A crash course. . .
Day 4: The Mock Theta Functions

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So far we've discussed...

- Partitions
  Generating functions, q-series identities, Dyson's rank
- Modular Forms
  Eisenstein series, Dimension Formula, τ function
- Elliptic Curves
  Congruence Number Problem, Mordell group, Rank
- Today: Tying it all together
  Looking for parallel results with another set of interesting functions

Elliptic Curves

Exercise:
Let $\sum b(n)q^n = \eta^2(4z)\eta^2(8z)$.
Let E be the elliptic curve $y^2 = x^3 - x$.
Find $a(p)$ for many primes p.

1. Do you notice a pattern?
2. Compare to $b(p)$. Do you notice a pattern?

Recall:

$$a(p) = p + 1 - \#E(\mathbb{F}_p)$$

where the latter means points for E modulo p. (Point at infinity.)

$E : y^2 = x^3 - x$ and $\sum b(n)q^n = \eta^2(4z)\eta^2(8z)$

Shimura-Taniyama Conjecture (Proved by Wiles, Taylor, Breuil, Conrad, Diamond) Let E be an elliptic curve. Let

$$\sum_{n\geq 1} \frac{a_E(n)}{n^s} = L(E,s) := \prod_{p \text{ good}} \frac{1}{1 - a(p)p^{-s} + p^{1-2s}} \prod_{p \text{ bad}} \frac{1}{1 - a(p)p^{-s}}.$$

This is a modular form of weight 2 for a certain group $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$.

- The N in the level depends on the conductor (bad primes).
- The form is actually a cusp form, moreover, a Hecke eigenform, moreover a newform.

Hecke Operators

Suppose $f(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.
For each prime p,

$$f(z)|T(p) := \sum_{n \geq 0} \left(a(pn) + p^{k-1} a(n/p) \right) q^n.$$

For $m \geq 1$,

$$f(z)|T(m) := \sum_{n \geq 0} \left(\sum_{d | \mathrm{gcd(m,n)}} d^{k-1} a(mn/d) \right) q^n.$$

- $a(n/p) = 0$ if $p \nmid n$.
- $T(mn) = T(m)T(n)$ if $(m, n) = 1$.
- Can relate these to sums over lattices.

$$f(z)|T(p) \in M_k(\mathrm{SL}_2(\mathbb{Z})).$$

The Hecke operators also take cuspforms to cuspforms.

Example

Recall,

$$\sum_{n=1}^{\infty} \tau(n) q^n = \Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$
$$= q - 24q^2 + 252q^4 - 1472q^4 + 4830q^5 - 6048q^6 - \cdots .$$

$\Delta(z) \in S_{12}$, a one dimensional space, so

$$\Delta(z)|T(p) = c(p)\Delta(z)$$

for some $c(p) \in \mathbb{C}$.

$$\Delta(z) = \sum \tau(n) q^n = q - 24q^2 + \cdots$$
$$\Delta(z)|T(p) = \sum_{n \geq 0} \left(\tau(pn) + p^{k-1} \tau(n/p) \right) q^n = \tau(p) q + \cdots$$

So $c(p) = \tau(p)$. This implies $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$.

Hecke Eigenforms

$\Delta(z)$ is an example of a Hecke Eigenform.

$f(z) \in M_k$ is a Hecke Eigenform if, for each $m \geq 2$,

$$f(z)|T(m) = c(m)f(z)$$

for some $c(m) \in \mathbb{C}$.

In general, if $f(z)$ is

- a newform
- a cuspform
- normalized so that $a(1) = 1$

then the coefficients of the Fourier expansion of $f(z)$ are multiplicative.

Eigenforms more generally

- Can define Hecke operators for $M_k(\Gamma_0(N), \chi)$ with $k \in \frac{1}{2}\mathbb{Z}$.
- These preserve the space, take cuspforms to cuspforms, and are multiplicative
- Can similarly define Hecke Eigenforms

(Shimura Correspondence)

More operators

For $f(z) = \sum a(n)q^n \in M_k(\Gamma_0(N), \chi)$, $d \geq 0$

- $f(z)|U(d) = \sum a(dn)q^n \in M_k(\Gamma_0(dN), \chi)$ (or N if $d \mid N$)
- $f(z)|V(d) = \sum a(n)q^{dn} \in M_k(\Gamma_0(dN), \chi)$

Note that $f(z)|U(p) \equiv f(z)|T(p) \pmod{p}$.

