A crash course in Partitions, Modular Forms, Elliptic Curves, and Mock Thetas – oh my!

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March 2008

Let's start simple...

$$4 = 4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

A partition of an integer n is an expression of n as a sum of positive integers where order does not matter.

Let $p(n)$ denote the number of partitions of n.

Example. $p(4) = 5$

The size of $p(n)$

Let's look at a few values:

- $p(0) = 1$
- $p(1) = 1$
- $p(2) = 2$
- $p(3) = 3$
- $p(4) = 5$
- $p(5) = 7$
- $p(10) = 42$
- $p(100) = 190{,}569{,}292$
- $p(1{,}000) = 15{,}658{,}181{,}104{,}580{,}771{,}094{,}597{,}751{,}280{,}645$

First 51 values of $p(n)$

$p(0), p(1), p(2), p(3), p(4) \ldots$

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 25015, 31185, 37388, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226 ...

First 51 values of $p(n)$

Divisible by 2

$p(0)$, $p(1)$, $p(2)$, $p(3)$, $p(4)$...

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 25015, 31185, 37388, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226 ...

First 51 values of $p(n)$

Divisible by 3

$p(0)$, $p(1)$, $p(2)$, $p(3)$, $p(4)$...

1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 25015, 31185, 37388, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226 ...

Out of the first 10,000 how often is $p(n)$...

- Divisible by 2?
  4996 times ($\sim$ 50%)
- Divisible by 3?
  3313 times ($\sim$ 33%)
- Divisible by 5?
  3615 times ($\sim$ 36%)

First few values of $p(n)$

Divisible by 5

$p(0)$, $p(1)$, $p(2)$, $p(3)$, $p(4)$ . . .

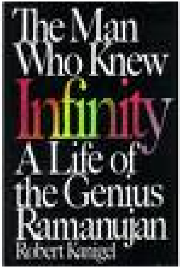
1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, 2436, 3010, 3718, 4565, 5604, 6842, 8349, 10143, 12310, 14883, 17977, 21637, 25015, 31185, 37388, 44583, 53174, 63261, 75175, 89134, 105558, 124754, 147273, 173525, 204226 . . .

First few values of $p(n)$

p(0)	p(1)	p(2)	p(3)	p(4)
1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310
14883	17977	21637	25015	31185
37388	44583	53174	63261	75175
89134	105558	124754	147273	173525

Ramanujan observed

$p(5n+4)$ is divisible by 5



Ramanujan's Congruences

- $p(5n + 4)$ is divisible by 5
- $p(7n + 5)$ is divisible by 7
- $p(11n + 6)$ is divisible by 11

Ramanujan's Congruences

- $p(5n+4) \equiv 0 \pmod{5}$
- $p(7n+5) \equiv 0 \pmod{7}$
- $p(11n+6) \equiv 0 \pmod{11}$

We say $a \equiv b \pmod{m}$ if a and b differ by a multiple of m.

Example. $-7 \equiv -2 \equiv 3 \equiv 8 \equiv \cdots \pmod{5}$

The **big** question

It is really easy to compute partitions and find patterns,

but how do we prove our observations?

One method we can use is Combinatorics:

the study of counting objects that meet certain criteria.

Partitions of 4

Dyson's rank: Largest part - Number of Parts

4	• • • •	4- 1 = 3	2	• •	2 - 3 = -1
			+1	•	
3	• • •	3- 2 = 1	+1	•	
+1	•				
2	• •	2- 2 = 0	1	•	1- 4 = -3
+2	• •		+1	•	
			+1	•	
			+1	•	

We have every possible rank modulo 5 an equal number of times!

$(0 \equiv 0 \pmod 5); \quad 1 \equiv 1 \pmod 5); \quad -3 \equiv 2 \pmod 5); \quad 3 \equiv 3 \pmod 5); \quad -1 \equiv 4 \pmod 5))$

Dyson's Conjecture

- In 1944, Physicist Freeman Dyson conjectured that this rank phenomenon divides the partitions of $5n+4$ into 5 equal classes for all n.
- Example. $p(9) = 30$
  There are 6 partitions of 30 with rank 1 (mod 5), 6 with rank 2 (mod 5), and so on. . . .
- This would explain combinatorially why 5 divides $p(5n+4)$.
- Atkin and Swinnerton-Dyer proved Dyson's conjecture in 1954.
- Works only for 5 and 7.
- The more complicated Andrews-Garvan crank works for 5, 7, and 11.

