A crash course...
Day 1: Partitions

Sharon Anne Garthwaite

Bucknell University

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Partitions

Recall, A partition of an integer n is an expression of n as a sum of positive integers where order does not matter.

Let $p(n)$ denote the number of partitions of n.

1. Find the number of partitions of $n = 1, 2, 3, 4, 5, 6, 7$ with only odd parts.
2. Find the number of partitions of $n = 1, 2, 3, 4, 5, 6, 7$ into distinct parts (non-repeating parts).

Generating Functions

Recall, the generating function for partitions is:

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} p(n)q^n.$$

- How can we modify this to generate partitions with odd parts?

Generating Functions

Let $p_{\mathcal{O}}(n)$ denote the number of partitions of n into odd parts.

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} &= \frac{1}{1-q^1} \cdot \frac{1}{1-q^3} \cdot \frac{1}{1-q^5} \cdots \\
&= \left(1 + q^1 + q^1 \cdot q^1 + q^1 \cdot q^1 \cdot q^1 + \cdots\right)\left(1 + q^3 + q^3 \cdot q^3 + \cdots\right) \\
&= 1 + q^1 + q^{1+1} + q^{1+1+1} + q^3 + q^{1+1+1+1} + q^{1+3} + \cdots \\
&= 1 + \sum_{n \geq 1} p_{\mathcal{O}}(n) q^n.
\end{aligned}$$

Generating Functions

Recall, the generating function for partitions is:

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- How can we modify this to generate partitions with distinct parts?

Generating Functions

Let $p_{\mathcal{D}}(n)$ denote the number of partitions of n into distinct parts.

$$\begin{aligned}
\prod_{n\geq 1}(1+q^n) &= (1+q^1)(1+q^2)(1+q^3)(1+q^4)(1+q^5)\cdots \\
&= 1+q^1+q^1q^2+q^1q^2+q^3+q^1q^3+q^4+q^1q^4+q^2q^3+q^5\cdots \\
&= 1+q+q^2+2q^3+2q^4+3q^5+\cdots \\
&= 1+\sum_{n\geq 1} p_{\mathcal{D}}(n)q^n.
\end{aligned}$$

Generating Functions

Compare:

$$1+\sum_{n\geq 1} p_{\mathcal{O}}(n)q^n = \prod_{n\geq 1}\frac{1}{1-q^{2n+1}}$$

$$1+\sum_{n\geq 1} p_{\mathcal{D}}(n)q^n = \prod_{n\geq 1}(1+q^n).$$

Notation

- $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{j=1}^{n}(1-aq^{j-1}).$
- $(a;q)_\infty = (1-a)(1-aq)\cdots = \prod_{j=1}^{\infty}(1-aq^{j-1}).$

Examples:

- $\sum_{n\geq 0} p(n)q^n = (q;q)_\infty^{-1}.$
- $\sum_{n\geq 0} p_{\mathcal{O}}(n)q^n = (q;q^2)_\infty^{-1}.$
- $\sum_{n\geq 0} p_{\mathcal{D}}(n)q^n = (-q;q)_\infty.$

Jacobi Triple Product

For $z \neq 0$ and $|q| < 1$,

$$\prod_{n\geq 0}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}) = \sum_{n\in\mathbb{Z}} z^n q^{n^2}.$$

Example: $q \mapsto q^{3/2}$, $z \mapsto -q^{1/2}$.

$$\prod_{n\geq 1}(1-q^n) = \prod_{n\geq 0}(1-q^{3n+3})(1-q^{3n+2})(1-q^{3n+1}) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/2}.$$

Finding $p(n)$ quickly

$$\left(1+\sum_{n=1}^{\infty} p(n)q^n\right)\prod_{n=1}^{\infty}(1-q^n) = 1$$

Euler's recursive formula:

$$p(n) = \sum_{k\in\mathbb{Z}-\{0\}} (-1)^{k+1} p(n-\omega(k)).$$

where

$$\omega(k) := \frac{1}{2}k(3k+1) = 1, 2, 5, 7, 12, 15, 22, 26, \cdots .$$

Example. $p(20) = p(19) + p(18) - p(15) - p(13) + p(8) + p(5)$

Ferrer's Graph

Take a partition. For each part, create a row of dots.
Example: 5 Partitions of 4.

