the same for $M(-q)$. For example, the transformation laws for the mf$_3$ $f(q)$ are given by

$$q^{-\frac{1}{24}} f(q) = \sqrt{\frac{8\pi}{\alpha}}\, q_1^{\frac{4}{3}}\omega(q_1^2) + \sqrt{\frac{24\alpha}{\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\sinh \alpha x}{\sinh \frac{3}{2}\alpha x}\, dx\,,$$

$$q^{-\frac{1}{24}} f(-q) = -\sqrt{\frac{\pi}{\alpha}}\, q_1^{-\frac{1}{24}} f(-q_1) + \sqrt{\frac{24\alpha}{\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh \frac{5}{2}\alpha x + \cosh \frac{1}{2}\alpha x}{\cosh 3\alpha x}\, dx\,,$$

where $\omega(q)$ is another mf$_3$. (The usual notation for modular forms $F(\tau)$ has $F(A\tau)$ on the left and $F^*(\tau)$ on the right.)

The transformation laws for $g_3(q^a, q^b)$ (where a and b are integers with $b > 0$ and $a \not\equiv 0 \pmod b$) are stated in [**GM2**]. In particular, we proved that for nonintegral rational numbers r,

$$q^{\frac{3}{2}r(1-r)-\frac{1}{24}} g_3(q^r, q) = \sqrt{\frac{\pi}{2\alpha}} \csc(\pi r) q_1^{-\frac{1}{6}} h_3(e^{2\pi i r}, q_1^4) - \sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \frac{\cosh(3r-1)\alpha x + \cosh(3r-2)\alpha x}{\cosh \frac{3}{2}\alpha x} \, dx \,, \quad (4.1)$$

where

$$h_3(y, q) = \sum_{n=0}^\infty \frac{q^{n^2}}{(yq\,;q)_n (y^{-1}q\,;q)_n} \,. \quad (4.2)$$

A transformation law with $g_3(q^a, q^b)$ on the left is obtained by putting $r = a/b$ and replacing q by q^b (hence q_1 becomes $q_1^{1/b}$ and α becomes $b\alpha$). This law shows that $g_3(q^a, q^b)$ and $h_3(e^{2\pi i a/b}, q)$ are mf's. The transformation law for $g_3((-q)^a, (-q)^b)$, also stated in [**GM2**], depends on the parity of a and b.

The transformation laws for the mf$_3$'s, mf$_5$'s and mf$_7$'s can be obtained from the transformation laws for $g_3(q^a, q^b)$ and $g_3((-q)^a, (-q)^b)$ using the mock theta "conjectures" of §3. The explicit laws are found in [**W1**] and [**GM2**].

We now outline a proof of (4.1). The first step is to show that

$$g_3(q^r, q) = \frac{1}{(q)_\infty} \sum_{n=-\infty}^\infty \frac{(-1)^n q^{\frac{3}{2}n(n+1)}}{1 - q^{n+r}} \quad (4.3)$$

and

$$h_3(e^{2\pi i r}, q) = \frac{4\sin^2 \pi r}{(q)_\infty} \sum_{n=-\infty}^\infty \frac{(-1)^n q^{\frac{1}{2}n(3n+1)}}{(1 - e^{2\pi i r}q^n)(1 - e^{-2\pi i r}q^n)} \,. \quad (4.4)$$

The series here are called *generalized Lambert series*. As in [**GM2**, pp. 196–198], equations (4.3) and (4.4) are obtained from the Watson-Whipple transformation [**GR**, p. 242, (III.17)]. The transformation law (4.1) is obtained from (4.3) using contour integration and the saddle-point method. This technique can be applied more generally to the series

$$u_k(x, q) = \sum_{n=-\infty}^\infty \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1 - xq^n} \,, \quad (4.5)$$

where $x = q^r$ and k is a positive integer. Application of the saddle-point method to the integral

$$\frac{1}{2\pi i} \left(\int_{-\infty-\epsilon i}^{+\infty-\epsilon i} + \int_{+\infty+\epsilon i}^{-\infty+\epsilon i} \right) \frac{\pi}{\sin \pi z} \, \frac{e^{-\frac{1}{2}k\alpha z(z+1)}}{1 - e^{-\alpha(z+r)}} \, dz$$

leads to the transformation law

$$q^{\frac{1}{2}kr(1-r)}u_k(q^r,q) = \frac{4\pi}{\alpha}\sin(\pi r)\,v_k(e^{2\pi ir},q_1^4) - \sqrt{\frac{k\alpha}{2\pi}}\sum_{m=1}^{k-1} q^{\frac{(k-2m)^2}{8k}} j(q^m,q^k)\int_0^\infty e^{-\frac{1}{2}k\alpha x^2}\frac{\cosh(kr-m)\alpha x}{\cosh\frac{1}{2}k\alpha x}\,dx\,, \tag{4.6}$$

where

$$v_k(y,q) = \sum_{n=-\infty}^{\infty}\frac{(-1)^{kn}q^{\frac{1}{2}n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)} \tag{4.7}$$

and $j(x,q)$ is defined in (1.5). From this law it follows that $u_k(q^r,q)/j(q^h,q^k)$ is an mf for integers h, k with $0<h<k$.

When $k=1$, 2 or 3, the Watson-Whipple transformation can be used to express $u_k(x,q)$ and $v_k(y,q)$ in Eulerian form:

$$\left.\begin{aligned}
u_1(x,q) &= \frac{(q)_\infty^3}{j(x,q)}\,,\\
u_2(x,q) &= \frac{(q)_\infty^2}{(q^2;q^2)_\infty}g_2(x,q) = j(q,q^2)g_2(x,q)\,,\\
u_3(x,q) &= (q)_\infty g_3(x,q) = j(q,q^3)g_3(x,q)\,,\\
v_1(y,q) &= \frac{(q)_\infty^2}{(y\,;q)_\infty(y^{-1};q)_\infty}\,,\\
v_2(y,q^2) &= \frac{(q^2;q^2)_\infty^2 h_2(y,q)}{(q)_\infty(1-y)(1-y^{-1})}\,,\\
v_3(y,q) &= \frac{(q)_\infty h_3(y,q)}{(1-y)(1-y^{-1})}\,,
\end{aligned}\right\} \tag{4.8}$$

where

$$g_2(x,q) = \sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n+1)}(-q\,;q)_n}{(x\,;q)_{n+1}(x^{-1}q\,;q)_{n+1}}\,, \tag{4.9}$$

$$h_2(y,q) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q\,;q^2)_n}{(yq^2;q^2)_n(y^{-1}q^2;q^2)_n}\,. \tag{4.10}$$

A more detailed study of the functions $g_2(x,q)$ and $h_2(y,q)$ is found in [**M3**].