How else can we study partitions?

Ramanujan did not prove his congruences combinatorially.

How did he do it?

We'll have to look at another way to study partitions, with q-series:

a method for generating the values of $p(n)$ using power-series.

Important Power Series:

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$= \frac{1}{1-x}$$

From multiplication to addition...

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{1-q^1} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdots$$

$$= \sum_{n=0}^{\infty} q^{1n} \cdot \sum_{n=0}^{\infty} q^{2n} \cdot \sum_{n=0}^{\infty} q^{3n} \ldots$$

$$= \left(1 + q^1 + q^1 \cdot q^1 + q^1 \cdot q^1 \cdot q^1 + \cdots\right) \left(1 + q^2 + q^2 \cdot q^2 + \cdots\right)$$
$$\times \left(1 + q^3 + q^3 \cdot q^3 + q^3 \cdot q^3 \cdot q^3 + \cdots\right) \cdots$$

From multiplication to addition...

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{1-q^1} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdots$$

$$= \sum_{n=0}^{\infty} q^{1n} \cdot \sum_{n=0}^{\infty} q^{2n} \cdot \sum_{n=0}^{\infty} q^{3n} \ldots$$

$$= \left(1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots\right)\left(1 + q^2 + q^{2+2} + \cdots\right) \times \left(1 + q^3 + q^{3+3} + q^{3+3+3} + \cdots\right)\cdots$$

$$= 1 + q^1 + q^2 + q^{1+1} + q^3 + q^{2+1} + q^{1+1+1} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} p(n) q^n$$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1q^0 + 1q^1 + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots$$

Ramanujan Congruences Revisited

Ramanujan found,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6} = 5 + 30q + 135q^2 \ldots,$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8} = 7 + 77q + \ldots$$

This gives us another way to prove that

5 divides $p(5n+4)$ and 7 divides $p(7n+5)$.

Can we find more congruences for $p(n)$?

- Powers of $5, 7, 11$
- Atkin: a few examples for $5, 7, 13, 17, 19, 23, 29, 31$

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13}$$

- Conjecture. (Erdös) If ℓ is prime then there is an n such that

$$p(n) \equiv 0 \pmod{\ell}.$$

Congruences Everywhere!

In 2000, Ken Ono proved:

For any prime $\ell \geq 5$, there exist infinitely many congruences of the form

$$p(An + B) \equiv 0 \pmod{\ell}$$

Examples:

$$p(17303 \cdot n + 237) \equiv 0 \pmod{13}$$
$$p(48037937 \cdot n + 1122838) \equiv 0 \pmod{17}$$
$$p(1977147619 \cdot n + 815655) \equiv 0 \pmod{19}$$
$$p(14375 \cdot n + 3474) \equiv 0 \pmod{23}$$
$$p(348104768909 \cdot n + 43819835) \equiv 0 \pmod{29}$$
$$p(4063467631 \cdot n + 30064597) \equiv 0 \pmod{31}$$

Ramanujan's Congruences

In 2003 Scott Ahlgren and Matt Boylan proved:

For primes $\ell \geq 5$ the only congruences $p(An + B) \equiv 0 \pmod{\ell}$ of the form

$$p(\ell n + B) \equiv 0 \pmod{\ell}$$

are for $\ell = 5, 7, 11$.

(This was conjectured by Ramanujan.)

