RAMANUJAN'S FORTY IDENTITIES FOR THE ROGERS–RAMANUJAN FUNCTIONS

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Abstract. Sir Arthur Conan Doyle's famous fictional detective Sherlock Holmes and his sidekick Dr. Watson go camping and pitch their tent under the stars. During the night, Holmes wakes his companion and says, "Watson, look up at the stars and tell me what you deduce." Watson says, "I see millions of stars, and it is quite likely that a few of them are planets just like Earth. Therefore there may also be life on these planets." Holmes replies, "Watson, you idiot. Somebody stole our tent."

When seeking proofs of Ramanujan's identities for the Rogers–Ramanujan functions, Watson, i.e., G. N. Watson, was not an "idiot." He, L. J. Rogers, and D. M. Bressoud found proofs for several of the identities. A. J. F. Biagioli devised proofs for most (but not all) of the remaining identities. Although some of the proofs of Watson, Rogers, and Bressoud are likely in the spirit of those found by Ramanujan, those of Biagioli are not. In particular, Biagioli used the theory of modular forms. Haunted by the fact that little progress has been made into Ramanujan's insights on these identities in the past 85 years, the present authors sought "more natural" proofs. Thus, instead of a missing tent, we have had missing proofs, i.e., Ramanujan's missing proofs of his forty identities for the Rogers–Ramanujan functions. In this paper, for 35 of the 40 identities, we offer proofs that are in the spirit of Ramanujan. Some of the proofs presented here are due to Watson, Rogers, and Bressoud, but most are new. We also establish several new identities for the Rogers–Ramanujan functions. However, we feel that we have still failed to discover most of Ramanujan's thinking about these identities.

The five remaining identities to be proved are:

Entry 28 (second part)
Entry 29
Entry 30
Entry 31 or Entry 32
Entry 35

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1. Introduction

The Rogers–Ramanujan functions in the title are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}, \tag{1.1}$$

where here and in the sequel we use the customary notation $(a;q)_0 := 1$,

$$\begin{aligned} (a;q)_n &:= \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1, \\ (a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n, \qquad |q| < 1. \end{aligned}$$

These functions satisfy the famous Rogers–Ramanujan identities [22], [17], [19, pp. 214–215]

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \qquad \text{and} \qquad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.2}$$

At the end of his brief communication [18], [19, p. 231] announcing his proofs of the Rogers–Ramanujan identities (1.2), Ramanujan remarks, "I have now found an algebraic relation between $G(q)$ and $H(q)$, viz.:

$$H(q)\{G(q)\}^{11} - q^2 G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6. \tag{1.3}$$

Another noteworthy formula is

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1. \tag{1.4}$$

Each of these formulae is the simplest of a large class." Ramanujan did not indicate how he had proved these two identities, which, as we shall see below, are two from a list of forty identities involving $G(q)$ and $H(q)$ that Ramanujan had compiled.

In his paper [23] establishing ten of the identities, Rogers remarks, "these [identities] were communicated privately to me in February 1919 ..." Rogers did not indicate if further identities were included in Ramanujan's communication to him.

In 1933, Watson [25] proved eight of the identities, but with two of them from the group that Rogers had proved. Watson confides, "Among the formulae contained in the manuscripts left by Ramanujan is a set of about forty which involve functions of the types $G(q)$ and $H(q)$; the beauty of these formulae seems to me to be comparable with that of the Rogers–Ramanujan identities. So far as I know, nobody else has discovered any formulae which approach them even remotely; if my belief is well-founded, the undivided credit for the discovery of these formulae is due to Ramanujan." This last statement appears to be so obvious, especially since the manuscript was evidently in Watson's possession, that one wonders why he wrote it.

Ramanujan's forty identities for $G(q)$ and $H(q)$ (which do not include (1.2)) were first brought in their entirety before the mathematical public by B. J. Birch [7], who in 1975 found Watson's handwritten copy of Ramanujan's list of forty identities in the Oxford University Library. Ramanujan's original manuscript was in Watson's possession for many years and now has evidently been lost. Watson's handwritten list was later published along with Ramanujan's lost notebook [21, pp. 236–237] in 1988. Certain pairs of the identities are linked, and so it is natural to place them, in fact, in 35 (not 40) separate entries.

D. Bressoud [10], in his Ph.D. thesis, proved fifteen from the list of forty. His published paper [11] contains proofs of the general identities from [10] which he developed in order to prove Ramanujan's identities. All the proofs of Rogers, Watson, and Bressoud employ classical means, although it would seem that in most cases the proofs are not like those found by Ramanujan.

After the work of Rogers, Watson, and Bressoud, nine remained to be proved. A. J. F. Biagioli [6] used modular forms to prove eight of them. At this moment then, only one of the forty identities has not been proved by any means, but it is clear that modular forms can be used to establish this last identity. About such proofs, Birch [7] opines, "A dull proof would have little value – in fact, all the functions involved in the identities are essentially theta functions, so modular forms of known level with poles of bounded order at known places, so the identities may presumably be verified by just checking that the first hundred or so powers of x are correct." It should be remarked that Biagioli's [6] proofs are more elegant than one might discern from Birch's remarks, for Biagioli used Fricke involutions and other properties of modular forms to drastically reduce the number of terms envisioned by Birch. In fact, in most cases, Biagioli required only a few terms.

In this paper, we offer proofs of 35 of the 40 identities. Furthermore, in the last section of this paper, Section 34, we establish several new identities involving the Rogers–Ramanujan functions. Some of the proofs that we present were found by either Rogers, Watson, or Bressoud. However, most of the proofs presented in this paper are new. Our goal has been to find proofs for all forty identities that Ramanujan might have given himself. Indeed, in several of our proofs, we utilize modular equations found by Ramanujan and left in his notebooks [20]. Although all the proofs offered here are in the spirit of Ramanujan's mathematics, it is to be omitted that for some proofs, knowing the identity beforehand was a distinct advantage to us in finding a proof. It is unfortunate that we have failed to find proofs of five of the identities, four of which were proved by Biagioli [6] and one of which has not been proved at all. Although some of our proofs may have been those found by Ramanujan, it is clear that all of us, including the aforementioned authors and the present authors, have not unveiled Ramanujan's secrets which remain hidden by an inpenetrable fog.

2. Definitions and Preliminary Results

We first recall Ramanujan's definitions for a general theta function and some of its important special cases. Set

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{2.1}$$

Basic properties satisfied by $f(a,b)$ include [3, p. 34, Entry 18]

$$f(a,b) = f(b,a), \tag{2.2}$$

$$f(1,a) = 2f(a,a^3), \tag{2.3}$$

$$f(-1,a) = 0, \tag{2.4}$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \tag{2.5}$$

The basic property (2.2) will be used many times in the sequel without comment. The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.6}$$

The three most important special cases of (2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.8}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty =: q^{-1/24}\eta(\tau), \tag{2.9}$$

where $q = \exp(2\pi i\tau)$, $\operatorname{Im} \tau > 0$, and η denotes the Dedekind eta–function. The product representations in (2.7)–(2.9) are special cases of (2.6). Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty. \tag{2.10}$$

Using (2.6) and (2.9), we can rewrite the Rogers–Ramanujan identities (1.2) in the forms

$$G(q) = \frac{f(-q^2,-q^3)}{f(-q)} \qquad \text{and} \qquad H(q) = \frac{f(-q,-q^4)}{f(-q)}. \tag{2.11}$$

We shall use (2.11) many times in the remainder of the paper. A useful consequence of (2.11) in conjunction with the Jacobi triple product identity (2.6) is

$$G(q)H(q) = \frac{f(-q^5)}{f(-q)}. \tag{2.12}$$

Basic properties of the functions (2.7)–(2.10) include [3, pp. 39–40, Entries 24, 25(iii)]

$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}}, \tag{2.13}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)}, \tag{2.14}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q), \qquad \chi(q)\chi(-q) = \chi(-q^2), \tag{2.15}$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \tag{2.16}$$

It is easy to conclude from (2.14) or (2.6) that

$$\psi(-q) = \chi(-q)f(-q^4) = \frac{f(-q)}{\chi(-q^2)}, \quad \chi(q)f(-q) = \varphi(-q^2). \tag{2.17}$$

We shall use the famous quintuple product identity, which, in Ramanujan's notation, takes the form (2.1) [3, p. 80, Entry 28(iv)]

$$\frac{f(-a^2,-a^{-2}q)}{f(-a,-a^{-1}q)} = \frac{1}{f(-q)}\Big\{f(-a^3q,-a^{-3}q^2) + af(-a^{-3}q,-a^3q^2)\Big\}, \tag{2.18}$$

where a is any complex number.

The function $f(a,b)$ also satisfies a useful addition formula. For each positive integer n, let

$$U_n := a^{n(n+1)/2}b^{n(n-1)/2} \qquad \text{and} \qquad V_n := a^{n(n-1)/2}b^{n(n+1)/2}.$$

Then [3, p. 48, Entry 31]

$$f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.19}$$

The Rogers–Ramanujan functions are intimately associated with the Rogers–Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots, \qquad |q|<1, \tag{2.20}$$

which first appeared in a paper by Rogers [22] in 1894. Using the Rogers–Ramanujan identities (1.2), Rogers proved that

$$R(q) = q^{1/5}\frac{H(q)}{G(q)} = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.21}$$

This was independently discovered by Ramanujan and can be found in his notebooks [20], [3, p. 79, Chap. 16, Entry 38(iii)].

3. The Forty Identities

Entry 3.1.

$$G^{11}(q)H(q) - q^2G(q)H^{11}(q) = 1 + 11qG^6(q)H^6(q). \tag{3.1}$$

Entry 3.1 was one of two identities stated by Ramanujan without proof in [18], [19, p. 231]. As related in the Introduction, Ramanujan [18] claims that, "Each of these formulae is the simplest of a large class." Ramanujan's remark is interesting, because Entry 3.1 is the only identity among the forty in which powers of $G(q)$ or $H(q)$ appear. It would seem from Ramanujan's remark that he had further identities involving powers of $G(q)$ or $H(q)$, but no further identities of this sort are known. The first published proof of (3.1) is by H. B. C. Darling [12] in 1921. A second proof by Rogers [23] appeared in the same year. One year later, L. J. Mordell [16] found another proof.