When $k=1$, there is no Mordell integral in (4.6), which says that

$$q^{\frac{1}{2}r(1-r)}u_1(q^r,q) = \frac{4\pi}{\alpha}\sin(\pi r)\,v_1(e^{2\pi ir},q_1^4)\,.$$

By (4.8), this becomes

$$\frac{q^{\frac{1}{2}r(1-r)}(q)_\infty^2}{(q^r\,;q)_\infty(q^{1-r}\,;q)_\infty} = \frac{4\pi}{\alpha}\sin(\pi r)\frac{(q_1^4;q_1^4)_\infty^2}{(e^{2\pi ir};q_1^4)_\infty(e^{-2\pi ir};q_1^4)_\infty}\,,$$

a transformation law for a θf.

When $k=2$, (4.6) simplifies to

$$q^{r(1-r)}u_2(q^r,q) = \frac{4\pi}{\alpha}\sin(\pi r)\,v_2(e^{2\pi ir},q_1^4) - \sqrt{\frac{\alpha}{\pi}}\,\frac{(q)_\infty^2}{(q^2;q^2)_\infty}\int_0^\infty e^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,dx\,. \tag{4.11}$$

By the functional equation of the Dedekind η-function we get

$$q^{\frac{1}{24}}(q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\,q_1^{\frac{1}{6}}(q_1^4;q_1^4)_\infty\,. \tag{4.12}$$

Hence

$$\frac{(q)_\infty^2}{(q^2;q^2)_\infty} = \sqrt{\frac{4\pi}{\alpha}}\,\frac{q_1^{\frac{1}{4}}(q_1^4;q_1^4)_\infty^2}{(q_1^2;q_1^2)_\infty}\,. \tag{4.13}$$

Dividing (4.11) by (4.13), and using (4.8), we obtain

$$q^{r(1-r)}g_2(q^r,q) = \sqrt{\frac{4\pi}{\alpha}}\sin(\pi r)\frac{q_1^{-\frac{1}{4}}h_2(e^{2\pi ir},q_1^2)}{(1-e^{2\pi ir})(1-e^{-2\pi ir})} - \sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,dx\,.$$

Since $(1-e^{2\pi ir})(1-e^{-2\pi ir}) = 2-2\cos 2\pi r = 4\sin^2\pi r$, the above formula simplifies to

$$q^{r(1-r)}g_2(q^r,q) = \sqrt{\frac{\pi}{4\alpha}}\csc(\pi r)\,q_1^{-\frac{1}{4}}h_2(e^{2\pi ir},q_1^2) - \sqrt{\frac{\alpha}{\pi}}\int_0^\infty e^{-\alpha x^2}\frac{\cosh(2r-1)\alpha x}{\cosh\alpha x}\,dx\,. \tag{4.14}$$

When $k=3$, (4.6) becomes

$$\begin{aligned} q^{\frac{3}{2}r(1-r)}u_3(q^r,q) &= \frac{4\pi}{\alpha}\sin(\pi r)\,v_3(e^{2\pi ir},q_1^4) \\ &\qquad - \sqrt{\frac{3\alpha}{2\pi}}\,q^{\frac{1}{24}}(q)_\infty\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x+\cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,dx\,, \end{aligned}$$

since $j(q,q^3)=j(q^2,q^3)=(q)_\infty$. Dividing by (4.12), and using (4.8), we obtain

$$\begin{aligned} q^{\frac{3}{2}r(1-r)-\frac{1}{24}}g_3(q^r,q) &= \sqrt{\frac{8\pi}{\alpha}}\sin(\pi r)\,\frac{q_1^{-\frac{1}{6}}h_3(e^{2\pi ir},q_1^4)}{(1-e^{2\pi ir})(1-e^{-2\pi ir})} \\ &\qquad - \sqrt{\frac{3\alpha}{2\pi}}\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x+\cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,dx \\ &= \sqrt{\frac{\pi}{2\alpha}}\csc(\pi r)\,q_1^{-\frac{1}{6}}h_3(e^{2\pi ir},q_1^4) \\ &\qquad - \sqrt{\frac{3\alpha}{2\pi}}\int_0^\infty e^{-\frac{3}{2}\alpha x^2}\frac{\cosh(3r-1)\alpha x+\cosh(3r-2)\alpha x}{\cosh\frac{3}{2}\alpha x}\,dx\,, \end{aligned}$$

which is (4.1).

5. The mock theta functions of even order

In §3 we observed that the mf's of odd order are related to the function $g_3(x,q)$ of (3.2). It turns out that the mf's of even order are similarly related to the function

$$g_2(x,q)=\sum_{n=0}^{\infty}\frac{q^{\frac12 n(n+1)}(-q\,;q)_n}{(x\,;q)_{n+1}(x^{-1}q\,;q)_{n+1}}$$

of (4.9).

We begin with the mf$_2$'s:

$$\left.\begin{aligned}
A(q)&=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q\,;q^2)_{n+1}}\,,\\
B(q)&=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(q\,;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{q^{n}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}}\,,\\
\mu(q)&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q\,;q^2)_n}{(-q^2;q^2)_n^2}\,.
\end{aligned}\right\}\tag{5.1}$$

The function $\mu(q)$ appears several times in the Lost Notebook [**R2**, (3.1), (3.4), (3.8), (3.9), (3.11), (3.13) with $a=1$]. It is related to $A(q)$ by the identity [**A1**, (3.28)]:

$$\mu(q)+4A(-q)=\frac{(q)_\infty^5}{(q^2;q^2)_\infty^4}\,.$$

The mock theta "conjectures" of order 2 are [**GM3**]:

$$\left.\begin{aligned}
A(q^2)&=qg_2(q,q^4)-q(-q^2;q^2)_\infty(-q^4;q^4)_\infty^2(q^8;q^8)_\infty\,,\\
B(q)&=g_2(q,q^2)\,,\\
\mu(q^4)&=-2qg_2(q,q^2)+\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^3(q^8;q^8)_\infty}{(q)_\infty^2(q^{16};q^{16})_\infty^2}\,.
\end{aligned}\right\}\tag{5.2}$$

These are not needed to prove the modular transformation laws, which are [**A1**, (4.7), (4.8)], [**M1**, p. ?]:

$$\left.\begin{aligned}
q^{-\frac18}A(q)&=\sqrt{\frac{\pi}{16\alpha}}\,q_1^{-\frac18}\mu(-q_1)-\sqrt{\frac{\alpha}{2\pi}}K(\alpha)\,,\\
q^{-\frac18}A(-q)&=\sqrt{\frac{\pi}{2\alpha}}\,q_1^{\frac12}B(-q_1)-\sqrt{\frac{\alpha}{8\pi}}J\Big(\frac{\alpha}{2}\Big)\,,\\
q^{-\frac18}\mu(q)&=\sqrt{\frac{8\pi}{\alpha}}\,q_1^{\frac12}B(q_1)+\sqrt{\frac{2\alpha}{\pi}}J\Big(\frac{\alpha}{2}\Big)\,,
\end{aligned}\right\}\tag{5.3}$$

where $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$ with $\alpha\beta = \pi^2$. The Mordell integrals J, K and their inversions are

$$J(\alpha) = \int_0^\infty \frac{e^{-\alpha x^2}}{\cosh \alpha x}\, dx\,, \qquad J(\beta) = \sqrt{\frac{\alpha^3}{\pi^3}}\, J(\alpha)\,,$$

$$K(\alpha) = \int_0^\infty e^{-\frac{1}{2}\alpha x^2} \frac{\cosh \frac{1}{2}\alpha x}{\cosh \alpha x}\, dx\,, \qquad K(\beta) = \sqrt{\frac{\alpha^3}{\pi^3}}\, K(\alpha)\,.$$