Another type of congruence

- $b_4 := 240$; $b_6 := -504$; $b_8 := 480$; $b_{10} := -264$; $b_{14} := -24$,
- $\sigma_k(n) := \sum_{1 \leq d | n} d^k$,
- $\delta_\ell := \frac{\ell^2 - 1}{24}$.
- $\omega(k) := \frac{1}{2}k(3k+1)$, $k \in \mathbb{Z}$

Theorem (G., Proc. Amer.. Math. Soc., 2007)
For each prime $\ell \geq 5$ and each $m \in \{4, 6, 8, 10, 14\}$ we have,

$$p(\ell^2 n - \delta_\ell) \equiv \sum_{k \in \mathbb{Z} - \{0\}} (-1)^{k+1} p(\ell^2(n - \omega(k)) - \delta_\ell)$$

$$+ b_m \sum_{\substack{j \geq 1 \\ k \in \mathbb{Z}}} \sigma_{m-1}(j)(-1)^{k+1} p(\ell^2(n - \omega(k) - j/\ell) - \delta_\ell) \pmod{\ell}.$$

Example $m = 4, \ell = 5$, $p(5^2 n + 24) \equiv 0 \pmod 5$

Newforms

For $f(z) = \sum a(n)q^n \in M_k(\Gamma_0(N), \chi)$, $d \geq 0$,

$$f(z)|V(d) = \sum a(n)q^{dn} \in M_k(\Gamma_0(dN), \chi).$$

Consider $S_6(\Gamma_0(8))$. This space has dimension 3.

$$\eta^{1}2(2z), \eta^{1}2(2z)|V(2) = \eta^{1}2(4z) \in S_6(\Gamma_0(8))$$

Note:

$$\eta^{1}2(2z) \in S_6(\Gamma_0(4))$$

is an eigenform. This sits in a lower level.
On the other hand,

$$\eta^8(z)\eta^4(4z) + 8\eta^{1}2(4z) \in S_6(\Gamma_0(8))$$

is an eigenform and does not come from a lower level.
We call this a newform.

Newforms

- Normalized cusp form
- Hecke Eigenform

Newforms have 'small' coefficients.
For $\sum a(n)q^n$,

$$|a(n)| = O_\epsilon(n^{\frac{k-1}{2}+\epsilon})$$

Example. $\Delta(z)$ is a newform for S_{12}.

In Shimura-Taniyama, $\sum a_E(n)q^n$ is a newform of weight 2 and level N, where N is the conductor of E.

Another operator

$$\Theta := q\frac{d}{dq},$$

$$\Theta \sum a(n)q^n = \sum na(n)q^n.$$

Example.

$$\Theta(E_6) = \frac{1}{2}E_2E_6 - \frac{1}{2}E_8,$$

so

$$\Theta(2E_6) \equiv E_6^2 - E_4^3 \equiv -3\Delta \pmod 5$$

.

Consequences:

- $n\sigma_5(n) \equiv \tau(n) \pmod 5$
- $$\sum p(n)q^{n+1}\prod(1-q^{5n})^5 = \frac{\eta^5(5z)}{\eta(z)} \equiv \Delta(z) \pmod 5,$$

 so $p(5n+4) \equiv 0 \pmod 5$.

Theta Series

For $f \in M_k$,

$$\Theta(f) = kfE_2/12 - f_\theta,$$

where f_θ is a meromorphic modular form of weight $k+2$.
Serre studied the action of Θ on $E_2(z), E_4(z), E_6(z)$.

Ramanujan's Congruences

Theorem (Ahlgren-Boylan, Invent. Math, 2003)

For primes $\ell \geq 5$ *the only congruences* $p(An + B) \equiv 0 \pmod{\ell}$ *of the form*

$$p(\ell n + B) \equiv 0 \pmod{\ell}$$

are for $\ell = 5, 7, 11$.

(This was conjectured by Ramanujan.)

Proof: Filtrations, Tate Cycle.

And now for something completely different. . . ?

In 1920 Ramanujan wrote about his discovery of "very interesting functions," such as

$$\begin{aligned} f(q) &:= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\ &= 1 + q - 2q^2 + 3q^3 - 3q^4 + 3q^5 - 5q^6 + \cdots ; \\ \omega(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2} \\ &= 1 + 2q + 3q^2 + 4q^3 + 6q^4 + 8q^5 + 10q^6 + \cdots . \end{aligned}$$

Here $q := e^{2\pi i z}$.

Big Question.

The mock theta functions have partition-theoretic interpretations. (See Andrews, Invent. Math, 2007).

We can use modular forms to study arithmetic/size of partitions.

Are there similar results for the mock thetas?