By (2.21), the identity (3.1) is equivalent to a famous identity for the Rogers–Ramanujan continued fraction (2.20), namely,

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{3.2}$$

This equality was found by Watson in Ramanujan's notebooks [20] and proved by him [24] in order to establish claims about the Rogers–Ramanujan continued fraction communicated by Ramanujan in his first two letters to Hardy [24]. A different proof of (3.2) can be found in Berndt's book [3, pp. 265–267]. The identity (3.2) can also be found in an unpublished manuscript of Ramanujan first appearing in handwritten form with his lost notebook [21, pp. 135–177, 283–243]. An annotated account of Ramanujan's manuscript with considerable commentary and numerous references has been prepared by Berndt and K. Ono [4].

Entry 3.2.

$$G(q)G(q^4) + qH(q)H(q^4) = \chi^2(q) = \frac{\varphi(q)}{f(-q^2)}. \tag{3.3}$$

Entry 3.2 was first proved in print by Rogers [23]; Watson [25] also found a proof. In fact, G. E. Andrews [1, p. 27] has shown that (3.3) follows from a very general identity in three variables found in Ramanujan's lost notebook.

Entry 3.3.

$$G(q)G(q^4) - qH(q)H(q^4) = \frac{\varphi(q^5)}{f(-q^2)}. \tag{3.4}$$

Watson [25] has given a proof of (3.4).

Entry 3.4.

$$G(q^{11})H(q) - q^2G(q)H(q^{11}) = 1. \tag{3.5}$$

Entry 3.4 is the second identity offered by Ramanujan without proof in [18], [19, p. 231]. The first published proof was given by Rogers [23]. Watson [25] also gave a proof. R. Blecksmith, J. Brillhart, and I. Gerst [8] have shown that (3.5) follows from a very general theta function identity established by the authors.

Proofs of the next seven entries were first given by Rogers [23].

Entry 3.5.

$$G(q^{16})H(q) - q^3G(q)H(q^{16}) = \chi(q^2). \tag{3.6}$$

Entry 3.6.

$$G(q)G(q^9) + q^2H(q)H(q^9) = \frac{f^2(-q^3)}{f(-q)f(-q^9)}. \tag{3.7}$$

Entry 3.7.

$$G(q^2)G(q^3) + qH(q^2)H(q^3) = \frac{\chi(-q^3)}{\chi(-q)}. \tag{3.8}$$

Entry 3.8.

$$G(q^6)H(q) - qG(q)H(q^6) = \frac{\chi(-q)}{\chi(-q^3)}. \tag{3.9}$$

Entry 3.9.

$$G(q^7)H(q^2) - qG(q^2)H(q^7) = \frac{\chi(-q)}{\chi(-q^7)}. \tag{3.10}$$

Entry 3.10.

$$G(q)G(q^{14}) + q^3H(q)H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}. \tag{3.11}$$

Entry 3.11.

$$G(q^8)H(q^3) - qG(q^3)H(q^8) = \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \tag{3.12}$$

Entry 3.12.

$$G(q)G(q^{24}) + q^5H(q)H(q^{24}) = \frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}. \tag{3.13}$$

Entry 3.13.

$$G(q^9)H(q^4) - qG(q^4)H(q^9) = \frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}. \tag{3.14}$$

Entry 3.14.

$$G(q^{36})H(q) - q^7G(q)H(q^{36}) = \frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)}. \tag{3.15}$$

Entries 3.12–3.14 were first proved by Bressoud in his doctoral dissertation [10].

Entry 3.15.

$$G(q^3)G(q^7) + q^2H(q^3)H(q^7) = G(q^{21})H(q) - q^4G(q)H(q^{21}) \tag{3.16}$$

$$= \frac{1}{2\sqrt{q}}\chi(q^{1/2})\chi(-q^{3/2})\chi(q^{7/2})\chi(-q^{21/2}) - \frac{1}{2\sqrt{q}}\chi(-q^{1/2})\chi(q^{3/2})\chi(-q^{7/2})\chi(q^{21/2}). \tag{3.17}$$

The only known proofs of (3.16) and (3.17) are by Biagioli [6], who used the theory of modular forms.

Entry 3.16.

$$G(q^2)G(q^{13}) + q^3H(q^2)H(q^{13}) = G(q^{26})H(q) - q^5G(q)H(q^{26}) \tag{3.18}$$

$$= \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}. \tag{3.19}$$

The only known proof of (3.18) is by Bressoud [10], while Biagioli, using the theory of modular forms, has established the only known proof of (3.19). Biagioli's [6] formulation of (3.19) contains two misprints; the formula is also misnumbered as #17 instead of #18.

Proofs of the next four identities, (3.20)–(3.23), have been given by Bressoud [10].

Entry 3.17.

$$G(q)G(q^{19})+q^4H(q)H(q^{19}) = \frac{1}{4\sqrt{q}}\chi^2(q^{1/2})\chi^2(q^{19/2}) - \frac{1}{4\sqrt{q}}\chi^2(-q^{1/2})\chi^2(-q^{19/2}) - \frac{q^2}{\chi^2(-q)\chi^2(-q^{19})}. \tag{3.20}$$

Entry 3.18.

$$G(q^{31})H(q)-q^6G(q)H(q^{31}) = \frac{1}{2q}\chi(q)\chi(q^{31}) - \frac{1}{2q}\chi(-q)\chi(-q^{31}) + \frac{q^3}{\chi(-q^2)\chi(-q^{62})}. \tag{3.21}$$

Entry 3.19.

$$\left\{G(q)G(q^{39})+q^8H(q)H(q^{39})\right\} f(-q)f(-q^{39})$$
$$=\left\{G(q^{13})H(q^3)-q^2G(q^3)H(q^{13})\right\} f(-q^3)f(-q^{13}) \tag{3.22}$$
$$=\frac{1}{2q}\left(\varphi(-q^3)\varphi(-q^{13})-\varphi(-q)\varphi(-q^{39})\right). \tag{3.23}$$

Entry 3.20.

$$G(q)H(-q)+G(-q)H(q) = \frac{2}{\chi^2(-q^2)} = \frac{2\psi(q^2)}{f(-q^2)}. \tag{3.24}$$

Entry 3.21.

$$G(q)H(-q)-G(-q)H(q) = \frac{2q\psi(q^{10})}{f(-q^2)}. \tag{3.25}$$

Watson [25] constructed proofs of both (3.24) and (3.25).

Entry 3.22.

$$G(-q)G(-q^4)+qH(-q)H(-q^4) = \chi(q^2). \tag{3.26}$$

Entry 3.23.

$$G(-q^2)G(-q^3)+qH(-q^2)H(-q^3) = \frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)}. \tag{3.27}$$

Entry 3.24.

$$G(-q^6)H(-q)-qH(-q^6)G(-q) = \frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)}. \tag{3.28}$$

Bressoud [10] established the three previous entries.

Entry 3.25.

$$G(-q)G(q^9)-q^2H(-q)H(q^9) = \frac{\chi(-q)\chi(q^9)}{\chi(-q^6)}. \tag{3.29}$$

Equality (3.29) is a corrected version of that given by Watson [21] and was first proved by Bressoud [10].

Entry 3.26.

$$G(q^{11})H(-q)+q^2G(-q)H(q^{11})=\frac{\chi(q^2)\chi(q^{22})}{\chi(-q^2)\chi(-q^{22})}-\frac{2q^3}{\chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44})}. \tag{3.30}$$

Watson [25] established (3.30). The minus sign in front of the second expression on the right side of (3.30) is missing in Watson's list [21].

Our formulations of Entries 3.27 and 3.28 are slightly different from those of Ramanujan, who had reversed the hypotheses in each entry. In other words, he intended that the formulas for U and V be the conclusions in each case, with the pairs of equations, (3.33), (3.34) and (3.35), (3.36) being the conditions under which the formulas for U and V should hold. Watson proved Entry 3.27 under the same interpretation as we have given.

Entry 3.27. *Define*

$$U:=U(q):=G(q)G(q^{44})+q^9H(q)H(q^{44}) \tag{3.31}$$

and

$$V:=V(q):=G(q^4)G(q^{11})+q^3H(q^4)H(q^{11}). \tag{3.32}$$

Then

$$U^2+qV^2=\chi^3(q)\chi^3(q^{11}) \tag{3.33}$$

and

$$UV+q=\chi^2(q)\chi^2(q^{11}). \tag{3.34}$$

Entry 3.28. *Define*

$$U:=G(q^{17})H(q^2)-q^3G(q^2)H(q^{17}) \qquad \textit{and} \qquad V:=G(q)G(q^{34})+q^7H(q)H(q^{34}).$$

Then

$$\frac{U}{V}=\frac{\chi(-q)}{\chi(-q^{17})} \tag{3.35}$$

and

$$U^4V^4-qU^2V^2=\frac{\chi^3(-q^{17})}{\chi^3(-q)}\left(1+q^2\frac{\chi^3(-q)}{\chi^3(-q^{17})}\right)^2. \tag{3.36}$$

Bressoud proved (3.35) in his thesis [10]. Biagioli claimed in [6] that he was going to prove (3.36), but a proof of (3.36) does not appear in his paper.

Entry 3.29.

$$\begin{aligned}&\left\{G(q^2)G(q^{23})+q^5H(q^2)H(q^{23})\right\}\left\{G(q^{46})H(q)-q^9G(q)H(q^{46})\right\}\\ =&\chi(-q)\chi(-q^{23})+q+\frac{2q^2}{\chi(-q)\chi(-q^{23})}.\end{aligned} \tag{3.37}$$

Entry 3.30.

$$\frac{G(q^{19})H(q^4)-q^3G(q^4)H(q^{19})}{G(q^{76})H(-q)+q^{15}G(-q)H(q^{76})}=\frac{\chi(-q^2)}{\chi(-q^{38})}. \tag{3.38}$$

Entry 3.31.

$$\frac{G(q^2)G(q^{33})+q^7H(q^2)H(q^{33})}{G(q^{66})H(q)-q^{13}H(q^{66})G(q)}=\frac{\chi(-q^3)}{\chi(-q^{11})}. \tag{3.39}$$

Entry 3.32.

$$\frac{G(q^3)G(q^{22})+q^5H(q^3)H(q^{22})}{G(q^{11})H(q^6)-qG(q^6)H(q^{11})}=\frac{\chi(-q^{33})}{\chi(-q)}. \tag{3.40}$$

Using the theory of modular forms, Biagioli [6] constructed proofs of Entries 3.29–3.32. No other proofs are known.