Another type of congruence

- $b_4 := 240$; $b_6 := -504$; $b_8 := 480$; $b_{10} := -264$; $b_{14} := -24$,
- $\sigma_k(n) := \sum_{1 \leq d | n} d^k$,
- $\delta_\ell := \frac{\ell^2 - 1}{24}$.
- $\omega(k) := \frac{1}{2}k(3k+1)$, $k \in \mathbb{Z}$

For each prime $\ell \geq 5$ and each $m \in \{4, 6, 8, 10, 14\}$ we have,

$$p(\ell^2 n - \delta_\ell) \equiv \sum_{k \in \mathbb{Z} - \{0\}} (-1)^{k+1} p(\ell^2(n - \omega(k)) - \delta_\ell)$$

$$+ b_m \sum_{\substack{j \geq 1 \\ k \in \mathbb{Z}}} \sigma_{m-1}(j)(-1)^{k+1} p(\ell^2(n - \omega(k) - j/\ell) - \delta_\ell) \pmod{\ell}.$$

Example $m = 4, \ell = 5$, $p(5^2 n + 24) \equiv 0 \pmod 5$

A little more complex$\cdots$

Modular Forms:

Special power series in $q = e^{2\pi i z}$.

Modular Forms

Example

- $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = e^{2\pi i z/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$
- $\eta\left(z + 1\right) = e^{2\pi i/24} \cdot \eta(z)$
- $\eta\left(\frac{-1}{z}\right) = \sqrt{-iz} \cdot \eta(z)$

Example

- $E_4(z) := 1 - 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad \sigma_k(n) = \sum_{1 \leq d | n} d^k$
- $E_4\left(z + 1\right) = E_4(z)$
- $E_4\left(\frac{-1}{z}\right) = z^4 \cdot E_4(z)$

Modular Forms

- $\mathrm{SL}_2(\mathbb{Z}) := \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \}$
- $\mathbb{H} := \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$
- For $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H}$,

$$Vz := \frac{az + b}{cz + d}.$$

- Cusps

Modular Forms

Definition (Modular Form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$)

- For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H} := \{z : \mathrm{Im}(z) > 0\}$,

  Transformation Property

- And

  Analytic Property

Modular Forms

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- For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H} := \{z : \mathrm{Im}(z) > 0\}$,

$$g\left(Vz\right) = \epsilon(a, b, c, d)(cz + d)^k g(z).$$

- And

Analytic Property

Modular Forms

Definition (Modular Form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$)

- For all $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $z \in \mathbb{H} := \{z : \mathrm{Im}(z) > 0\}$,

$$g\left(Vz\right) = \epsilon(a, b, c, d)(cz + d)^k g(z).$$

- Fourier expansion:

$$g(z) = \sum_{n \geq n_0} a_g(n) q^n,$$

where $q := e^{2\pi iz}$.

The size of $p(n)$: An exact formula

Hardy-Ramanujan-Rademacher formula (1917,1922):

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

- $I_s(x)$ is an I-Bessel function.
- $A_k(n)$ is a "Kloosterman-type" sum.

$$A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{x}\right) \cdot e\left(\frac{x}{12k}\right),$$

where $e(x) := e^{2\pi i x}$.

More q-series

Dyson's Rank: Largest Part - Number of Parts

$$\sum_{n\in\mathbb{Z}}(\text{partitions with even rank } + \text{ partitions with odd rank})q^n$$
$$= \sum_{n\in\mathbb{Z}} p(n)q^n$$
$$= \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}.$$

$$\sum_{n\in\mathbb{Z}}(\text{partitions with even rank } - \text{ partitions with odd rank})q^n$$
$$= \sum_{n\geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

Ramanujan, 1920:

$$
\begin{aligned}
f(q) &:= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} \\
&= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \cdots ; \\
\omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2 \cdots (1-q^{2n+1})^2} \\
&= 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 10q^6 + \cdots .
\end{aligned}
$$

Here $q := e^{2\pi i z}$.

Very elusive functions

- Not modular!
- Satisfy very complicated transformation properties.
- Recent break through in understanding these functions.

SUMmary

- Partitions are seemingly simple objects with not so simple properties.
- It is easy to collect data and make observations, but how do we prove our observations?
- The variety of methods we have for studying partitions makes this a subject accessible at a variety of levels.
- These simple observations can lead to the development of deep theory.

Plan for the Workshop

- Rest of Today: Partitions
- Tomorrow: Partitions, Modular Forms
- Day 3: Elliptic Curves
- Day 4: Tying it all together.
  - Modular Forms
  - Elliptic Curves + Modular Forms
  - Mock Thetas + Modular Forms