History

Define $\alpha_f(n)$ and $\alpha_\omega(n)$ by

$$f(q) = \sum_{n\geq 0} \alpha_f(n) q^n; \quad \omega(q) = \sum_{n\geq 0} \alpha_\omega(n) q^n.$$

Andrews-Dragonette Conjecture (1952,1966,2003):

$$\alpha_f(n) = \pi(24n-1)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{1/2}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

- $A_k(n)$ is the $p(n)$ "Kloosterman-type" sum.
- $I_{1/2}(z)$ satisfies

$$I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh(z).$$

Compare this to Hardy-Ramanujan-Rademacher formula:

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

The Circle Method

- $$P(z) = \sum_{n \geq 0} p(n) z^n = \prod_{n=1}^{\infty} \frac{1}{1 - z^n}; \qquad |z| < 1$$
- $$p(n) = \frac{1}{2\pi i} \int_C \frac{P(z)dz}{z^{n+1}}$$
- $P(z)$ has singularities at each root of unity.
- Estimate values using transformation properties.
- Show the error converges to zero.

Recent work

Dyson (1987): There should be a coherent theory based on the few examples of mock thetas.

Zwegers (Contemp. Math., 2003) :

- Vector-valued modular forms

Bringmann and Ono (Invent. Math., 2006) :

- Harmonic Weak Maass forms
- Andrews-Dragonette conjecture

Gordon and McIntosh:

- Transformation Properties with Mordell Integrals

Real analytic vector-valued modular forms

Define the following:

$$F(z) := \left(q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}) \right)^T.$$

$$G(z) := 2i\sqrt{3} \int_{-\bar{z}}^{i\infty} \frac{(g_1(\tau),\ g_0(\tau),\ -g_2(\tau))^T}{\sqrt{-i(\tau+z)}}\, d\tau.$$

- The $g_i(\tau)$ are the cuspidal weight 3/2 theta functions.
- The components of $G(z)$ are 'period integrals'

$$\text{Ex. } g_0(\tau) := \sum_{n=-\infty}^{\infty} (-1)^n \left(n + \frac{1}{3}\right) e^{3\pi i \left(n+\frac{1}{3}\right)^2 \tau}.$$

$$H(z) := (H_0(z), H_1(z), H_2(z)) = F(z) - G(z)$$

Real analytic vector-valued modular forms

Theorem (Zwegers)

The function $H(z)$ is a vector-valued real analytic modular form of weight $1/2$ satisfying

$$H(z+1) = \begin{pmatrix} e(-1/24) & 0 & 0 \\ 0 & 0 & e(1/3) \\ 0 & e(1/3) & 0 \end{pmatrix} H(z),$$

$$H(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(z),$$

where $e(x) := e^{2\pi i x}$.

Harmonic Weak Maass forms

Theorem (Bringmann-Ono)

- *$H_0(24z)$ is a harmonic weak Maass form of weight $1/2$ on $\Gamma_0(144)$ with Nebentypus $\left(\frac{12}{\bullet}\right)$.*

Definition (Harmonic weak Maass form)

1. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$

$$f(Mz) = \begin{cases} \psi(d)(cz+d)^k f(z) & \text{if } k \in \mathbb{Z}, \\ \psi(d)\left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k}(cz+d)^k f(z) & \text{if } k \in \frac{1}{2}+\mathbb{Z}. \end{cases}$$

2. $\Delta_k f = 0$, where

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

3. $f(z)$ has at most linear exponential growth at all the cusps of $\Gamma_0(N)$.

Weak Maass forms

- $H_0(24z) = P_{\frac{1}{2}}\left(\frac{3}{4}; 24z\right)$, where

$$P_k(s;z) := \frac{2}{\sqrt{\pi}} \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(2)} \chi(M)^{-1}(cz+d)^{-k}\varphi_{s,k}(Mz).$$

Here

$$\varphi_{s,k}(Mz) = |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|)\left(-\frac{\pi y}{6}\right) e\left(-\frac{x}{24}\right).$$

The coefficients of $\omega(q)$

Theorem (G, IJNT)
The coefficients $\alpha_\omega(n)$ of $\omega(q)$ are

$$\frac{\pi(3n+2)^{-1/4}}{2\sqrt{2}} \sum_{\substack{k=1\\(k,2)=1}}^{\infty} \frac{(-1)^{\frac{k-1}{2}} A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} I_{1/2}\left(\frac{\pi\sqrt{3n+2}}{3k}\right).$$

Define $c(n,m)$ by formula for $\alpha_\omega(n)$ truncated at $k = 2m-1$.

n	$\alpha_\omega(n)$	$c(n,1)$	$c(n,2)$	$c(n,1000)$
1	2	1.9949	2.2428	1.9963
5	8	7.8769	8.0420	7.9958
10	29	28.6164	29.0178	29.0000
100	1995002	1994993.7262	1995001.6972	1995001.9987

Finding the coefficients of $\omega(q)$

- Construct a real analytic weight $1/2$ vector-valued modular form reflecting transformations of $P_{\frac{1}{2}}(\frac{3}{4}, z)$ on $\mathrm{SL}_2(\mathbb{Z})$
- Express the Fourier expansions of the component functions
- Put it all together: Zwegers, Bringmann and Ono, and the constructed vector-valued modular form.