Entry 3.33.

$$\frac{G(q)G(q^{54})+q^{11}H(q)H(q^{54})}{G(q^{27})H(q^2)-q^5G(q^2)H(q^{27})}=\frac{\chi(-q^3)\chi(-q^{27})}{\chi(-q)\chi(-q^9)}. \tag{3.41}$$

Entry 3.34.

$$\begin{aligned}&\left\{G(q)G(-q^{19})-q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19})-q^4H(-q)H(q^{19})\right\}\\=&G(q^2)G(q^{38})+q^8H(q^2)H(q^{38}).\end{aligned} \tag{3.42}$$

Proofs of (3.41) and (3.42) have been found by Bressoud [10], who corrected a misprint in Watson's [21] formulation of (3.42).

Entry 3.35.

$$\begin{aligned}&\left\{G(q)G(q^{94})+q^{19}H(q)H(q^{94})\right\}\left\{G(q^{47})H(q^2)-q^9G(q^2)H(q^{47})\right\}\\=&\chi(-q)\chi(-q^{47})+2q^2+\frac{2q^4}{\chi(-q)\chi(-q^{47})}+q\sqrt{4\chi(-q)\chi(-q^{47})+9q^2+\frac{8q^4}{\chi(-q)\chi(-q^{47})}}.\end{aligned} \tag{3.43}$$

The only known proof [6]of Entry 3.35 employs the theory of modular forms.

Ramanujan's identities are remarkable for several reasons. The Rogers–Ramanujan functions are associated with modular equations of degree 5 and q-products with base q^5. However, the "5" is missing on all the right sides of the identities, except for Entries 3.3 and 3.21. One would expect to see in such identities theta functions with arguments q^{5n}, for certain positive integers n, but such functions do not appear! Moreover, the right sides in almost all of the identities are expressed entirely in terms of the modular function χ with no other theta function appearing. We have no explanation for this phenomenon. It seems likely that the function χ played a more important role in Ramanujan's thinking than we are able to discern. As we shall see in the proofs throughout the paper, some of the identities are amenable to general techniques established either by Watson, Rogers, or the authors. However, for those identities that are more difficult to prove (and there are many), these ideas do not appear to be useable. It was unsettling for us to find a proof of a certain identity with a great deal of effort and then discover that our ideas were inapplicable to any of the remaining identites that we sought to prove. In other words, each of the "hard" identities required an argument that seems to apply to only that identity. This is one reason why the authors feel that Ramanujan had at least one key idea that all researchers to date have missed. It seems likely that the function χ played an important role in Ramanujan's primary idea(s). Each of the forty identities, in principal, can be associated with modular equations of a certain degree. It happens that for each such degree, Ramanujan recorded at least one modular equation of that degree in his notebooks [20], [3]. We conjecture that Ramanujan utilized modular equations to prove some of the forty identities in manners that we have not been able to discern.

4. The Principal Ideas Behind the Proofs

In this section, we describe the main ideas behind the proofs given by Watson [25], Rogers [23], Bressoud [10], and the present authors.

We first discuss an idea of Watson [25]. In these proofs, one expresses the left sides of the identities in terms of theta functions by using (2.11). In some cases, after clearing fractions, the right side can be expressed as a product of two theta functions, say with summations indices m and n. One then tries to find a change of indices of the form

$$\alpha m + \beta n = 5M + a \qquad \text{and} \qquad \gamma m + \delta n = 5N + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side. We emphasize that this method only works when the right side is a product of two theta functions, and even then, in only some cases, does this kind of change of variables produce the desired equality. This method was probably not that used by Ramanujan, because it would seem that the identity to be proved must be explicitly known in advance.

We next present a modest generalization of Rogers's method [23]. We let p and m denote odd positive integers with $p > 1$, and let α, β, and λ be real numbers such that

$$\alpha m^2 + \beta = \lambda p. \tag{4.1}$$

The special case when α, β, and λ are integers is given by Rogers [23]. Consider, for any real number v, the product

$$\begin{aligned} &q^{p\alpha m^2 v^2} f(-q^{p\alpha+2p\alpha mv}, -q^{p\alpha-2p\alpha mv}) q^{p\beta v^2} f(-q^{p\beta+2p\beta v}, -q^{p\beta-2p\beta v}) \\ &= \sum_{r=-\infty}^{\infty} (-1)^r q^{p\alpha(r+mv)^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{p\beta(s+v)^2} = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (-1)^{r+s} q^I, \end{aligned} \tag{4.2}$$

where

$$I = p\alpha(r + mv)^2 + p\beta(s + v)^2.$$

For fixed s, write $r = ms + t$. Then, by (4.1),

$$\begin{aligned} I =& p\alpha\left\{(s+v)m+t\right\}^2 + p\beta(s+v)^2 \\ =& \lambda p^2(s+v)^2 + 2p\alpha mt(s+v) + p\alpha t^2 \\ =& \lambda\left\{p(s+v)+\frac{\alpha mt}{\lambda}\right\}^2 - \frac{\alpha^2m^2t^2}{\lambda} + p\alpha t^2 \\ =& \lambda\left\{p(s+v)+\frac{\alpha mt}{\lambda}\right\}^2 + \frac{\alpha\beta}{\lambda}t^2. \end{aligned} \tag{4.3}$$

Note also that, since m is odd,

$$(-1)^{r+s} = (-1)^t. \tag{4.4}$$

Now let

$$S_p := \left\{\frac{1}{2p}, \frac{3}{2p}, \ldots, \frac{2p-1}{2p}\right\}. \tag{4.5}$$

Thus, using (4.2)–(4.5), we find that

$$\begin{aligned} &\sum_{v\in S_p} q^{p\alpha m^2v^2} f(-q^{p\alpha+2p\alpha mv}, -q^{p\alpha-2p\alpha mv}) q^{p\beta v^2} f(-q^{p\beta+2p\beta v}, -q^{p\beta-2p\beta v}) \\ &= \sum_{v\in S_p}\sum_{r=-\infty}^{\infty}(-1)^r q^{p\alpha(r+mv)^2}\sum_{s=-\infty}^{\infty}(-1)^s q^{p\beta(s+v)^2} = \sum_{k=1}^{p}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^I, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} I = I(r,s,t) :=& \lambda\left\{p\left(s+\frac{2k-1}{2p}\right)+\frac{\alpha mt}{\lambda}\right\}^2 + \frac{\alpha\beta}{\lambda}t^2 \\ =& \lambda\left\{ps+k-1+\frac{1}{2}+\frac{\alpha mt}{\lambda}\right\}^2 + \frac{\alpha\beta}{\lambda}t^2 \\ =& \lambda\left\{u+\frac{1}{2}+\frac{\alpha mt}{\lambda}\right\}^2 + \frac{\alpha\beta}{\lambda}t^2, \end{aligned} \tag{4.7}$$

upon letting $u := ps + k - 1$. Hence, (4.6) can now be expressed as

$$\begin{aligned} &\sum_{v\in S_p} q^{p\alpha m^2v^2} f(-q^{p\alpha+2p\alpha mv}, -q^{p\alpha-2p\alpha mv}) q^{p\beta v^2} f(-q^{p\beta+2p\beta v}, -q^{p\beta-2p\beta v}) \\ &= \sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^I, \end{aligned} \tag{4.8}$$

with I as given in (4.7).

The strategy of Rogers is to find two sets of parameters $\{\alpha_1, \beta_1, m_1, p_1, \lambda_1\}$ and $\{\alpha_2, \beta_2, m_2, p_2, \lambda_2\}$ both giving rise to the same function on the right-hand side of (4.8). This would establish an identity between two sums of products of two theta functions each of the form (4.2). For instance, if we choose the two sets of parameters such that

$$\alpha_1\beta_1 = \alpha_2\beta_2, \qquad \lambda_1 = \lambda_2, \qquad \text{and} \qquad \frac{\alpha_1 m_1}{\lambda_1} \pm \frac{\alpha_2 m_2}{\lambda_2} \qquad \text{is an integer,} \tag{4.9}$$

then both sets of parameters would satisfy the formula for I in (4.7), thus giving rise to the same function on the right-hand side of (4.8).

We next show that the contributions of the terms with indices k and $p-k+1$ are identical. Applying (2.5) with $n=-m$, we find that

$$\begin{aligned}
&q^{\alpha m^2(2k-1)^2/(4p)}f(-q^{p\alpha+\alpha m(2k-1)},-q^{p\alpha-\alpha m(2k-1)})\\
=&q^{\alpha m^2(2k-1)^2/(4p)+m^2p\alpha-m^2\alpha(2k-1)}f(-q^{p\alpha+\alpha m(2k-1)-2p\alpha m},-q^{p\alpha-\alpha m(2k-1)+2p\alpha m})\\
=&q^{\alpha m^2(2p-2k+1)^2/(4p)}f(-q^{p\alpha+\alpha m(2p-2k+1)},-q^{p\alpha-\alpha m(2p-2k+1)}), \qquad (4.10)
\end{aligned}$$

where we have used the fact that p is odd. The same argument holds for the other theta function in (4.2). This establishes our claim.

Next, we show that the contribution of the term with $k=(p+1)/2$, i.e., $v=1/2$, equals 0. Thus, we examine

$$\sum_{r=-\infty}^{\infty}(-1)^r q^{p\alpha(r+m/2)^2}=q^{p\alpha m^2/4}f(-q^{p\alpha(1-m)},-q^{p\alpha(1+m)}). \qquad (4.11)$$

To the theta function in (4.11), we apply (2.5) with $n=(m-1)/2$. Thus, for some constant c,

$$f(-q^{p\alpha(1-m)},-q^{p\alpha(1+m)})=q^c f(-1,-q^{2p\alpha})=0, \qquad (4.12)$$

by (2.4). The same argument shows that the other theta function appearing in (4.6) also vanishes when $v=1/2$.

Using (4.10) and (4.12) in (4.6), we deduce that, when p is odd,

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2}F(\alpha,\beta,m,p,\lambda,k):=&\sum_{k=1}^{(p-1)/2}\sum_{r=-\infty}^{\infty}(-1)^r q^{p\alpha(r+m(2k-1)/(2p))^2}\sum_{s=-\infty}^{\infty}(-1)^s q^{p\beta(s+(2k-1)/(2p))^2}\\
=&\sum_{k=1}^{(p-1)/2}q^{\alpha m^2(2k-1)^2/(4p)}f(-q^{p\alpha+\alpha m(2k-1)},-q^{p\alpha-\alpha m(2k-1)})\\
&\times q^{\beta(2k-1)^2/(4p)}f(-q^{p\beta+\beta(2k-1)},-q^{p\beta-\beta(2k-1)})\\
=&\frac{1}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^I, \qquad (4.13)
\end{aligned}$$

where I is given in (4.7).