We turn next to mf$_6$'s, ten of which have been identified and studied thus far [**AH**], [**M4**]. They are the following:

$$\beta(q) = \sum_{n=0}^\infty \frac{q^{3n^2+3n+1}}{(q\,;q^3)_{n+1}(q^2;q^3)_{n+1}} = q g_3(q, q^3)\,, \tag{5.4}$$

$$\gamma(q) = \sum_{n=0}^\infty \frac{q^{n^2}(q)_n}{(q^3;q^3)_n} = h_3(q, q^3)\,, \tag{5.5}$$

$$\left.\begin{aligned}
\phi(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^{n^2}(q\,;q^2)_n}{(-q\,;q)_{2n}}\,, &\quad \psi(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^{(n+1)^2}(q\,;q^2)_n}{(-q\,;q)_{2n+1}}\,,\\
\rho(q) &= \sum_{n=0}^\infty \frac{q^{n(n+1)/2}(-q\,;q)_n}{(q\,;q^2)_{n+1}}\,, &\quad \sigma(q) &= \sum_{n=0}^\infty \frac{q^{(n+1)(n+2)/2}(-q\,;q)_n}{(q\,;q^2)_{n+1}}\,,\\
\lambda(q) &= \sum_{n=0}^\infty \frac{(-1)^n q^n(q\,;q^2)_n}{(-q\,;q)_n}\,, &\quad \mu(q) &= \sum_{n=0}^\infty \frac{(-1)^n (q\,;q^2)_n}{(-q\,;q)_n}\,,
\end{aligned}\right\} \tag{5.6}$$

$$\nu(q) = \sum_{n=0}^\infty \frac{q^{n+1}(-q\,;q)_{2n+1}}{(q\,;q^2)_{n+1}}\,, \qquad \xi(q) = \sum_{n=0}^\infty \frac{q^{n+1}(-q\,;q)_{2n}}{(q\,;q^2)_{n+1}}\,. \tag{5.7}$$

The series defining $\mu(q)$ converges in the Cesàro (C,1) sense. In fact the sequence of its even partial sums converges, as does the sequence of its odd partial sums; $\mu(q)$ is the average of their limits. The functions (5.5) and (5.6) are in the Lost Notebook, while (5.4) and (5.7) arise in the modular transformation laws [**M4**]. In view of their expressions in terms of $g_3(x, q)$ and $h_3(x, q)$, a case can be made for designating $\beta(q)$ and $\gamma(q)$ as mf$_3$'s.

Ramanujan listed five linear relations connecting mf$_6$'s:

$$\left.\begin{aligned}
q^{-1}\psi(q^2) + \rho(q) &= (-q\,;q^2)_\infty^2 j(-q, q^6)\,,\\
\phi(q^2) + 2\sigma(q) &= (-q\,;q^2)_\infty^2 j(-q^3, q^6)\,,\\
2\phi(q^2) - 2\mu(-q) &= (-q\,;q^2)_\infty^2 j(-q^3, q^6)\,,\\
2q^{-1}\psi(q^2) + \lambda(-q) &= (-q\,;q^2)_\infty^2 j(-q, q^6)\,,\\
3\phi(q) - 2\gamma(q) &= \frac{j(q, q^2)^2}{j(-q, q^3)}\,.
\end{aligned}\right\} \tag{5.8}$$

Additional relations include [**M4**]

$$\left.\begin{aligned}\nu(q^2)-\sigma(-q)&=\frac{q(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty}\,,\\ 2q^{-1}\xi(q^2)+\rho(-q)&=\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^3(q^{12};q^{12})_\infty}\,,\\ \phi(q^3)+2q^{-1}\psi(q^3)+2\beta(q)&=\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^5}{(q)_\infty^2(q^6;q^6)_\infty^3}\,,\end{aligned}\right\}\tag{5.9}$$

and the mock theta “conjectures”

$$\left.\begin{aligned}\phi(q^4)&=\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty^3}{(q)_\infty^2(q^6;q^6)_\infty^3(q^8;q^8)_\infty(q^{24};q^{24})_\infty}-2qg_2(q,q^6)\,,\\ \psi(q^4)&=\frac{q^3(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^{24};q^{24})_\infty^2}{(q)_\infty(q^3;q^3)_\infty(q^8;q^8)_\infty^2}-q^3g_2(q^3,q^6)\,.\end{aligned}\right\}\tag{5.10}$$

With the aid of (5.8)–(5.10), all the mf$_6$'s (5.4)–(5.7) can be expressed in terms of $g_2(x,q)$ and θf's.

The modular transformation laws for $\beta(q)$ and $\gamma(q)$ are:

$$q^{-\frac{1}{8}}\beta(q)=\sqrt{\frac{2\pi}{9\alpha}}\,q_1^{-\frac{1}{18}}\gamma(q_1^{\frac{4}{3}})-\sqrt{\frac{81\alpha}{8\pi}}\left[J\Big(\frac{9\alpha}{2}\Big)+\frac{1}{9}J\Big(\frac{\alpha}{2}\Big)\right],$$

$$q^{-\frac{1}{8}}\beta(-q)=-\sqrt{\frac{\pi}{9\alpha}}\,q_1^{-\frac{1}{72}}\gamma(-q_1^{\frac{1}{3}})+\sqrt{\frac{81\alpha}{2\pi}}\left[K(9\alpha)-\frac{1}{9}K(\alpha)\right],$$

$$q^{-\frac{1}{24}}\gamma(q)=\sqrt{\frac{6\pi}{\alpha}}\,q_1^{-\frac{1}{6}}\beta(q_1^{\frac{4}{3}})+\sqrt{\frac{27\alpha}{2\pi}}J_1\Big(\frac{3\alpha}{2}\Big),$$

$$q^{-\frac{1}{24}}\gamma(-q)=-\sqrt{\frac{3\pi}{\alpha}}\,q_1^{-\frac{1}{24}}\beta(-q_1^{\frac{1}{3}})+\sqrt{\frac{27\alpha}{2\pi}}K_1(3\alpha)\,,$$

where

$$J_1(\alpha)=\frac{1}{2}J(\alpha)+\frac{1}{6}J\Big(\frac{\alpha}{9}\Big)=\int_0^\infty e^{-\alpha x^2}\frac{\cosh\frac{2}{3}\alpha x}{\cosh\alpha x}\,dx$$

and

$$K_1(\alpha)=\frac{1}{3}K\Big(\frac{\alpha}{9}\Big)-K(\alpha)=\int_0^\infty e^{-\frac{1}{2}\alpha x^2}\,\frac{\cosh\frac{5}{6}\alpha x-\cosh\frac{1}{6}\alpha x}{\cosh\alpha x}\,dx\,.$$