Constructing the modular form

Definition

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, define,

$$\chi_0(M) := \begin{cases} i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} e\left(\frac{3dc}{8} - \frac{(a+d)}{24c} - \frac{a}{4}\right)\omega_{-d,c}^{-1} & \text{if } c > 0,\ c \text{ even}, \\ e\left(\frac{-b}{24}\right) & \text{if } c = 0; \end{cases}$$

$$\chi_1(M) := i^{-1/2}(-1)^{\frac{c-1}{2}} e\left(\frac{3dc}{8} - \frac{(a+d)}{24c}\right)\omega_{-d,c}^{-1} \qquad \text{if } c > 0,\ d \text{ even},$$

$$\chi_2(M) := i^{-1/2}(-1)^{\frac{c-1}{2}} e\left(\frac{3dc}{8} - \frac{(a+d)}{24c}\right)\omega_{-d,c}^{-1} \qquad \text{if } c > 0,\ c, d \text{ odd},$$

where the $\omega_{-d,c}^{-1}$ come from Dedekind sums.

Constructing the modular form

Definition

$$\mathcal{P}(z) := (P_0(z), P_1(z), P_2(z))^T ,$$

where,

$$P_0(z) := \frac{2}{\sqrt{\pi}} \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(2)} \chi_0(M)^{-1}(cz+d)^{-1/2} \varphi_{3/4,1/2}(Mz);$$

$$P_1(z) := \frac{2}{\sqrt{\pi}} \sum_{\substack{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M'S \\ M' \in \Gamma_\infty \backslash \Gamma_0(2)}} \chi_1(M)^{-1}(cz+d)^{-1/2} \varphi_{3/4,1/2}(Mz);$$

$$P_2(z) := \frac{2}{\sqrt{\pi}} \sum_{\substack{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = M'ST \\ M' \in \Gamma_\infty \backslash \Gamma_0(2)}} \chi_2(M)^{-1}(cz+d)^{-1/2} \varphi_{3/4,1/2}(Mz).$$

Connection to $H(z)$

Theorem (G., IJNT)

The function $\mathcal{P}(z)$ is a vector-valued real analytic modular form of weight $1/2$ satisfying

$$\mathcal{P}(z+1) = \begin{pmatrix} e(-1/24) & 0 & 0 \\ 0 & 0 & e(1/3) \\ 0 & e(1/3) & 0 \end{pmatrix} \mathcal{P}(z),$$

$$\mathcal{P}(-1/z) = \sqrt{-iz} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathcal{P}(z).$$

The coefficients of $\omega(q)$

$$P_1(24z) = H_1(24z) = 2q^8\omega(q^{12}) + \text{ Non-holomorphic part.}$$

$P_1(z) = \sum_{n\geq 0} \alpha(n) q^{\frac{n}{2}+\frac{1}{3}} + \sum_{n<0} \beta_y(n) q^{\frac{n}{2}+\frac{1}{3}}$,
where,

$$\alpha(n) = \frac{\pi}{\sqrt{2}}(3n+2)^{-\frac{1}{4}} \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{3n+2}}{3k}\right),$$

$$\beta_y(n) = \frac{\pi^{\frac{1}{2}}}{\sqrt{2}}|3n+2|^{-\frac{1}{4}} \cdot \Gamma\left(\frac{1}{2}, \frac{\pi|3n+2|\cdot y}{3}\right) \sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{A_k\left(\frac{n(k+1)}{2} - \frac{3(k^2-1)}{8}\right)}{k} \cdot J_{\frac{1}{2}}\left(\frac{\pi\sqrt{|3n+2|}}{3k}\right).$$

Familes of Mock Thetas

- Weight $1/2$, Dyson's Rank
  Bringmann - Ono (Annals), B-O-Rhoades (JAMS)

Let $N(m, n)$ denote the number of partitions of n with m parts.

Holomorphic part from letting z be a root of unity in:

$$1 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} N(m, n) z^m q^n = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(zq; q)_n (z^{-1} q; q)_n}.$$

Non-holomorphic part from period integrals of weight $3/2$ theta series.

- Weight $3/2$, Hypergeometric series and period integrals of weight $1/2$ theta series
  B-Folsom-O

Automorphic Forms

Modular forms are well understood.

- Finite dimensionality
- Operators
- Size of coefficients
- Congruences (Galois Representations)

Now want parallel theory for harmonic weak Maass forms.