If p is even and if α is even, then the same argument shows that the terms with indices k and $p-k+1$ are identical. Hence, for p even,

$$\begin{aligned}\sum_{k=1}^{p/2} F(\alpha,\beta,m,p,\lambda,k) :=&\sum_{k=1}^{p/2}\sum_{r=-\infty}^{\infty}(-1)^r q^{p\alpha(r+m(2k-1)/(2p))^2}\sum_{s=-\infty}^{\infty}(-1)^s q^{p\beta(s+(2k-1)/(2p))^2}\\ =&\sum_{k=1}^{p/2} q^{\alpha m^2(2k-1)^2/(4p)} f(-q^{p\alpha+\alpha m(2k-1)},-q^{p\alpha-\alpha m(2k-1)})\\ &\times q^{\beta(2k-1)^2/(4p)} f(-q^{p\beta+\beta(2k-1)},-q^{p\beta-\beta(2k-1)})\\ =&\frac{1}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^I, \end{aligned}\tag{4.14}$$

where I is given in (4.7).

For later applications, we record some special cases of (4.13) and (4.14). If $p=5$ and $m=1$,

$$\begin{aligned}\sum_{k=1}^{2} F(\alpha,\beta,1,5,\lambda,k) =& q^{(\alpha+\beta)/20} f(-q^{4\alpha},-q^{6\alpha}) f(-q^{4\beta},-q^{6\beta})\\ &+ q^{9(\alpha+\beta)/20} f(-q^{2\alpha},-q^{8\alpha}) f(-q^{2\beta},-q^{8\beta}).\end{aligned}\tag{4.15}$$

If $p=5$ and $m=3$,

$$\begin{aligned}\sum_{k=1}^{2} F(\alpha,\beta,3,5,\lambda,k) =& q^{(9\alpha+\beta)/20} f(-q^{2\alpha},-q^{8\alpha}) f(-q^{4\beta},-q^{6\beta})\\ &- q^{(\alpha+9\beta)/20} f(-q^{4\alpha},-q^{6\alpha}) f(-q^{2\beta},-q^{8\beta}),\end{aligned}\tag{4.16}$$

where we applied (2.5) with $n=1$. If $p=3$ and $m=1$,

$$\sum_{k=1}^{1} F(\alpha,\beta,1,3,\lambda,k) = q^{(\alpha+\beta)/12} f(-q^{2\alpha}) f(-q^{2\beta}).\tag{4.17}$$

If $p=2$ and $m=1$, by (2.8),

$$\sum_{k=1}^{1} F(\alpha,\beta,1,2,\lambda,k) = q^{(\alpha+\beta)/8} f(-q^{\alpha},-q^{3\alpha}) f(-q^{\beta},-q^{3\beta}) = q^{(\alpha+\beta)/8}\psi(-q^{\alpha})\psi(-q^{\beta}).\tag{4.18}$$

In our proofs of 35 of the 40 identities, we have borrowed some of Rogers's proofs that use his method described above. Although we have given credit to Rogers in Section 3 for each of the entries he proved, for brevity, we do not repeat these acknowledgments in the ensuing sections where proofs are given. Rogers's ideas were extended by Bressoud [10], but we have not employed Bressoud's more general theorems in this paper. We have used Rogers's method, however, in proving further identities in Ramanujan's list.

A third approach is a method of *elimination*. Here one sets, $T(q)$, say, equal to the left side of the identity to be proved. By changes of variable, if necessary, one records two further (previously proved) identities involving $G(q)$ and $H(q)$, each involving a pair of the same Rogers–Ramanujan functions appearing in the identity to be proved. Thus, we have three equations involving the same three Rogers–Ramanujan functions, which we proceed to eliminate from the three equations. There remains then an identity involving $T(q)$ and (usually) theta functions to be proved. It must be emphasized that this method can only be applied if one can find two identities related to the one to be proved. In particular, the method cannot be utilized in those cases where Ramanujan offered only one or two identities of a given degree. The theta function identity to be verified is usually difficult, and generally one should convert it to a modular equation. Hopefully, the modular equation is a known one, in particular, one of the couple hundred that Ramanujan found, but, of course, it may not be. For completeness, we next define a modular equation.

We give the definition of a modular equation, as understood by Ramanujan. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, ℓ, and $\ell' := \sqrt{1-\ell^2}$, respectively, where $0 < k, \ell < 1$. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{4.19}$$

for some positive rational integer n. A relation between k and ℓ induced by (4.19) is called a *modular equation of degree n*. Following Ramanujan, set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = \ell^2.$$

We often say that β has degree n over α. If

$$q = \exp(-\pi K'/K), \tag{4.20}$$

one of the most fundamental relations in the theory of elliptic functions is given by the formula [3, pp. 101–102],

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} =: \frac{2}{\pi}K(k). \tag{4.21}$$

The first equality in (4.21) and elementary theta function identities make it possible to write each modular equation as a theta function identity. (The second equality in (4.21) arises from expanding the integrand in a binomial series and integrating termwise.) Lastly, the multiplier m of degree n is defined by

$$m = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{4.22}$$

Ramanujan derived an extensive "catalogue" of formulas [3, pp. 122–124] giving the "evaluations" of $f(-q)$, $\varphi(q)$, $\psi(q)$, and $\chi(q)$ at various powers of the arguments in terms of

$$z := z_1 := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha), \qquad \alpha, \qquad \text{and} \qquad q.$$

If q is replaced by q^n, then the evaluations are given in terms of

$$z_n := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta), \qquad \beta, \qquad \text{and} \qquad q^n,$$

where β has degree n over α.

In [5], Berndt and Yesilyurt utilized a fourth approach in which $G(q)$ and $H(q)$ are expressed as linear combinations of G and H with arguments q^n for certain positive integers n. Watson [25] discovered the first pair of formulas of this sort, but used them to prove only one of the forty identities. Berndt and Yesilyurt [5] developed several formulas of this kind and employed them in proving over a dozen of the forty identities.

We provide here statements and proofs of the lemmas from [5] that we use in the sequel to establish several of Ramanujan's forty identities. Some of our proofs below *actually use some of Ramanujan's forty identities.* Indeed, some of our arguments are circular. However, in all such instances, we exhibit at least one further proof of each particular entry, which is independent of the other entries. Moreover, our arguments then show that certain pairs of entries are equivalent; for example, Entries 3.7 and 3.12 are equivalent.

We begin with Watson's lemma [25], Lemma 4.1. Watson's proof of (4.23) [25, p. 60] is based on Entries 3.2 and 3.3. Here, we provide a direct proof.

Lemma 4.1. *With* $f(-q)$ *defined by* (2.9),

$$G(q) = \frac{f(-q^8)}{f(-q^2)}\big(G(q^{16}) + qH(-q^4)\big), \tag{4.23}$$

$$H(q) = \frac{f(-q^8)}{f(-q^2)}\big(q^3H(q^{16}) + G(-q^4)\big). \tag{4.24}$$

Proof. Employing (2.18) with q replaced by q^{10} and a replaced by q, we find that

$$\frac{f(-q^2,-q^8)f(-q^{10})}{f(-q,-q^9)} = f(-q^{13},-q^{17}) + qf(-q^7,-q^{23}). \tag{4.25}$$

The left hand side of (4.25), by (2.6) and (2.11), is easily seen to be equal to $f(-q^2)G(q)$, and so we conclude that

$$f(-q^2)G(q) = f(-q^{13},-q^{17}) + qf(-q^7,-q^{23}). \tag{4.26}$$

Similarly replacing q by q^{10}, q^5, q^5, and a by $q^7, -q, -q^2$, respectively, in (2.18) and using (2.6) and (2.11), we find that

$$f(-q^2)H(q) = f(-q^{11},-q^{19}) + q^3f(-q,-q^{29}), \tag{4.27}$$

$$f(-q)G(q^2) = f(q^7,q^8) - qf(q^2,q^{13}), \tag{4.28}$$

and

$$f(-q)H(q^2) = f(q^4,q^{11}) - qf(q,q^{14}). \tag{4.29}$$

Using (2.19) twice with $n=2$, and with $U_n = (-1)^n q^{15n^2-2n}, V_n = (-1)^n q^{15n^2+2n}$ and $U_n = (-1)^n q^{15n^2-8n}, V_n = (-1)^n q^{15n^2+8n}$, respectively, we separate each term on the right side of (4.26) into its even and odd parts and so find that

$$\begin{aligned} f(-q^2)G(q) &= f(q^{56},q^{64}) - q^{13}f(q^4,q^{116}) + q\left(f(q^{44},q^{76}) - q^7f(q^{16},q^{104})\right)\\ &= f(q^{56},q^{64}) - q^8f(q^{16},q^{104}) + q\left(f(q^{44},q^{76}) - q^{12}f(q^4,q^{116})\right)\\ &= f(-q^8)G(q^{16}) + qf(-q^8)H(-q^4), \end{aligned}$$

where in the last step we used (4.28) and (4.27) with q replaced by q^8 and $-q^4$, respectively. This proves (4.23). The related identity (4.24) is proved in a similar way, and so we omit the details. □

Lemma 4.2. *With χ defined by* (2.10),

$$\chi(-q)\chi(q^3)G(q) = \frac{\chi(q^6)}{\chi(-q^4)}G(-q^6) - q^5\frac{\chi(q^2)}{\chi(-q^{12})}H(q^{24}), \tag{4.30}$$

$$\chi(-q)\chi(q^3)H(q) = -q\frac{\chi(q^6)}{\chi(-q^4)}H(-q^6) + \frac{\chi(q^2)}{\chi(-q^{12})}G(q^{24}). \tag{4.31}$$