The transformation laws for (5.6) and (5.7) are more complex than those for $\beta(q)$ and $\gamma(q)$. They are:

$$q^{-\frac{1}{36}}\,\phi(q^{\frac{2}{3}}) = \sqrt{\frac{4\pi}{\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\rho(q_1^3) + \sigma(q_1^3)] + \sqrt{\frac{4\alpha}{\pi}}\,J_1(\alpha)\,,$$

$$q^{-\frac{1}{72}}\,\phi(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\phi(-q_1^3) - 2\psi(-q_1^3)] + \sqrt{\frac{2\alpha}{\pi}}\,K_1(\alpha)\,,$$

$$q^{-\frac{1}{/4}}\,\psi(q^{\frac{2}{3}}) = -\sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\rho(q_1^3) - 2\sigma(q_1^3)] + \sqrt{\frac{\alpha}{\pi}}\,J(\alpha)\,,$$

$$q^{-\frac{1}{8}}\,\psi(-q^{\frac{1}{3}}) = -\sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\phi(-q_1^3) + \psi(-q_1^3)] + \sqrt{\frac{2\alpha}{\pi}}\,K(\alpha)\,,$$

$$q^{\frac{1}{6}}\,\rho(q^{\frac{2}{3}}) = \sqrt{\frac{\pi}{2\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\phi(q_1^3) - \psi(q_1^3)] - \sqrt{\frac{2\alpha}{\pi}}\,J(2\alpha)\,,$$

$$q^{\frac{1}{12}}\,\rho(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\rho(-q_1^3) + 2\sigma(-q_1^3)] + \sqrt{\frac{\alpha}{\pi}}\,J(\alpha)\,,$$

$$q^{-\frac{1}{18}}\,\sigma(q^{\frac{2}{3}}) = \sqrt{\frac{\pi}{8\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\phi(q_1^3) + 2\psi(q_1^3)] - \sqrt{\frac{2\alpha}{\pi}}\,J_1(2\alpha)\,,$$

$$q^{-\frac{1}{36}}\,\sigma(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\rho(-q_1^3) - \sigma(-q_1^3)] - \sqrt{\frac{\alpha}{\pi}}\,J_1(\alpha)\,,$$

$$q^{\frac{1}{6}}\,\lambda(q^{\frac{2}{3}}) = \sqrt{\frac{8\pi}{\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\nu(q_1^3) - \xi(q_1^3)] + \sqrt{\frac{\alpha}{8\pi}}\,J(2\alpha)\,,$$

$$q^{\frac{1}{12}}\,\lambda(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\lambda(-q_1^3) + 2\mu(-q_1^3)] - \sqrt{\frac{4\alpha}{\pi}}\,J(\alpha)\,,$$

$$q^{-\frac{1}{18}}\,\mu(q^{\frac{2}{3}}) = \sqrt{\frac{2\pi}{\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\nu(q_1^3) + 2\xi(q_1^3)] + \sqrt{\frac{8\alpha}{\pi}}\,J_1(2\alpha)\,,$$

$$q^{-\frac{1}{36}}\,\mu(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\lambda(-q_1^3) - \mu(-q_1^3)] + \sqrt{\frac{4\alpha}{\pi}}\,J_1(\alpha)\,,$$

$$q^{-\frac{1}{36}}\,\nu(q^{\frac{2}{3}}) = \sqrt{\frac{\pi}{4\alpha}}\,q_1^{-\frac{1}{4}}\,[q_1\lambda(q_1^3) + \mu(q_1^3)] - \sqrt{\frac{\alpha}{\pi}}\,J_1(\alpha)\,,$$

$$q^{-\frac{1}{72}}\,\nu(-q^{\frac{1}{3}}) = \sqrt{\frac{\pi}{\alpha}}\,q_1^{-\frac{9}{8}}\,[q_1\nu(-q_1^3) - 2\xi(-q_1^3)] - \sqrt{\frac{\alpha}{2\pi}}\,K_1(\alpha)\,,$$

$$q^{-\frac{1}{4}}\xi(q^{\frac{2}{3}}) = -\sqrt{\frac{\pi}{16\alpha}}\, q_1^{-\frac{1}{4}}[q_1\lambda(q_1^3) - 2\mu(q_1^3)] - \sqrt{\frac{\alpha}{4\pi}}\, J(\alpha)\,,$$

$$q^{-\frac{1}{8}}\xi(-q^{\frac{1}{3}}) = -\sqrt{\frac{\pi}{\alpha}}\, q_1^{-\frac{9}{8}}[q_1\nu(-q_1^3) + \xi(-q_1^3)] - \sqrt{\frac{\alpha}{2\pi}}\, K(\alpha)\,.$$

Continuing on, we come next to mf$_8$'s [**GM1**]:

$$\left.\begin{aligned} S_0(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(-q^2;q^2)_n}\,, \qquad & S_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q\,;q^2)_n}{(-q^2;q^2)_n}\,, \\ T_0(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q\,;q^2)_{n+1}}\,, \qquad & T_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q\,;q^2)_{n+1}}\,. \end{aligned}\right\} \tag{5.11}$$

They satisfy the linear relations ([**GM1**], [**GM2**, p. 222])

$$S_0(q) + 2T_0(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(-q^2;q^2)_n} = \frac{1}{2}\Big[(-q^{\frac{1}{2}};q)_\infty^3 + (q^{\frac{1}{2}};q)_\infty^3\Big]\theta_4(0,q)\,,$$

$$S_1(q) + 2T_1(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n(n+2)}(-q\,;q^2)_n}{(-q^2;q^2)_n} = \frac{1}{2}q^{-\frac{1}{2}}\Big[(-q^{\frac{1}{2}};q)_\infty^3 - (q^{\frac{1}{2}};q)_\infty^3\Big]\theta_4(0,q)\,,$$

and the mock theta "conjectures" [**GM3**]

$$S_0(-q^2) = \frac{j(-q,q^2)j(q^6,q^{16})}{j(q^2;q^8)} - 2qg_2(q,q^8)\,,$$

$$S_1(-q^2) = \frac{j(-q,q^2)j(q^2,q^{16})}{j(q^2,q^8)} - 2qg_2(q^3,q^8)\,.$$

The transformation laws for (5.11) involve the functions

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(-q^4;q^4)_n} = S_0(q^2) + qS_1(q^2)\,,$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(-q^2;q^4)_{n+1}} = T_0(q^2) + qT_1(q^2)\,,$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(q\,;q^2)_n}\,,$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}}\,.$$

