A crash course. . .
Day 2: Modular Forms

Sharon Anne Garthwaite

Bucknell University

March 2008

Recall,

$$\sum p(n)q^n = \prod (1-q^n)^{-1}.$$

$$\sum p(n)q^{24n-1} = \left(q\prod(1-q^{24n})\right)^{-1}$$
$$= \eta^{-1}(24z), \quad q := e^{2\pi i z}.$$

This is an example of a modular form.

Basic Definition

$f(z)$, a meromorphic function on the upper half plane

$$\mathbb{H} = \{z := x + iy \in \mathbb{C} : x, y \in \mathbb{R}, y > 0\},$$

is a modular form of weight k if:

- $f(z+1) = f(z)$.
- $f(-1/z) = z^k f(z)$ for some $k \in \mathbb{Z}$.
- $f(z)$ has a Fourier expansion

$$f(z) := \sum_{n \geq n_0}^{\infty} a_f(n) q^n; \qquad q := e^{2\pi i z}.$$

"Behavior not too bad as $z \to i\infty$"

Transformation Property

$\mathrm{SL}_2(\mathbb{Z}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$

- Example: $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
- S, T generate $\mathrm{SL}_2(\mathbb{Z})$.

Transformation Property

For $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H}$,

$$Vz := \frac{az+b}{cz+d}.$$

Fractional Linear Transformations

- Inflations/Rotations $z \to az$
- Translations $z \to z + b$
- Inversions $z \to -1/z$

"Conformal maps" that map circles and lines to circles and lines.

Transformation Property

For $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H}$,

$$Vz := \frac{az+b}{cz+d}.$$

Examples:

- $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $T : z \mapsto z+1$
  $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $S : z \mapsto 1/z$.
- $ST : z \mapsto -1/(z+1)$, $TS : z \mapsto -1/z + 1 = (z-1)/z$.
  Check: If $z \in \mathbb{H}$, $Vz \in \mathbb{H}$.
- Fundamental domain $\mathcal{F}$

More General Transformation Property

For $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H}$,

$$Vz := \frac{az + b}{cz + d}.$$

$$f\,(Vz) = (cz + d)^k f(z).$$

Definition ("Slash" operator)

$$f|_k V(z) := (cz + d)^{-k} f(Vz).$$

Transformation property: $f|_k V(z) = f(z)$

Cusps

- Want to "complete" $\mathbb{H} = \{z := x + iy \in \mathbb{C} : x, y \in \mathbb{R}, y > 0\}$.
- Add point at infinity : ∞.
- $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \infty = a/c$.
- Cusps: $\infty \cup \mathbb{Q}$
- Same equivalency class in $\mathrm{SL}_2(\mathbb{Z})$.

Simple Complex Analysis

Let $f(z) : \mathbb{C} \to \mathbb{C}$.

- z_0 is a zero of f if $f(z_0) = 0$.
- z_0 is a singularity of f if $f(z_0)$ is undefined.
- $f(z)$ is analytic if it is differentiable with respect to z.
- $f(z)$ is meromorphic if it is analytic except an isloated singularities which are "poles."

Singularities

1. Removable
   Example $\dfrac{z^2-1}{z-1}$.
2. Poles
   Example $\dfrac{1}{\sin(z)} = \dfrac{1}{z} + f(z)$, $f(z)$ analytic. (Taylor series)
3. Essential
   Example $e^{1/z}$.

Meromorphic vs. Holomorphic

On $\mathrm{SL}_2(\mathbb{Z})$, if $f(z)$ has a Fourier expansion

$$f(z) := \sum_{n \geq n_0}^{\infty} a_f(n) q^n; \qquad q := e^{2\pi i z},$$

$f(z)$ is holomorphic if $n_0 \geq 0$. "$f(z)$ does not blow up at the cusps."

Let M_k denote the space of weight k holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$.

Cauchy's Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{z - w} \, dw.$$

where

- D is a bounded domain with a smooth boundary.
- $z \in D$.
- $f(z)$ analytic on D.
- $f(z)$ extends smootly to ∂D.

Applications

For $f(z)$ with Fourier expansion (about infinity)

$$f(z) := \sum_{n \geq n_0}^{\infty} a_f(n) q^n = g(q); \qquad q := e^{2\pi i z},$$

$$a_f(n) = \frac{1}{2\pi i} \int_\gamma \frac{g(q)}{q^{n+1}} \, dq.$$

Residue Theorem

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{a \in D} Res(f)|_{z=a}.$$

Examples of Modular Forms: Eisenstein series

Let

$$G_k(z) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

This sum is absolutely convergent for $k > 2$.

Normalized Eisenstein series

For even $k \geq 4$,

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where the rational (Bernoulli) numbers B_k are

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 + \cdots ,$$

and

$$\sigma_{k-1}(n) = \sum_{1 \leq d | n} d^{k-1}.$$

This is $G_k(z)$ normalized to have constant coefficient 1.

Eisenstein Series

- $E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$
- $E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$
- $E_8(z) := 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$
- $E_{10}(z) := 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n$
- $E_{12}(z) := 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$
- $E_{14}(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n$

If $\ell \geq 5$ is prime then $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$.