Proof. By two applications of Entry 3.7, the second with q replaced by $-q$, and by Entry 3.20 with q replaced by q^3,

$$\begin{aligned}
&\frac{\chi(-q^3)}{\chi(-q)}G(-q^3) - \frac{\chi(q^3)}{\chi(q)}G(q^3)\\
&= \left(G(q^2)G(q^3) + qH(q^2)H(q^3)\right)G(-q^3) - \left(G(q^2)G(-q^3) - qH(q^2)H(-q^3)\right)G(q^3)\\
&=qH(q^2)\left\{H(q^3)G(-q^3) + H(-q^3)G(q^3)\right\} = 2q\frac{H(q^2)}{\chi^2(-q^6)},
\end{aligned} \tag{4.32}$$

which, by (2.15), simplifies to

$$\chi(q)\chi(-q^3)G(-q^3) - \chi(-q)\chi(q^3)G(q^3) = 2q\frac{\chi(-q^2)}{\chi^2(-q^6)}H(q^2). \tag{4.33}$$

By employing (4.23) with q replaced by $-q^3$ and q^3, respectively, in (4.33), we find that

$$\begin{aligned}
L(q) :=&\chi(q)\chi(-q^3)\left\{G(q^{48}) - q^3H(-q^{12})\right\} - \chi(-q)\chi(q^3)\left\{G(q^{48}) + q^3H(-q^{12})\right\}\\
=&2q\frac{f(-q^6)\chi(-q^2)}{f(-q^{24})\chi^2(-q^6)}H(q^2) = 2q\chi(-q^2)\chi(q^6)H(q^2),
\end{aligned} \tag{4.34}$$

by (2.14). Collecting terms on the left side of (4.34) and using (11.10) below, we find that

$$\begin{aligned}
L(q) &= \left\{\chi(q)\chi(-q^3) - \chi(-q)\chi(q^3)\right\}G(q^{48}) - q^3\left\{\chi(q)\chi(-q^3) + \chi(-q)\chi(q^3)\right\}H(-q^{12})\\
&=2q\frac{\chi(q^4)}{\chi(-q^{24})}G(q^{48}) - 2q^3\frac{\chi(q^{12})}{\chi(-q^8)}H(-q^{12}).
\end{aligned} \tag{4.35}$$

Hence, by (4.34) and (4.35),

$$2q\frac{\chi(q^4)}{\chi(-q^{24})}G(q^{48}) - 2q^3\frac{\chi(q^{12})}{\chi(-q^8)}H(-q^{12}) = 2q\chi(-q^2)\chi(q^6)H(q^2).$$

Dividing both sides by $2q$ and then replacing q^2 by q, we deduce (4.31). The companion equality (4.30) is proved in a similar way, and so we omit the details. □

Lemma 4.3. *We have*

$$\chi(q)\chi(-q^3)G(q^9) - \chi(-q)\chi(q^3)G(-q^9) = 2q\frac{G(q^4)}{\chi(-q^{18})} \tag{4.36}$$

and

$$\chi(q)\chi(-q^3)H(q^9)+\chi(-q)\chi(q^3)H(-q^9)=2\frac{H(q^4)}{\chi(-q^{18})}. \tag{4.37}$$

Proof. The proofs of (4.36) and (4.37) are very similar to the proofs of (4.30) and (4.31), except that Entry 3.13 is used instead of Entry 3.20. We only prove (4.37), since the proof of (4.36) follows along the same lines.

By two applications of Entry 3.13 and one application of Entry 3.20 with q replaced by q^3,

$$\begin{aligned}&\frac{\chi(q)\chi(-q^3)}{\chi(q^3)\chi(-q^{18})}H(q^9)+\frac{\chi(-q)\chi(-q^3)}{\chi(-q^3)\chi(-q^{18})}H(-q^9)\\ =&\left\{G(-q^9)H(q^4)+qG(q^4)H(-q^9)\right\}H(q^9)+\left\{G(q^9)H(q^4)-qG(q^4)H(q^9)\right\}H(-q^9)\\ =&H(q^4)\left\{G(-q^9)H(q^9)+G(q^9)H(-q^9)\right\}H(q^9)=2\frac{H(q^4)}{\chi^2(-q^{18})}.\end{aligned}$$

Using (2.15) above, we complete the proof of (4.37). □

Lemma 4.4. *If*

$$a(q)=\frac{\chi^2(q)\chi(-q^2)}{\chi(-q^6)} \qquad \textit{and} \qquad b(q)=\frac{\chi(-q)\chi(-q^2)}{\chi(-q^3)\chi(-q^6)}, \tag{4.38}$$

then

$$G(q)=a(q)G(q^6)-qb(q)H(q^4), \tag{4.39}$$

$$H(q)=qa(q)H(q^6)+b(q)G(q^4). \tag{4.40}$$

First Proof of Lemma 4.4. The equality (4.39) can be rewritten in the form

$$\frac{\chi(-q^6)}{\chi(-q^2)}G(q)=-q\frac{\chi(-q)}{\chi(-q^3)}H(q^4)+\chi^2(q)G(q^6). \tag{4.41}$$

When the identities for $\frac{\chi(-q^6)}{\chi(-q^2)}$, $\frac{\chi(-q)}{\chi(-q^3)}$, and $\chi^2(q)$ are substituted from (3.8), (3.9), and (3.3), respectively, it is easy to see that (4.41) is trivially satisfied. The proof of (4.40) follows along the same lines. □

Second Proof of Lemma 4.4. Define

$$B(q):=G(q)+qH(q^4) \qquad \text{and} \qquad qA(q):=-H(q)+G(q^4). \tag{4.42}$$

Let us also define

$$s(q):=\frac{\chi(-q^3)}{\chi(-q)}. \tag{4.43}$$

From the definition (4.42) and (3.3), we see that

$$-q^2A(q)H(q^4)+B(q)G(q^4)=G(q)G(q^4)+qH(q)H(q^4)=\chi^2(q). \tag{4.44}$$

Similarly, by (4.42), (3.8), (3.9), and (4.43), we find that

$$\begin{aligned}
&qA(q)G(q^6) + qB(q)H(q^6) \\
&= -H(q)G(q^6) + qG(q)H(q^6) + \left\{G(q^4)G(q^6) + q^2H(q^4)H(q^6)\right\} \\
&= -\frac{1}{s(q)} + s(q^2). \qquad (4.45)
\end{aligned}$$

By (3.8) and (4.43), we solve for $B(q)$ and $qA(q)$ in (4.44) and (4.45) and find that

$$B(q) = \frac{\chi^2(q)}{s(q^2)}G(q^6) - q\frac{1}{s(q)s(q^2)}H(q^4) + qH(q^4),$$

$$qA(q) = -\frac{1}{s(q)s(q^2)}G(q^4) - q\frac{\chi^2(q)}{s(q^2)}H(q^6) + G(q^4),$$

which, by (4.42), immediately yields (4.39) and (4.40). □

Our fifth approach uses a formula of R. Blecksmith, J. Brillhart, and I. Gerst [9] providing a representation for a product of two theta functions as a sum of m products of pairs of theta functions, under certain conditions. This formula generalizes formulas of H. Schröter [3, pp. 65–72], which have been enormously useful in establishing many of Ramanujan's modular equations [3].

Define, for $\epsilon \in \{0,1\}$ and $|ab| < 1$,

$$f_\epsilon(a,b) = \sum_{n=-\infty}^{\infty} (-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}.$$

Theorem 4.5. *Let a, b, c, and d denote positive numbers with $|ab|, |cd| < 1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^\beta = (cd)^{\alpha(m-\alpha\beta)}. \qquad (4.46)$$

Let $\epsilon_1, \epsilon_2 \in \{0,1\}$, and define $\delta_1, \delta_2 \in \{0,1\}$ by

$$\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 \pmod 2 \qquad \text{and} \qquad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 \pmod 2, \qquad (4.47)$$

respectively, where $p = m - \alpha\beta$. Then, if R denotes any complete residue system modulo m,

$$\begin{aligned}
f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = &\sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^\alpha}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^\alpha}\right) \\
&\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^p}, \frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^p}\right). \qquad (4.48)
\end{aligned}$$

Proof. Setting $s = k - \alpha n$, we find that

$$\begin{aligned}
f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) &= \sum_{n,s=-\infty}^{\infty}(-1)^{\epsilon_1 n+\epsilon_2 s}(ab)^{n^2/2}(a/b)^{n/2}(cd)^{s^2/2}(c/d)^{s/2} \\
&= \sum_{n,k=-\infty}^{\infty}(-1)^{\epsilon_1 n+\epsilon_2(k-\alpha n)}(ab)^{n^2/2}(a/b)^{n/2}(cd)^{(k-\alpha n)^2/2}(c/d)^{(k-\alpha n)/2}.
\end{aligned}$$

Expand into residue classes modulo m and set $k = tm + r, -\infty < t < \infty, r \in R$, to deduce that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{n,t=-\infty}^{\infty}(-1)^{\epsilon_1 n+\epsilon_2(tm+r-\alpha n)} \times (ab)^{n^2/2}(a/b)^{n/2}(cd)^{(tm+r-\alpha n)^2/2}(c/d)^{(tm+r-\alpha n)/2}.$$

Next, setting $n = \ell + \beta t, -\infty < \ell < \infty$, we find that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{\ell,t=-\infty}^{\infty}(-1)^{\epsilon_1(\ell+\beta t)+\epsilon_2(tm+r-\alpha(\ell+\beta t))} \times (ab)^{(\ell+\beta t)^2/2}(a/b)^{(\ell+\beta t)/2}(cd)^{(tm+r-\alpha(\ell+\beta t))^2/2}(c/d)^{(tm+r-\alpha(\ell+\beta t))/2}.$$

Recalling that $p = m - \alpha\beta$ and noting that $tm + r - \alpha(\ell + \beta t) = tp + r - \alpha\ell$, we find that

$$f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) = \sum_{r\in R}\sum_{\ell,t=-\infty}^{\infty}(-1)^{\epsilon_1(\ell+\beta t)}(-1)^{\epsilon_2(tp+r-\alpha\ell)} \times (ab)^{(\ell+\beta t)^2/2}(a/b)^{(\ell+\beta t)/2}(cd)^{(tp+r-\alpha\ell)^2/2}(c/d)^{(tp+r-\alpha\ell)/2}.$$

Now, by (4.46) and the definition $p = m - \alpha\beta$, we find that

$$\begin{aligned}(ab)^{\beta(\ell t+\beta t^2/2)}(cd)^{t^2p^2/2-tp\alpha\ell} &= (cd)^{\alpha p(\ell t+\beta t^2/2)}(cd)^{t^2p^2/2-tp\alpha\ell}\\ &= (cd)^{t^2p(\alpha\beta+p)/2}\\ &= (cd)^{t^2pm/2}.\end{aligned}$$