$$
\begin{array}{rl}
4 & \bullet\;\bullet\;\bullet\;\bullet \\
\\
3 & \bullet\;\bullet\;\bullet \\
+1 & \bullet \\
\\
2 & \bullet\;\bullet \\
+2 & \bullet\;\bullet
\end{array}
\qquad
\begin{array}{rl}
2 & \bullet\;\bullet \\
+1 & \bullet \\
+1 & \bullet \\
\\
1 & \bullet \\
+1 & \bullet \\
+1 & \bullet \\
+1 & \bullet
\end{array}
$$

Conjugate partitions

Example. Partitions of 5 with at most 3 parts.

Read down the columns of the Ferrer's graph.

•	•	•	•	•	5	1+1+ 1+1+ 1
•	•	•	•		4	2+1+ 1+1
•					+1	
•	•	•			3	2+2+ 1
•	•				+2	
•	•	•			3	3+1+ 1
•					+1	
•					+1	
•	•				2	2+2+ 1
•	•				+2	
•					+1	

Conjugate partitions

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Read down the columns of the Ferrer's graph.

•	•	•	•	•	5	1+1+ 1+1+ 1
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•					+1	
•	•	•			3	2+2+ 1
•	•				+2	
•	•	•			3	3+1+ 1
•					+1	
•					+1	
•	•				2	2+2+ 1
•	•				+2	
•					+1	

Generating Function

$p_k(n) :=$ number of partitions of n into at most k parts.
$p_{\leq k}(n) :=$ number of partitions of n into parts at most k.

$$\begin{aligned}
\sum_{n\geq 0} p_k(n)q^n &= \sum_{n\geq 0} p_{\leq k}(n)q^n \\
&= \prod \frac{1}{(1-q^1)\cdots(1-q^k)} \\
&= (q,q)_k^{-1}.
\end{aligned}$$

Durfee's square

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} p(n) q^n = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(q;q)_n^2}.$$

Durfee's square

Can add in other parameters:

Let $p(m,n)$ denote the number of partitions of n with m parts.

$$1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}p(m,n)z^{m}q^{n}.$$

$$=1+\sum_{n\geq 1}\frac{z^{n}q^{n^2}}{(zq;q)_n(q;q)_n}.$$

The Ramanujan Congruences

- $p(5n+4) \equiv 0 \pmod{5}$
- $p(7n+5) \equiv 0 \pmod{7}$
- $p(11n+6) \equiv 0 \pmod{11}$

Recall,

- Dyson's rank gives a combinatorial proof for $5, 7$.
  Rank:=Largest part - Number of Parts.
- The Andrews-Garvan crank gives a combinatorial proof for $5, 7, 11$.
- Karl Mahlburg's work shows the crank also explains Ono's (complicated) congruences for all primes at least 5.

Proof of Ramanujan congruence modulo 5

For $|q| < 1$:

- $\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n\in\mathbb{Z}}(-1)^n(2n+1)q^{n(n+1)/2}$,
- $\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/2}$.

- Let $\sum a(n)q^n = \sum p(n)q^n \prod(1-q^{5n}) \pmod 5$.
- Prove by induction that

$$p(5n+4) \equiv 0 \pmod 5 \Leftrightarrow a(5n+4) \equiv 0 \pmod 5.$$

- Express $\prod(1-q^n)^4$ as a product of sums.
- Confirm coefficients vanish modulo 5 for $n \equiv 4 \pmod 5$.

Problems:

1. Prove that the number of partitions of n where only the odd parts may repeat is equal to the the number of partitions of n where each part can appear at most three times.
2. Prove that the number of partitions of n into distinct odd parts is equal to the number of partitions of n that are self-conjugate.
3. For any positive integer k, prove that the number of partitions of n into parts that repeat at most $k-1$ times is equal to the number of partitions of n into parts that are not divisible by k.
4. Prove
$$\prod_{n\geq 1}(1+(z)q^n) = 1 + \sum_{n\geq 1} \frac{(z^n)q^{n(n+1)/2}}{(1-q^1)\cdots(1-q^n)}.$$
5. Fill in the details of the sketch of the q-series proof of Ramanujan's congruence modulo 5.
6. Adapt the proof of Ramanujan's congruence modulo 5 to modulo 7.