Eisenstein Series

What about $E_2(z) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n$?

- $E_2(z+1) = E_2(z)$.
- $E_2(-1/z) = z^2 E_2(z) + \frac{12z}{2\pi i}$

"Quasi-modular form".

Note: For $p \geq 3$ prime, $E_2(z) \equiv E_{p+1}(z) \pmod{p}$.

More Modular Forms

How can we make new modular forms?

- Add and subtract? Only if the same weight.
- Multiplication? Yes

Example: $E_4(z)E_6(z) \in M_{10}$.

Can we get all modular forms from Eisenstein series?

How 'big' is M_k?

Finite dimensional $\mathbb{C}$ vector space.
For $k \geq 4$ even, dimension of M_k is

$$d(k) := \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$

If k odd, $d(k) = 0$.

Dimension formula consequence of the Valence formula.

Valence Formula

If $f(z) \in M_k$,

$$\frac{k}{12} = v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\omega(f) + \sum_{\substack{\tau \in \mathcal{F} \\ \tau \notin \{i,\omega\}}} v_\tau(f).$$

- $\omega := e^{2\pi i/3}$.
- $\mathcal{F}$ fundamental domain of $\mathbb{H}$.
- $v_\tau(f)$ the order of vanishing of f at τ

The Valence formula can be computed with complex integration.

$$d(k) := \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases}$$

- For which weights is M_k one dimensional?
- What is the first k with $d(k) = 2$?

Delta Function

$E_4(z)^3, E_6(z)^2 \in M_{12}$. Check: $E_4(z)^3 \neq cE_6(z)^2$ for any $c \in \mathbb{C}$.

Define

$$\begin{aligned}\Delta(z) :&= \frac{E_4^2(z) - E_6^3(z)}{1728} \\ &= q\prod_{n=1}^{\infty}(1-q^n)^{24} \\ &= q - 24q^2 + 252q^3 - \cdots \in M_{12} \cap \mathbb{Z}[[q]].\end{aligned}$$

- $\Delta(z) := \eta^{24}(z)$ where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$.
- First example of a cusp form.

Cusp Forms

If $f(z) \in M_k$ has Fourier expansion:

$$f(z) := \sum_{n \geq n_0}^{\infty} a_f(n) q^n,$$

$f(z)$ is a cusp form if $a_f(0) = 0$.

Let S_k denote the space of cusp forms of weight k.

Dimension formula

- Can use the valence formula to prove $d(k) = 1$ for $k = 4, 6, 8, 10, 14$.
- Can check by valence formula that only zero of $\Delta(z)$ is at ∞.
- Can set up a correspondence between M_{k-12} and S_k using multiplication/division by $\Delta(z)$.
  - $S_k = \Delta M_{k-12}, \qquad k \geq 14.$
  - $M_k = \mathbb{C}E_k \oplus S_k, \qquad k \geq 4.$
- It is true that M_k is generated by $\{E_4^i(z)E_6^j(z) : 4i + 6j = k\}$.

Ramanujan's τ function

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$
$$= q - 24q^2 + 252q^4 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \cdots$$

Define $\tau(n)$ by

$$q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum \tau(n) q^n.$$

Questions?

- For which n is $\tau(n) = 0$?
- How big are the values of $\tau(n)$?

Ramanujan's τ function

- For which n is $\tau(n) = 0$?
  Lehmer's Conjecture (1947): $\tau(n) \neq 0$ for any $n \geq 1$.
- How big are the values of $\tau(n)$?
  Deligne (1974) $|\tau(p)| \leq 2p^{11/2}$ if p is prime.

How quickly does $p(n)$ grow?

Hardy-Ramanujan (1918):

$$p(n) = \sum_{k<\alpha\sqrt{n}} P_k(n) + O(n^{-1/4})$$

α a constant, $P_k(n)$ an exponential function.

- Infinite sum diverges for all n (Lehmer, 1937)
- Gives

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$

- Get exact value if n large enough to make error less than $1/2$.
- Marks the start of the circle method.

Circle Method Idea

1. Let $P(q) := \prod(1-q^n)^{-1}$ be generating function for $p(n)$. For each $n \geq 0$,

$$\frac{P(q)}{q^{n+1}} = \sum_{k=0}^{\infty} \frac{p(k)q^k}{q^{n+1}} \quad 0 < |q| < 1$$

2. Cauchy's residue theorem:

$$p(n) = \frac{1}{2\pi i} \int_C \frac{P(q)}{q^{n+1}} dq$$

   C closed counter-clockwise contour around origin inside unit disk.

3. Want to use information about singularities of $P(q)$. These occur at each root of unity ($x^k = 1$)

Circle Method Idea

1. Choose circular contour C with radius close to 1.
2. Divide C into disjoint arcs $C_{h,k}$ about roots of unity $e^{2\pi ih/k}$ for $1 \leq h < k \leq N$, $(h,k) = 1$.
3. On each arc $C_{h,k}$ replace $P(x)$ that has the same behavior near $e^{2\pi ih/k}$. (This is where modular transformation properties of $P(x)$ comes into play.)
4. Evaluate these integrals along each arc. Make distinction between major/minor arcs for main terms and error.