Hence, recalling the definitions of δ_1 and δ_2 from (4.47), we find that

$$\begin{aligned}f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) &= \sum_{r\in R}\sum_{\ell,t=-\infty}^{\infty}(-1)^{\delta_1\ell}(-1)^{\delta_2 t}(-1)^{\epsilon_2 r}(cd)^{r^2/2}(c/d)^{r/2}\\ &\times \left(ab(cd)^{\alpha^2}\right)^{\ell^2/2}\left(\frac{a}{b}\left(\frac{c}{d}\right)^{-\alpha}(cd)^{-2r\alpha}\right)^{\ell/2}((cd)^{mp})^{t^2/2}\left(\left(\frac{a}{b}\right)^{\beta}\left(\frac{c}{d}\right)^{p}(cd)^{2pr}\right)^{t/2}\\ &= \sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2}f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^{\alpha}}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^{\alpha}}\right)\\ &\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{(m-\alpha\beta)(m+1-2r)/2}}{c^{m-\alpha\beta}}, \frac{(a/b)^{\beta/2}(cd)^{(m-\alpha\beta)(m+1+2r)/2}}{d^{m-\alpha\beta}}\right),\end{aligned}$$

after some elementary algebra and elementary manipulation. □

5. Proof of Entry 3.1

We begin by proving the following identity from Chapter 19 of Ramanujan's second notebook [20], [3, p. 80, Entry 38(iv); p. 262, Entry 10(iii)]. Our proof is taken from [3, pp. 81–82].

Lemma 5.1. *We have*

$$f^2(-q^2,-q^3)-q^{2/5}f^2(-q,-q^4)=f(-q)\left\{f(-q^{1/5})+q^{1/5}f(-q^5)\right\}. \tag{5.1}$$

Proof. Apply (2.19) with $a=-q$, $b=-q^2$, and $n=5$. Then

$$U_n=(-1)^nq^{n(3n-1)/2} \qquad \text{and} \qquad V_n=(-1)^nq^{n(3n+1)/2}.$$

Thus, by (2.9) and (2.19),

$$\begin{aligned} f(-q)=f(-q,-q^2)=&f(-q^{35},-q^{40})-qf(-q^{50},-q^{25})+q^5f(-q^{65},-q^{10})\\ &-q^{12}f(-q^{80},-q^{-5})+q^{22}f(-q^{95},-q^{-20})\\ =&-qf(-q^{25})+\left\{f(-q^{35},-q^{40})+q^5f(-q^{10},-q^{65})\right\}\\ &-q^2\left\{f(-q^{20},-q^{55})+q^{10}f(-q^{-5},-q^{80})\right\}, \end{aligned} \tag{5.2}$$

where we applied (2.5). We now invoke the quintuple product identity (2.18) twice, with q replaced by q^{25} and $a=q^5,q^{10}$, respectively. We therefore find that (5.2) can be written as

$$f(-q)+qf(-q^{25})=f(-q^{25})\left\{\frac{f(-q^{15},-q^{10})}{f(-q^{20},-q^5)}-q^2\frac{f(-q^5,-q^{20})}{f(-q^{15},-q^{10})}\right\}. \tag{5.3}$$

By (2.6),

$$f(-q,-q^4)f(-q^2,-q^3)=f(-q)f(-q^5). \tag{5.4}$$

Multiplying both sides of (5.3) by $f(-q)$, but with q replaced by $q^{1/5}$, and using (5.4), we deduce that

$$\begin{aligned} f(-q)\left\{f(-q^{1/5})+q^{1/5}f(-q^5)\right\}=&f(-q)f(-q^5)\left\{\frac{f(-q^2,-q^3)}{f(-q,-q^4)}-q^{2/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}\right\}\\ =&f^2(-q^2,-q^3)-q^{2/5}f^2(-q,-q^4), \end{aligned}$$

which completes the proof. □

Proof of Entry 3.1. Replace $q^{1/5}$ by $q^{1/5}\zeta$ in (5.1), where ζ is a fifth root of unity, and multiply all five identities together. We then find that

$$f^5(-q)\prod_{\zeta}f(-q^{1/5}\zeta)=\prod_{\zeta}\left\{f^2(-q^2,-q^3)-q^{2/5}\zeta^2f^2(-q,-q^4)-q^{1/5}\zeta f(-q)f(-q^5)\right\}. \tag{5.5}$$

Multiplying out the products on each side of (5.5), we find that

$$\begin{aligned} \frac{f^{11}(-q)}{f(-q^5)}=&f^{10}(-q^2,-q^3)-q^2f^{10}(-q,-q^4)-qf^5(-q)f^5(-q^5)\\ &-5qf^2(-q^2,-q^3)f^2(-q,-q^4)f^3(-q)f^3(-q^5)\\ &-5qf^4(-q^2,-q^3)f^4(-q,-q^4)f(-q)f(-q^5). \end{aligned} \tag{5.6}$$

By (5.4), (5.6) simplifies to

$$\begin{aligned}\frac{f^{11}(-q)}{f(-q^5)} =&f^{10}(-q^2,-q^3) - q^2 f^{10}(-q,-q^4) - qf^5(-q)f^5(-q^5)\\ &- 5qf^5(-q)f^5(-q^5) - 5qf^5(-q)f^5(-q^5)\\ =&f^{10}(-q^2,-q^3) - q^2 f^{10}(-q,-q^4) - 11qf^5(-q)f^5(-q^5). \qquad (5.7)\end{aligned}$$

Multiplying both sides of (5.7) by $f(-q^5)/f^{11}(-q)$, using (5.4), and lastly employing (2.11), we conclude that

$$\begin{aligned}1 =&\frac{f^{11}(-q^2,-q^3)f(-q,-q^4)}{f^{12}(-q)} - q^2\frac{f^{11}(-q,-q^4)f(-q^2,-q^3)}{f^{12}(-q)}\\ &- 11q\frac{f^6(-q,-q^4)f^6(-q^2,-q^3)}{f^{12}(-q)}\\ =&G^{11}(q)H(q) - q^2H^{11}(q)G(q) - 11qG^6(q)H^6(q),\end{aligned}$$

which completes the proof of Entry 3.1. □

6. Proofs of Entry 3.2

First Proof of Entry 3.2. Using (4.23) and (4.24) in (3.6), we find that

$$\begin{aligned}\chi(q^2) =&G(q^{16})H(q) - q^3H(q^{16})G(q)\\ =&\frac{f(-q^8)}{f(-q^2)}\Big\{G(q^{16})\big(q^3H(q^{16}) + G(-q^4)\big) - q^3H(q^{16})\big(G(q^{16}) + H(-q^4)\big)\Big\}\\ =&\frac{f(-q^8)}{f(-q^2)}\big\{G(-q^4)G(q^{16}) - q^4H(-q^4)H(q^{16})\big\}.\end{aligned}$$

Therefore, by (2.14) and (2.16), we deduce that

$$G(-q^4)G(q^{16}) - q^4H(-q^4)H(q^{16}) = \frac{f(-q^2)}{f(-q^8)}\chi(q^2) \quad = \frac{\chi(-q^2)f(-q^4)\chi(q^2)}{\dfrac{f(-q^4)}{\chi(-q^4)}} = \chi^2(-q^4),$$

which is Entry 3.2 with q replaced by $-q^4$. □

Our second proof is a moderate modification of the proof given by Watson [25].

Second Proof of Entry 3.25. Using (2.11), we may write (3.3) in the alternative form

$$f(-q^2,-q^3)f(-q^8,-q^{12}) + qf(-q,-q^4)f(-q^4,-q^{16}) = f(-q)f(-q^4)\chi^2(q). \qquad (6.1)$$

Using elementary product manipulations and (2.10), (2.8), and (2.7), we find that (6.1) can be rewritten as

$$f(-q^2,-q^3)f(-q^8,-q^{12}) + qf(-q,-q^4)f(-q^4,-q^{16}) = \psi(q)\varphi(-q^2). \qquad (6.2)$$

We prove (6.2).

By (2.7) and (2.8),

$$2\psi(q)\varphi(-q^2) = \sum_{m,n=-\infty}^{\infty} (-1)^n q^{m(m+1)/2+2n^2}. \qquad (6.3)$$

Set

$$m + n = 5M + a \qquad \text{and} \qquad m - 4n = 5N + b,$$

where a and b have values from the set $\{0, \pm 1, \pm 2\}$. Hence,

$$m = 4M + N + (4a + b)/5 \qquad \text{and} \qquad n = M - N + (a - b)/5.$$

It follows that $a = b$, and so $m = 4M + N + a$ and $n = M - N$, where $-2 \le a \le 2$. Then there is a one-to-one correspondence between the set of all pairs of integers (m, n), $-\infty < m, n < \infty$, and the triples of integers (M, N, a), $-\infty < M, N < \infty$, $-2 \le a \le 2$. From (6.3), we then find that

$$\begin{aligned}
2\psi(q)\varphi(-q^2) &= \sum_{a=-2}^{2} q^{a(a+1)/2} \sum_{M=-\infty}^{\infty} (-1)^M q^{10M^2+2M+4Ma} \sum_{N=-\infty}^{\infty} (-1)^N q^{5N^2/2+N+Na} \\
&= \sum_{a=-2}^{2} q^{a(a+1)/2} f(-q^{12+4a}, -q^{8-4a}) f(-q^{3+a}, -q^{2-a}) \\
&= 2f(-q^{12}, -q^8) f(-q^3, -q^2) + 2q f(-q^{16}, -q^4) f(-q^4, -q) \\
&\quad + q^3 f(-q^{20}, -1) f(-q^5, -1) \\
&= 2f(-q^{12}, -q^8) f(-q^3, -q^2) + 2q f(-q^{16}, -q^4) f(-q^4, -q), \qquad (6.4)
\end{aligned}$$

by (2.4). We immediately see that (6.2) has been proved.