The complete set of laws reads as follows [**GM1**]:

$$q^{-\frac{1}{16}}S_0(q) = \sqrt{\frac{\pi}{4\alpha}}V_0(q_1) + \sqrt{\frac{2\pi}{\alpha}}q_1^{-\frac{1}{4}}V_1(q_1) + \sqrt{\frac{4\alpha}{\pi}}K_3(\alpha)\,,$$

$$q^{\frac{7}{16}}S_1(q) = \sqrt{\frac{\pi}{4\alpha}}V_0(q_1) - \sqrt{\frac{2\pi}{\alpha}}q_1^{-\frac{1}{4}}V_1(q_1) - \sqrt{\frac{4\alpha}{\pi}}K_2(\alpha)\,,$$

$$q^{-\frac{1}{16}}T_0(q) = \sqrt{\frac{\pi}{16\alpha}}V_0(-q_1) - \sqrt{\frac{\pi}{2\alpha}}q_1^{-\frac{1}{4}}V_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}K_3(\alpha)\,,$$

$$q^{\frac{7}{16}}T_1(q) = \sqrt{\frac{\pi}{16\alpha}}V_0(-q_1) + \sqrt{\frac{\pi}{2\alpha}}q_1^{-\frac{1}{4}}V_1(-q_1) + \sqrt{\frac{\alpha}{\pi}}K_2(\alpha)\,,$$

$$q^{-\frac{1}{16}}S_0(-q) = \sqrt{\frac{\pi(2-\sqrt{2})}{\alpha}}q_1^{-\frac{1}{16}}T_0(-q_1) + \sqrt{\frac{\pi(2+\sqrt{2})}{\alpha}}q_1^{\frac{7}{16}}T_1(-q_1) + \sqrt{\frac{4\alpha}{\pi}}J_3(\alpha)\,,$$

$$q^{\frac{7}{16}}S_1(-q) = \sqrt{\frac{\pi(2+\sqrt{2})}{\alpha}}q_1^{-\frac{1}{16}}T_0(-q_1) - \sqrt{\frac{\pi(2-\sqrt{2})}{\alpha}}q_1^{\frac{7}{16}}T_1(-q_1) + \sqrt{\frac{4\alpha}{\pi}}J_2(\alpha)\,,$$

$$q^{-\frac{1}{16}}T_0(-q) = \sqrt{\frac{\pi(2-\sqrt{2})}{16\alpha}}q_1^{-\frac{1}{16}}S_0(-q_1) + \sqrt{\frac{\pi(2+\sqrt{2})}{16\alpha}}q_1^{\frac{7}{16}}S_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}J_3(\alpha)\,,$$

$$q^{\frac{7}{16}}T_1(-q) = \sqrt{\frac{\pi(2+\sqrt{2})}{16\alpha}}q_1^{-\frac{1}{16}}S_0(-q_1) - \sqrt{\frac{\pi(2-\sqrt{2})}{16\alpha}}q_1^{\frac{7}{16}}S_1(-q_1) - \sqrt{\frac{\alpha}{\pi}}J_2(\alpha)\,,$$

$$q^{-\frac{1}{8}}U_0(q) = \sqrt{\frac{\pi}{2\alpha}}V_0(q_1^{\frac{1}{2}}) + \sqrt{\frac{\alpha}{2\pi}}J\Big(\frac{\alpha}{2}\Big)\,,$$

$$q^{-\frac{1}{8}}U_1(q) = \sqrt{\frac{\pi}{8\alpha}}V_0(-q_1^{\frac{1}{2}}) - \sqrt{\frac{\alpha}{8\pi}}J\Big(\frac{\alpha}{2}\Big)\,,$$

$$V_0(q) = \sqrt{\frac{\pi}{\alpha}}q_1^{-\frac{1}{16}}U_0(q_1^{\frac{1}{2}}) - \sqrt{\frac{16\alpha}{\pi}}J(4\alpha)\,,$$

$$q^{-\frac{1}{4}}V_1(q) = \sqrt{\frac{\pi}{8\alpha}}q_1^{-\frac{1}{16}}U_0(-q_1^{\frac{1}{2}}) - \sqrt{\frac{\alpha}{\pi}}K(2\alpha)\,,$$

$$q^{-\frac{1}{8}}U_0(-q) = \sqrt{\frac{4\pi}{\alpha}}q_1^{-\frac{1}{8}}V_1(q_1^{\frac{1}{2}}) + \sqrt{\frac{2\alpha}{\pi}}K(\alpha)\,,$$

$$q^{-\frac{1}{8}}U_1(-q) = -\sqrt{\frac{\pi}{\alpha}}q_1^{-\frac{1}{8}}V_1(-q_1^{\frac{1}{2}}) - \sqrt{\frac{\alpha}{2\pi}}K(\alpha)\,,$$

$$V_0(-q)=\sqrt{\frac{4\pi}{\alpha}}q_1^{-\frac{1}{16}}U_1(q_1^{\frac12})+\sqrt{\frac{16\alpha}{\pi}}J(4\alpha)\,,$$

$$q^{-\frac14}V_1(-q)=-\sqrt{\frac{\pi}{2\alpha}}q_1^{-\frac{1}{16}}U_1(-q_1^{\frac12})-\sqrt{\frac{\alpha}{\pi}}K(2\alpha)\,.$$

Here the Mordell integrals J_2, J_3, K_2, K_3 are

$$J_2(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\cosh\frac12\alpha x}{\cosh 2\alpha x}\,dx\,,\qquad J_3(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\cosh\frac32\alpha x}{\cosh 2\alpha x}\,dx\,,$$

$$K_2(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\sinh\frac12\alpha x}{\sinh 2\alpha x}\,dx\,,\qquad K_3(\alpha)=\int_0^\infty e^{-\alpha x^2}\frac{\sinh\frac32\alpha x}{\sinh 2\alpha x}\,dx\,.$$

In view of the last eight transformation laws, a case can be made for regarding $U_0(q)$, $U_1(q)$, $V_0(q)$, $V_1(q)$ as mf$_2$'s. Indeed, the relevant Mordell integrals are the same as those in (5.3).

In [**M1**] it is shown that $V_1(q)$ is equal to the function

$$\lambda(q)=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q)_{2n}}{(-q^2;q^4)_{n+1}}\,,$$

found on page 8 of the Lost Notebook (see also [**A1**, (3.21)] and [**AB**, (12.5.3)]).