Rademacher, 1937: Change the contour C:

Convergent formula (exact formula) for $p(n)$:

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6k} \right),$$

- $I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz} \left(\frac{\sinh z}{z} \right)$
- $A_k(n)$ is a "Kloosterman-type" sum.

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{x} \right) \cdot e\left(\frac{x}{12k} \right),$$

where $e(x) := e^{2\pi ix}$.

Other types of Modular Forms

Recall that

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

$$\eta^{24}(z) = q \prod (1 - q^n)^{24} = \Delta(z) \in S_{12}.$$

What about $\eta(24z) = q \prod (1 - q^{24n})$?

By Euler's identity,

$$\begin{aligned}\eta(24z) &= q\prod(1-q^{24n})\\ &= \sum_{k\in\mathbb{Z}}(-1)^k q^{(6k+1)^2}\end{aligned}$$

Can write this as

$$\sum_{n\in\mathbb{Z}}\chi_{12}(n)q^{n^2}$$

Where

$$\chi_{12}(n) := \begin{cases}1 & n\equiv 1,11 \pmod{12}\\ -1 & n\equiv 5,7 \pmod{12}\\ 0 & \text{else.}\end{cases}$$

$\chi_{12}(n)$ is a called a Dirichlet character.

Dirichlet characters

Definition

A function $\psi : \mathbb{Z} \to \mathbb{C}$ is a Dirichlet character modulo m if

- $\psi(n) = \psi(n + m)$.
- $\psi(1) = 1$.
- $\psi(n_1 n_2) = \psi(n_1)\psi(n_2)$.
- $\psi(n) = 0$ if $\gcd(n, m) = 1$.

Transformation Property

$N(z) := \eta(24z)$ is not a modular form on $SL_2(\mathbb{Z})$.

However, if $576 | c$ then

$$N\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z\right) = \chi_{12}(d)\left(\frac{c}{d}\right)\epsilon_d (cz+d)^{1/2} N(z).$$

Here:

- $\left(\frac{c}{d}\right)$ is an extended Legendre symbol
- $$\epsilon_d := \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv -1 \pmod 4. \end{cases}$$

General Definitions

Definition (Modular Form of weight $k \in \mathbb{Z}$ on $\Gamma_0(N)$)

- For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, $z \in \mathbb{H} := \{z : \mathrm{Im}(z) > 0\}$,

$$g(Vz) = \chi(d)(cz+d)^k g(z).$$

- For all $V \in \mathrm{SL}_2(\mathbb{Z})$,

$$(cz+d)^{-k} g(Vz) = \sum_{n \geq n_V} a_{g,V}(n) q^{\frac{n}{N}}$$

where $q := e^{2\pi i z}$.

- $\chi(d)$ is a Dirichlet character modulo N called the Nebentypus
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) : N \mid c\}$.

Notation: $M_k(\Gamma_0(N), \chi)$. $SL_2(\mathbb{Z}) = \Gamma(1)$.

General Definitions

Definition (Modular Form of weight $k \in \mathbb{Z} + 1/2$ on $\Gamma_0(4N)$)

- For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, $z \in \mathbb{H} := \{z : \mathrm{Im}(z) > 0\}$,

$$g\left(Vz\right) = \chi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k g(z).$$

- For all $V \in \mathrm{SL}_2(\mathbb{Z})$,

$$(cz+d)^{-k} g(Vz) = \sum_{n \geq n_V} a_{g,V}(n) q^{\frac{n}{N}}$$

where $q := e^{2\pi i z}$.

Notation: $M_k(\Gamma_0(4N), \chi)$. Example: $\eta(24z) \in M_{1/2}(\Gamma_0(576), \chi_{12})$.

A note about cuspforms

For $f(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansion

$$f(z) = \sum_{n \geq 1} a_f(n) q^n, \quad q = e^{2\pi iz}$$

we said $f(z)$ was a cuspform if $a_f(0) = 0$.
This says that $f(z)$ vanishes at the cusps $\infty, \mathbb{Q}$.

For the general case we have to be more careful:
The cusps form more than one equivalence class.

Problems

1. Prove $G_k(z)$ satisfies the modular transformation property for $k \geq 3$.
2. Use the transformation property to prove that there are no non-zero modular forms of odd weight on $\mathrm{SL}_2(\mathbb{Z})$. Verify that $G_{2k+1}(z) = 0$ for $k \geq 1$.
3. Prove that $E_6(i) = 0$.
4. For which weights k will $E_k(\omega) = 0$ where $\omega = e^{2\pi i/3}$?
5. Prove that for each $n \geq 1$

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i).$$

6. Prove $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ using $E_6^2(z)$, $E_{12}(z)$ and $\Delta(z)$.
7. Compute the first 1000 coefficients of $\tau(n)$ and make a conjecture about what you observe.