The second equality of (3.3) follows immediately from the product representations of $\chi(q)$, $\varphi(q)$, and $f(-q^2)$ in (2.10), (2.7), and (2.9), respectively. $\square$

In his lost notebook [21, p. 27], Ramanujan offers the beautiful identity

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(q^4;q^4)_n} \sum_{n=0}^{\infty} \frac{a^{-2n} q^{4n^2}}{(bq^4;q^4)_n} + \sum_{n=0}^{\infty} \frac{a^n b^n q^{(n+1)^2}}{(q^4;q^4)_n} \sum_{n=0}^{\infty} \frac{a^{-2n-1} q^{4n^2+4n}}{(bq^4;q^4)_n} \\
= \frac{f(aq, q/a)}{(bq^4;q^4)_\infty} - (1-b) \sum_{n=0}^{\infty} a^{n+1} q^{(n+1)^2} \sum_{j=0}^{n} \frac{b^j}{(q^4;q^4)_j}. \qquad (6.5)
\end{aligned}$$

If we set $a = b = 1$ in (6.5) and multiply both sides by $(-q^2;q^2)_\infty$, we see that (6.5) reduces to

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} G(q^4) + (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^4;q^4)_n} H(q^4) = \frac{\varphi(q)}{f(-q^2)}. \qquad (6.6)$$

However, Rogers [22] proved that

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = G(q)$$

and

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = H(q),$$

and so (6.6) reduces to (3.3). A proof of (6.5) has been given by G. E. Andrews [1, pp. 28–33].

7. Proof of Entry 3.3

Entry 3.3 follows from combining Entry 3.2 with the following lemma.

Lemma 7.1. *We have*

$$\varphi^2(q) - \varphi^2(q^5) = 4qf^2(-q^{10})\frac{\chi(q)}{\chi(q^5)}. \tag{7.1}$$

Proof. By Entry 10(iv) in Chapter 19 of Ramanujan's second notebook [20], [3, p. 262] and the Jacobi triple product identity (2.6),

$$\begin{aligned}\varphi^2(q) - \varphi^2(q^5) =& 4qf(q,q^9)f(q^3,q^7)\\ =&4q(-q;q^{10})_\infty(-q^9;q^{10})_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^{10};q^{10})^2_\infty\\ =&4qf^2(-q^{10})\frac{(-q;q^2)_\infty}{(-q^5;q^{10})_\infty}\\ =&4qf^2(-q^{10})\frac{\chi(q)}{\chi(q^5)},\end{aligned}$$

and the proof is complete. □

The identity (7.1) is an analogue of

$$\varphi^2(q) - 5\varphi^2(q^5) = -4f^2(-q^2)\frac{\chi(q^5)}{\chi(q)},$$

which is found in Ramanujan's lost notebook [21] and was first proved by S.–Y. Kang [14, Thm. 2.2(i)].

Proof of Entry 3.3. The proof which follows is due to Watson [25]. From Entry 3.2, (2.11), and (5.4), we find that

$$\begin{aligned}\left\{G(q)G(q^4) - qH(q)H(q^4)\right\}^2 &= \left\{G(q)G(q^4) + qH(q)H(q^4)\right\}^2 - 4qG(q)H(q)G(q^4)H(q^4)\\ &=\frac{\varphi^2(q)}{f^2(-q^2)} - 4q\frac{f(-q^5)f(-q^{10})}{f(-q)f(-q^4)}. \end{aligned} \tag{7.2}$$

A straightforward calculation shows that

$$\chi(q) = \frac{f^2(-q^2)}{f(-q)f(-q^4)}. \tag{7.3}$$

Using (7.3) twice, we find that (7.2) can be written in the form

$$\left\{G(q)G(q^4) - qH(q)H(q^4)\right\}^2 = \frac{\varphi^2(q) - 4qf^2(-q^{10})\dfrac{\chi(q)}{\chi(q^5)}}{f^2(-q^2)} = \frac{\varphi^2(q^5)}{f^2(-q^2)},$$

where we applied Lemma 7.1. Taking the square root of both sides above, we complete the proof. □

8. Proof of Entry 3.4

By employing (2.11), we easily find that the proposed identity is equivalent to the identity

$$f(-q,-q^4)f(-q^{22},-q^{33}) - q^2 f(-q^2,-q^3)f(-q^{11},-q^{44}) = f(-q)f(-q^{11}). \tag{8.1}$$

To prove Entry (8.1), we apply the ideas of Rogers, in particular, (4.13) with the two sets of parameters $\alpha_1 = 1, \beta_1 = 11, m_1 = 3, p_1 = 5, \lambda_1 = 4$ and $\alpha_2 = 1, \beta_2 = 11, m_2 = 1, p_2 = 3, \lambda_2 = 4$. The requisite conditions (4.9) are readily seen to be satisfied. Using (4.16) and (4.17), we derive the identity

$$f(-q^2,-q^8)f(-q^{44},-q^{66}) - q^4 f(-q^4,-q^6)f(-q^{22},-q^{88}) = f(-q^2)f(-q^{22}),$$

which is the same as (8.1), but with q replaced by q^2.

9. Proof of Entry 3.5

The proof of Entry 3.5 is very similar to that for Entry 3.22 below. In fact, we reduce the desired equality to the same new modular equation (25.11) of degree 5. Remarkably, Ramanujan derived 27 modular equations of degree 5, although several are "reciprocals" of others [3, pp. 280–282, Entry 13]. In Ramanujan's terminology, let β have degree 5 over α.

Lemma 9.1. *If β has degree 5 over α, then*

$$(1-\beta)^{1/4} - (1-\alpha)^{1/4} = 2^{2/3}(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/24}. \tag{9.1}$$

Proof. Let

$$m = \frac{\varphi^2(q)}{\varphi^2(q^5)}$$

denote the *multiplier* of degree 5. As in [3, p. 284, eq. (13.3)], define

$$\rho := \sqrt{m^3 - 2m^2 + 5m}. \tag{9.2}$$

We shall need the following parameterizations for certain algebraic functions of α and β, namely [3, pp. 285–286, eqs. (13.8), (13.10), (13.11)],

$$\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = \frac{(m-1)(5-m)}{4m}, \tag{9.3}$$

$$\left\{(1-\alpha)^3(1-\beta)\right\}^{1/8} = \frac{\rho - 3m + 5}{4m}, \tag{9.4}$$

and

$$\left\{(1-\alpha)(1-\beta)^3\right\}^{1/8} = \frac{\rho - m^2 + 3m}{4m}, \tag{9.5}$$

where ρ is defined by (9.2). Using (9.3)–(9.5), we find that

$$\begin{aligned}\frac{(1-\beta)^{1/4}-(1-\alpha)^{1/4}}{2^{2/3}(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/24}} &= \frac{\left\{(1-\alpha)(1-\beta)^3\right\}^{1/8}-\left\{(1-\alpha)^3(1-\beta)\right\}^{1/8}}{2^{2/3}\left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/6}}\\ &= \frac{(\rho-m^2+3m)-(\rho-3m+5)}{(m-1)(5-m)}\\ &= \frac{m^2-6m+5}{(m-1)(m-5)}=1,\end{aligned}$$

which completes the proof. □

We begin the proof of Entry 3.5 with the system of equations

$$\begin{aligned}G(q^{16})H(q)-q^3G(q)H(q^{16}) &=: T(q),\\ G(q^{16})G(q^4)+q^4H(q^4)H(q^{16}) &= \frac{\varphi(q^4)}{f(-q^8)},\\ G(q^{16})G(q^4)-q^4H(q^4)H(q^{16}) &= \frac{\varphi(q^{20})}{f(-q^8)}.\end{aligned}$$

From the first equation above, we see that our task is to prove that $T(q)=\chi(q^2)$. The second and third equations are simply (3.3) and (3.4), respectively, but with q replaced by q^4. Regarding this system in the variables $G(q^{16})$, $q^3H(q^{16})$, and -1, we see that

$$\begin{vmatrix} H(q) & -G(q) & T(q)\\ G(q^4) & qH(q^4) & \dfrac{\varphi(q^4)}{f(-q^8)}\\ G(q^4) & -qH(q^4) & \dfrac{\varphi(q^{20})}{f(-q^8)}\end{vmatrix}=0. \tag{9.6}$$

Expanding the determinant in (9.6) along the last column, we find that

$$\begin{aligned}-2qG(q^4)H(q^4)T(q)-&\frac{\varphi(q^4)}{f(-q^8)}\left\{G(q)G(q^4)-qH(q)H(q^4)\right\}\\ &+\frac{\varphi(q^{20})}{f(-q^8)}\left\{G(q)G(q^4)+qH(q)H(q^4)\right\}=0.\end{aligned} \tag{9.7}$$

Using (2.12), (3.3), (3.4), and

$$\frac{f(-q^4)}{f(-q^8)}=\chi(-q^4)=\chi(-q^2)\chi(q^2), \tag{9.8}$$

which arises from (2.15), we can write (9.7) in the form

$$-2q\frac{f(-q^{20})}{f(-q^4)}T(q)-\frac{\varphi(q^4)}{f(-q^8)}\frac{\varphi(q^5)}{f(-q^2)}+\frac{\varphi(q^{20})}{f(-q^8)}\frac{\varphi(q)}{f(-q^2)}=0. \tag{9.9}$$

Rearranging (9.9) while using (9.8), we find that

$$2qT(q)=\frac{\chi(-q^2)\chi(q^2)}{f(-q^2)f(-q^{20})}\left\{\varphi(q)\varphi(q^{20})-\varphi(q^5)\varphi(q^4)\right\}. \tag{9.10}$$

Recall the representations [3, pp. 122–124, Entries 10(i), (v), (ii), 11(v), 12(iii), (iv), (viii)]

$$\varphi(q)=\sqrt{z_1}, \qquad \varphi(q^4)=\tfrac{1}{2}\sqrt{z_1}\left\{1+(1-\alpha)^{1/4}\right\}, \tag{9.11}$$

$$\varphi(-q)=\sqrt{z_1}(1-\alpha)^{1/4}, \qquad \psi(q^8)=\frac{1}{4q}\sqrt{z_1}\left\{1-(1-\alpha)^{1/4}\right\}, \tag{9.12}$$

$$f(-q^2)=\sqrt{z_1}2^{-1/3}\left(\frac{\alpha(1-\alpha)}{q}\right)^{1/12}, \qquad f(-q^4)=\sqrt{z_1}2^{-2/3}\left(\frac{(1-\alpha)\alpha^4}{q^4}\right)^{1/24}, \tag{9.13}$$

and

$$\chi(-q^2)=2^{1/3}\left(\frac{(1-\alpha)q^2}{\alpha^2}\right)^{1/24}, \tag{9.14}$$

where

$$z_n:=\varphi(q^n).$$

Recall from the theory of modular equations that, if n is the degree of the modular equation, then (9.13) also holds with z_1, q, and α replaced by z_n, q^n, and β, respectively, where β has degree n over α. Hence, from (9.13) and (9.14), we find that, after simplification,

$$\frac{\chi(-q^2)}{f(-q^2)f(-q^{20})}=\frac{2^{4/3}q}{\sqrt{z_1z_5}(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}}. \tag{9.15}$$

Employing (9.15) and (9.11) in (9.10), we deduce that

$$\begin{aligned}
2qT(q)=&\frac{\chi(q^3)2^{4/3}q}{\sqrt{z_1z_5}(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}}\\
&\times\sqrt{z_1z_5}\left\{\tfrac{1}{2}\left\{1+(1-\beta)^{1/4}\right\}-\tfrac{1}{2}\left\{1+(1-\alpha)^{1/4}\right\}\right\}\\
=&\frac{\chi(q^2)2^{1/3}q\left\{(1-\beta)^{1/4}-(1-\alpha)^{1/4}\right\}}{(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}}\\
=&2q\chi(q^2),
\end{aligned} \tag{9.16}$$

by Lemma 9.1. Equation (9.16) is trivially equivalent to (3.6), and so the proof is complete.