The functions $U_i(q)$ and $V_j(q)$ satisfy the linear relations ([**GM1**] and [**M1**]):

$$\left.\begin{aligned}U_0(q)+2U_1(q)&=(q)_\infty(-q\,;q^2)^4_\infty\,,\\ V_0(q)+V_0(-q)&=2(-q^2;q^4)^4_\infty(q^8;q^8)_\infty\,,\\ V_1(q)-V_1(-q)&=2q(-q^2;q^2)_\infty(-q^4;q^4)^2_\infty(q^8;q^8)_\infty\,,\end{aligned}\right\}\tag{5.12}$$

and are connected to the mf$_2$'s (5.1) by

$$\left.\begin{aligned}U_0(q)-2U_1(q)&=\mu(q)\,,\\ V_0(q)-V_0(-q)&=4qB(q^2)\,,\\ V_1(q)+V_1(-q)&=2A(q^2)\,,\end{aligned}\right\}\tag{5.13}$$

proved in [**M1**]. Combining (5.12) and (5.13), we obtain

$$\left.\begin{aligned}2U_0(q)&=(q)_\infty(-q\,;q^2)^4_\infty+\mu(q)\,,\\ 4U_1(q)&=(q)_\infty(-q\,;q^2)^4_\infty-\mu(q)\,,\\ V_0(q)&=(-q^2;q^4)^4_\infty(q^8;q^8)_\infty+2qB(q^2)\,,\\ V_1(q)&=q(-q^2;q^2)_\infty(-q^4;q^4)^2_\infty(q^8;q^8)_\infty+A(q^2)\,.\end{aligned}\right\}\tag{5.14}$$

Identities (5.14), together with (5.2), yield mock theta "conjectures" for $U_i(q)$ and $V_j(q)$. For example, $V_1(q)=qg_2(q,q^4)$, proved in [**GM1**. pp. 322–324].

Finally, we turn to mf_{10}'s, four of which appear on page 9 of the Lost Notebook:

$$\left.\begin{aligned} \phi(q) &= \sum_{n=0}^{\infty} \frac{q^{\frac12 n(n+1)}}{(q\,;q^2)_{n+1}}\,, \qquad & \psi(q) &= \sum_{n=0}^{\infty} \frac{q^{\frac12 (n+1)(n+2)}}{(q\,;q^2)_{n+1}}\,, \\ X(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q)_{2n}}\,, \qquad & \chi(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}}\,. \end{aligned}\right\} \tag{5.15}$$

Ramanujan listed eight linear relations connecting these functions:

$$\phi(q) - q^{-1}\psi(-q^4) + q^{-2}\chi(q^8) = \cdots\,,$$

$$\psi(q) + q\phi(-q^4) + X(q^8) = \cdots\,.$$

These relations are proved in [**C1**]–[**C4**].

The mock theta "conjectures" of order 10 are:

$$\left.\begin{aligned} \phi(q) &= \frac{(q^{10};q^{10})_\infty^2 j(-q^2,q^5)}{(q^5;q^5)_\infty j(q^2,q^{10})} + 2qg_2(q^2,q^5)\,, \\ \psi(q) &= -\frac{q(q^{10};q^{10})_\infty^2 j(-q,q^5)}{(q^5;q^5)_\infty j(q^4,q^{10})} + 2qg_2(q,q^5)\,, \\ X(-q^2) &= \frac{(q^4;q^4)_\infty^2\big(j(-q^2,q^{20})^2 j(q^{12},q^{40}) + 2q(q^{40};q^{40})_\infty^3\big)}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^8,q^{40})} \\ &\quad - 2qg_2(q,q^{20}) + 2q^5 g_2(q^9,q^{20})\,, \\ \chi(-q^2) &= \frac{q^2(q^4;q^4)_\infty^2\big(2q(q^{40};q^{40})_\infty^3 - j(-q^6,q^{20})^2 j(q^4,q^{40})\big)}{(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^{16},q^{40})} \\ &\quad - 2q^3 g_2(q^3,q^{20}) - 2q^5 g_2(q^7,q^{20})\,. \end{aligned}\right\} \tag{5.16}$$

The first two are proved in [**C**, pp. 533–534] and the last two in [**GM3**].

The transformation laws for (5.15) are:

$$q^{\frac15}\phi(q) = \sqrt{\frac{\pi(5+\sqrt5)}{10\alpha}}\,q_1^{-\frac{1}{20}}X(q_1^2) - \sqrt{\frac{\pi(5-\sqrt5)}{10\alpha}}\,q_1^{-\frac{9}{20}}\chi(q_1^2) - \sqrt{\frac{20\alpha}{\pi}}\,J_4(\alpha)\,,$$

$$q^{\frac15}\phi(-q) = \sqrt{\frac{\pi(5+\sqrt5)}{10\alpha}}\,q_1^{\frac15}\phi(-q_1) + \sqrt{\frac{\pi(5-\sqrt5)}{10\alpha}}\,q_1^{-\frac15}\psi(-q_1) + \sqrt{\frac{20\alpha}{\pi}}\,K_4(\alpha)\,,$$

$$q^{-\frac{1}{5}}\psi(q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\, q_1^{-\frac{1}{20}} X(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\, q_1^{-\frac{9}{20}} \chi(q_1^2) - \sqrt{\frac{20\alpha}{\pi}}\, J_5(\alpha)\,,$$

$$q^{-\frac{1}{5}}\psi(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\, q_1^{\frac{1}{5}} \phi(-q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\, q_1^{-\frac{1}{5}} \psi(-q_1) - \sqrt{\frac{20\alpha}{\pi}}\, K_6(\alpha)\,,$$

$$q^{-\frac{1}{40}} X(q) = \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\, q_1^{\frac{2}{5}} \phi(q_1^2) + \sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\, q_1^{-\frac{2}{5}} \psi(q_1^2) + \sqrt{\frac{10\alpha}{\pi}}\, K_7\Big(\frac{\alpha}{2}\Big)\,,$$

$$q^{-\frac{1}{40}} X(-q) = \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\, q_1^{-\frac{1}{40}} X(-q_1) - \sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\, q_1^{-\frac{9}{40}} \chi(-q_1) + \sqrt{\frac{40\alpha}{\pi}}\, J_6(\alpha)\,,$$

$$q^{-\frac{9}{40}} \chi(q) = -\sqrt{\frac{\pi(5-\sqrt{5})}{5\alpha}}\, q_1^{\frac{2}{5}} \phi(q_1^2) + \sqrt{\frac{\pi(5+\sqrt{5})}{5\alpha}}\, q_1^{-\frac{2}{5}} \psi(q_1^2) + \sqrt{\frac{10\alpha}{\pi}}\, K_5\Big(\frac{\alpha}{2}\Big)\,,$$

$$q^{-\frac{9}{40}} \chi(-q) = -\sqrt{\frac{\pi(5+\sqrt{5})}{10\alpha}}\, q_1^{-\frac{1}{40}} X(-q_1) - \sqrt{\frac{\pi(5-\sqrt{5})}{10\alpha}}\, q_1^{-\frac{9}{40}} \chi(-q_1) + \sqrt{\frac{40\alpha}{\pi}}\, J_7(\alpha)\,,$$

where as usual $q = e^{-\alpha}$ and $q_1 = e^{-\beta}$ with $\alpha\beta = \pi^2$. The Mordell integrals J_n and K_n $(n = 4, 5, 6, 7)$ are:

$$J_4(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\cosh \alpha x}{\cosh 5\alpha x}\, dx\,, \qquad J_5(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\cosh 3\alpha x}{\cosh 5\alpha x}\, dx\,,$$

$$J_6(\alpha) = \int_0^\infty e^{-10\alpha x^2}\, \frac{\cosh 9\alpha x - \cosh \alpha x}{\cosh 10\alpha x}\, dx\,,$$

$$J_7(\alpha) = \int_0^\infty e^{-10\alpha x^2}\, \frac{\cosh 7\alpha x + \cosh 3\alpha x}{\cosh 10\alpha x}\, dx\,,$$

$$K_4(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh \alpha x}{\sinh 5\alpha x}\, dx\,, \qquad K_5(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh 2\alpha x}{\sinh 5\alpha x}\, dx\,,$$

$$K_6(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh 3\alpha x}{\sinh 5\alpha x}\, dx\,, \qquad K_7(\alpha) = \int_0^\infty e^{-5\alpha x^2} \frac{\sinh 4\alpha x}{\sinh 5\alpha x}\, dx\,.$$

6. General relations between mock theta functions

In this section we consider how relations between mf's can be found. One method is to compare the Mordell integrals in their transformation laws. If two functions transform with the same Mordell integral, their difference may well be a θf. If this difference is a θ-product, an explicit formula for it can be found by computer algebra. Sometimes the difference is not itself a θ-product, but the even and odd parts of its Taylor series are. This phenomenon leads to the last two mock theta "conjectures" in (5.16).

Comparison of the Mordell integrals in (4.1) and (4.11) suggests that

$$g_3(q^{4r},q^4) - q^{1-2r}g_2(q^{6r+1},q^6) - q^{2r-1}g_2(q^{6r-1},q^6)$$

is a θf. Computer algebra leads to the specific conjecture that

$$g_3(x^4,q^4) = \frac{qg_2(x^6q,q^6)}{x^2} + \frac{x^2g_2(x^6q^{-1},q^6)}{q} - \frac{x^2(q^2;q^2)^3_\infty(q^{12};q^{12})_\infty j(x^2q,q^2)j(x^{12}q^6,q^{12})}{q(q^4;q^4)_\infty(q^6;q^6)^2_\infty j(x^4,q^2)j(x^6q^{-1},q^2)}\,. \tag{6.1}$$

It was noted in §3 that the mf's of odd order are related to $g_3(x,q)$, and in §5 that the mf's of even order are related to $g_2(x,q)$. By (6.1) and its limiting cases (discussed in §7) we can express all of the classical mf's in terms of $g_2(x,q)$. For this reason we can regard $g_2(x,q)$ as a *universal mock θ-function.*

Identity (6.1) has a broad generalization, which we now develop. Recall the generalized Lambert series

$$u_k(x,q) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac12 kn(n+1)}}{1-xq^n}\,,\qquad v_k(y,q) = \sum_{n=-\infty}^{\infty}\frac{(-1)^{kn}q^{\frac12 n(kn+1)}}{(1-yq^n)(1-y^{-1}q^n)}$$

of (4.5), (4.7). It can be shown algebraically that

$$u_k(x,q) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac12 kn(n+1)}}{(1-xq^n)(1-x^{-1}q^{n+1})}\,;$$

hence

$$u_k(x,q) = u_k(x^{-1}q,q)\,,\qquad v_k(y,q) = v_k(y^{-1},q).$$

From the definition of $u_k(x,q)$ it is easily seen that

$$u_{2k}(x,q) + u_{2k}(-x,q) = 2u_k(x^2,q^2)\,.$$

Somewhat more difficult to prove is the functional equation

$$u_k(xq,q) = -x^k u_k(x,q) - \sum_{m=1}^{k-1} x^m j(q^m,q^k)\,. \tag{6.2}$$

For odd values of k, $v_k(x,q)$ is related to $u_k(x,q)$ by

$$(1-x)v_k(x,q) = -x^{\frac12(k+1)}u_k(x,q) - \sum_{m=1}^{\frac12(k-1)} x^{\frac12(k+1)-m} j(q^m,q^k)\,. \tag{6.3}$$

When $k=3$, (6.3) says that

$$(1-x)v_3(x,q) = -x^2u_3(x,q) - x(q)_\infty\,,$$

which is equivalent to

$$h_3(x,q) = (1-x)(1+xg_3(x,q))\,, \tag{6.4}$$

by (4.8).

Another useful identity, valid for all positive integers k, is

$$\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n q^{\frac{1}{2}kn(n+1)}}{1-q^n} = \sum_{m=1}^{k-1} \frac{1-j(q^m,q^k)}{2}\,, \tag{6.5}$$

where the dash indicates that the term with $n=0$ is to be omitted.

We can now state the general identities of which (6.1) is the special case $k=3$. These identities express $u_k(x,q)$ in terms of $u_2(x,q)$ (and hence in terms of the universal mf $g_2(x,q)$). They are as follows:

$$\begin{aligned} u_k(x^4,q^4) = &-\frac{x^2(q^2;q^2)^3_\infty j(x^{2k-4}q,q^2)j(x^{4k}q^{2k},q^{4k})}{qj(x^4,q^2)j(x^{2k}q^{-1},q^2)j(q^{2k},q^{4k})} \\ &+\sum_{m=1}^{k-1}\frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})}\,u_2(x^{2k}q^{k-2m},q^{2k})\,, \quad k \text{ odd}\,, \end{aligned} \tag{6.6}$$

$$\begin{aligned} u_k(x^4,q^4) = &-\frac{x^4(q^2;q^2)^3_\infty j(x^{2k-4},q^2)j(x^{4k}q^{2k},q^{4k})}{q^2j(x^4,q^2)j(x^{2k}q^{-2},q^2)j(q^{2k},q^{4k})} \\ &+\sum_{m=1}^{k-1}\frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})}\,u_2(x^{2k}q^{k-2m},q^{2k})\,, \quad k \text{ even}\,. \end{aligned} \tag{6.7}$$

These identities are proved by showing that both sides satisfy the same functional equation, and that their difference has only removable singularities for q fixed and $x \neq 0$.

When $k=3$, (6.6) becomes

$$\begin{aligned} u_3(x^4,q^4) = &\frac{qj(q^4,q^{12})}{x^2j(q^6,q^{12})}\,u_2(x^6q,q^6) + \frac{x^2j(q^4,q^{12})}{qj(q^6,q^{12})}\,u_2(x^6q^{-1},q^6) \\ &-\frac{x^2(q^2;q^2)^3_\infty j(x^2q,q^2)j(x^{12}q^6,q^{12})}{qj(x^4,q^2)j(x^6q^{-1},q^2)j(q^6,q^{12})}\,, \end{aligned} \tag{6.8}$$

since $j(q^8,q^{12}) = j(q^4,q^{12})$. It follows from (4.8) and (4.12) that

$$u_2(x^6q^{\pm1},q^6) = \frac{(q^6;q^6)^2_\infty}{(q^{12};q^{12})_\infty}g_2(x^6q^{\pm1},q^6) = j(q^6,q^{12})g_2(x^6q^{\pm1},q^6)$$

and

$$u_3(x^4,q^4)=(q^4;q^4)_\infty g_3(x^4,q^4)=j(q^4,q^{12})g_3(x^4,q^4).$$

Substituting these expressions for u_2 and u_3 into (6.8), we obtain (6.1).

7. Singular cases of the general relations

As already remarked, identities (6.6) and (6.7) hold whenever all the terms are defined. These terms, regarded as functions of x with q fixed, are meromorphic for $x\neq 0$. At their poles, which are all simple, (6.6), (6.7) become identities between Laurent series. By equating the constant terms in the Laurent series of the two sides, we obtain a set of identities which we call *singular cases.*

The left sides of (6.6) and (6.7) are defined when $x\neq 0$ and $x^4q^{4n}\neq 1$ for any $n\in\mathbf{Z}$. For such x, the right sides of (6.6) and (6.7) are undefined when $x=\mu q^{\frac{m_0}{k}-\frac{1}{2}+n}$, where $\mu^{2k}=1$, $1\le m_0\le k-1$ and $n\in\mathbf{Z}$. In this case the product on the right side of either (6.6) or (6.7) and the m'th term of the sum have simple poles of equal residues. The constant terms of their Laurent series can be determined. For the m'th term of the sum this is done using (6.5), and for the product, by logarithmic differentiation. This results in identities of the form

$$u_k(x^4,q^4)=T(x,q)+\sum_{\substack{m=1\\ m\neq m_0}}^{k-1}\frac{q^{k-2m}j(q^{4m},q^{4k})}{x^{2k-4m}j(q^{2k},q^{4k})}\,u_2(x^{2k}q^{k-2m},q^{2k})\,, \tag{7.1}$$

where $x=\mu q^{\frac{m_0}{k}-\frac{1}{2}+n}$, and $T(x,q)$ is a θf.

For example, when $k=3$ and $m_0=2$, identity (7.1) becomes

$$u_3(q^{\frac{2}{3}},q^4)=\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty}+\frac{(q^4;q^4)^2_\infty(q^6;q^6)^4_\infty}{2q^{\frac{2}{3}}(q^2;q^2)^2_\infty(q^{12};q^{12})^2_\infty}-\frac{(q^4;q^4)_\infty}{2q^{\frac{2}{3}}}$$
$$+\frac{q^{\frac{2}{3}}j(q^4,q^{12})}{j(q^6,q^{12})}\,u_2(q^2,q^6)\,,$$

or equivalently,

$$g_3(q,q^6)=\frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty}+\frac{(q^6;q^6)_\infty(q^9;q^9)^4_\infty}{2q(q^3;q^3)^2_\infty(q^{18};q^{18})^2_\infty}-\frac{1}{2q}+qg_2(q^3,q^9)\,. \tag{7.2}$$

Ramanujan's letter includes the identity

$$4h_3(e^{\frac{\pi i}{3}},q)-h_3(-1,q)=\frac{3\theta_4^2(0,q^3)}{(q)_\infty}\,,$$

proved by Watson [**W1**]. After modular transformation, this becomes

$$1+2qg_3(q,q^6)-q^2g_3(q^3,q^6)=\frac{(q^2;q^2)^4_\infty}{(q)^2_\infty(q^6;q^6)_\infty}\,. \tag{7.3}$$

Using (6.1) with q replaced by $q^{\frac{3}{2}}$ and $z = q^{\frac{3}{4}}$, (7.3) can be written as

$$g_3(q,q^6) = -\frac{q(q^{18};q^{18})_\infty^4}{2(q^6;q^6)_\infty(q^9;q^9)_\infty^2} + \frac{(q^2;q^2)_\infty^4}{2q(q)_\infty^2(q^6;q^6)_\infty} - \frac{1}{2q} + qg_2(q^3,q^9)\,. \tag{7.4}$$

Equality of the θfs in (7.2) and (7.4) is not hard to prove. This alternate derivation of (7.2) does not extend to a proof of (7.1) for $k > 3$.

It was noted in §3 that the classical mf's of odd order can be expressed in terms of $g_3(x,q)$ and θf's. In §5, it was found that the functions of even order can be expressed in terms of $g_2(x,q)$ and θf's. By (6.6), (6.7), (7.1), all the classical mf's have such an expression using only $g_2(x,q)$ and θf's. For this reason we can regard $g_2(x,q)$ as a universal mf.

Another family of singular cases is obtained from (6.6) and (6.7) when $x = \mu q^{-n}$, where $\mu^4 = 1$ and $n \in \mathbf{Z}$. Using (6.5) and logarithmic differentiation to calculate the constant terms of their Laurent series, we get identities of the form

$$\sum_{m=1}^{k-1} \frac{q^{k-2m} j(q^{4m},q^{4k})}{x^{2k-4m} j(q^{2k},q^{4k})}\, u_2(x^{2k}q^{k-2m},q^{2k}) = T(x,q)\,, \quad k \text{ odd},$$

$$\sum_{\substack{m=1\\ m\neq k/2}}^{k-1} \frac{q^{k-2m} j(q^{4m},q^{4k})}{x^{2k-4m} j(q^{2k},q^{4k})}\, u_2(x^{2k}q^{k-2m},q^{2k}) = T(x,q)\,, \quad k \text{ even},$$

where $T(x,q)$ is a θf. These identities can also be proved using (6.2). More precisely, repeated application of (6.2) shows that

$$\begin{aligned}&\frac{q^{k-2m}}{x^{2k-4m}}\, u_2(x^{2k}q^{k-2m},q^{2k}) + \frac{q^{k-2(k-m)}}{x^{2k-4(k-m)}}\, u_2(x^{2k}q^{k-2(k-m)},q^{2k})\\ &\quad = \frac{q^{k-2m}}{x^{2k-4m}}\, u_2(x^{2k}q^{k-2m},q^{2k}) + \frac{x^{2k-4m}}{q^{k-2m}}\, u_2(x^{2k}q^{2m-k},q^{2k})\end{aligned}$$

is a θf whenever $x = \mu q^{-n}$.

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