10. Proofs of Entry 3.6

First Proof of Entry 3.6. By using (2.11), we find that in order to prove Entry 3.6, it suffices to prove that

$$f(-q^2,-q^3)f(-q^{18},-q^{27})+q^2f(-q,-q^4)f(-q^9,-q^{36})=f^2(-q^3). \tag{10.1}$$

We apply (4.13) with $\alpha_1=1,\beta_1=9,m_1=1,p_1=5,\lambda_1=2$ and with $\alpha_2=3,\beta_2=3,m_2=1,p_2=3,\lambda_2=2$. We easily check that these two sets of parameters satisfy conditions (4.9). From (4.13) and (4.15), we then deduce the identity

$$f(-q^4,-q^6)f(-q^{36},-q^{54})+q^4f(-q^2,-q^8)f(-q^{18},-q^{72})=f^2(-q^6),$$

which is precisely (10.1), but with q replaced by q^2. This then completes the proof of Entry 3.6. □

Second Proof of Entry 3.6. We rewrite (10.1) in the form

$$\sum_{\substack{m=-\infty \\ m\equiv 0 \pmod 3}}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{(5m^2-3m+5n^2-n)/2} + q^2 \sum_{\substack{m=-\infty \\ m\equiv 0 \pmod 3}}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{(5m^2-9m+5n^2-3n)/2}$$
$$= \sum_{\substack{m,n=-\infty \\ m\equiv 0, n\equiv 0 \pmod 3}}^{\infty} (-1)^{m+n} q^{(m^2-m+n^2-n)/2} := F(q). \tag{10.2}$$

Now, for $(a, b) \in \{0, \pm 1, \pm 2\}$, set

$$2m + n = 5M + a \qquad \text{and} \qquad -m + 2n = 5N + b.$$

Hence,

$$m = 2M - N + (2a - b)/5 \qquad \text{and} \qquad n = M + 2N + (a + 2b)/5,$$

where the parameters a and b are given in the first table below. The corresponding values of m and n are given in the table which follows.

a	0	± 1	± 2
b	0	± 2	∓ 1

m	$2M - N$	$2M - N$	$2M - N$	$2M - N + 1$	$2M - N - 1$
n	$M + 2N$	$M + 2N + 1$	$M + 2N - 1$	$M + 2N$	$M + 2N$

Recalling that $m, n \equiv 0 \pmod 3$, for the five cases in the table above, we find that, respectively,

$$M \equiv N \equiv 0 \pmod 3, \qquad M \equiv 1, N \equiv -1 \pmod 3, \qquad M \equiv -1, N \equiv 1 \pmod 3,$$
$$M \equiv N \equiv -1 \pmod 3, \qquad M \equiv N \equiv 1 \pmod 3.$$

Calculating the corresponding values of $m^2 + n^2 - m - n$, we find that

$$F(q) = \sum_{\substack{M,N=-\infty \\ M\equiv 0, N\equiv 0 \pmod 3}}^{\infty} (-1)^{M+N} q^{(5M^2-3M+5N^2-N)/2}$$
$$+ \sum_{\substack{M,N=-\infty \\ M\equiv 1, N\equiv -1 \pmod 3}}^{\infty} (-1)^{M+N+1} q^{(5M^2-M+5N^2+3N)/2}$$

$$+\sum_{\substack{M,N=-\infty\\M\equiv-1,N\equiv1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N+1}q^{(5M^2-5M+5N^2-5N+2)/2}$$
$$+\sum_{\substack{M,N=-\infty\\M\equiv-1,N\equiv-1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N+1}q^{(5M^2+M+5N^2-3N)/2}$$
$$+\sum_{\substack{M,N=-\infty\\M\equiv1,N\equiv1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N-1}q^{(5M^2-7M+5N^2+N+2)/2}$$
$$=:S_1+S_2+S_3+S_4+S_5. \tag{10.3}$$

First, setting $M=3m-1$, we find that

$$\sum_{\substack{M=-\infty\\M\equiv-1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^Mq^{5(M^2-M)/2}=-\sum_{m=-\infty}^{\infty}(-1)^mq^{45m(m-1)/2}=-f(-1,q^{45})=0,$$

by (2.4). Hence,

$$S_3=0. \tag{10.4}$$

Replacing M by $M+1$, and then changing the signs of M and N, we readily find that

$$S_5=\sum_{\substack{M,N=-\infty\\M\equiv0,N\equiv-1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N}q^{(5M^2+5N^2-3M-N)/2}.$$

Changing the signs of M and N and replacing M by $M+1$, we deduce that

$$S_2=q^2\sum_{\substack{M,N=-\infty\\M\equiv0,N\equiv1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N}q^{(5M^2+5N^2-9M-3N)/2}.$$

Replacing M by $M-1$, we easily see that

$$S_4=q^2\sum_{\substack{M,N=-\infty\\M\equiv0,N\equiv-1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N}q^{(5M^2+5N^2-9M-3N)/2}.$$

Hence,

$$S_1+S_5=\sum_{\substack{M,N=-\infty\\M\equiv0,N\equiv0,1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N}q^{(5M^2+5N^2-3M-N)/2} \tag{10.5}$$

and

$$S_2+S_4=\sum_{\substack{M,N=-\infty\\M\equiv0,N\equiv\pm1\,(\mathrm{mod}\,3)}}^{\infty}(-1)^{M+N}q^{(5M^2+5N^2-9M-3N)/2}. \tag{10.6}$$

Substituting (10.5)–(10.4) in (10.3) and comparing this with (10.2), we see that in order prove (10.2), we need to show that

$$\sum_{\substack{m,n=-\infty\\ m\equiv 0, n\equiv 1 \pmod 3}}^{\infty}(-1)^{m+n}q^{(5m^2+5n^2-3m-n)/2} + q^2\sum_{\substack{m,n=-\infty\\ m,n\equiv 0 \pmod 3}}^{\infty}(-1)^{m+n}q^{(5m^2+5n^2-9m-3n)/2}=0. \tag{10.7}$$

If we set $m=3k$, $n=3l+1$ and $m=-3l$, $n=3k$ in the first and second sums of (10.7), respectively, we easily deduce (10.7), and so the proof is complete. ☐

Entry 3.6 is a natural companion to Entry 3.13; in Section 17, a third proof of Entry 3.6 will be concomitantly given with a proof of Entry 3.13.

11. Proofs of Entry 3.7

First Proof of Entry 3.7. Using (2.11), we can write (3.8) in the alternative form

$$f(-q^4,-q^6)f(-q^6,-q^9)+qf(-q^2,-q^8)f(-q^3,-q^{12})=f(-q^2)f(-q^3)\frac{\chi(-q^3)}{\chi(-q)}. \tag{11.1}$$

Using

$$\chi(-q^3)=\frac{\varphi(-q^3)}{f(-q^3)} \qquad \text{and} \qquad \chi(-q)=\frac{f(-q^2)}{\psi(q)} \tag{11.2}$$

from (2.14), we rewrite (11.1) as

$$f(-q^4,-q^6)f(-q^6,-q^9)+qf(-q^2,-q^8)f(-q^3,-q^{12})=\psi(q)\varphi(-q^3). \tag{11.3}$$

For a and b in the set $\{0,\pm1,\pm2\}$, let

$$m+3n=5M+a \qquad \text{and} \qquad m-2n=5N+b,$$

from which it follows that

$$n=M-N+(a-b)/5 \qquad \text{and} \qquad m=2M+3N+(2a+3b)/5.$$

It follows easily that $a=b$, and so $m=2M+3N+a$ and $n=M-N$, where $-2\le a\le 2$. Thus, there is a one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty<m,n<\infty$, and triples of integers (M,N,a), $-\infty<M,N<\infty$, $-2\le a\le 2$.From the definitions (2.8) and (2.7) of $\psi(q)$ and $\varphi(-q^3)$, the indicated changes of indices of summation, and (2.4),

$$\begin{aligned}
2\psi(q)\varphi(-q^3) =& \sum_{m,n=-\infty}^{\infty}(-1)^n q^{m(m+1)/2+3n^2}\\
=& \sum_{a=-2}^{2} q^{a(a+1)/2}\sum_{M=-\infty}^{\infty}(-1)^M q^{5M^2+(1+2a)M}\sum_{N=-\infty}^{\infty}(-1)^N q^{15N^2/2+3N/2+3aN}\\
=& \sum_{a=-2}^{2} q^{a(a+1)/2} f(-q^{4-2a},-q^{6+2a})f(-q^{6-3a},-q^{9+3a})\\
=&2f(-q^4,-q^6)f(-q^6,-q^9)+2qf(-q^2,-q^8)f(-q^3,-q^{12})\\
&+q^3f(-1,-q^{10})f(-1,-q^{15})\\
=&2f(-q^4,-q^6)f(-q^6,-q^9)+2qf(-q^2,-q^8)f(-q^3,-q^{12}),
\end{aligned}$$

which is (11.3). So we complete our proof. □

Second Proof of Entry 3.7. Using (4.23) and (4.24) in (3.13), we arrive at

$$\begin{aligned}
\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)} =&G(q)G(q^{24})+q^5H(q)H(q^{24})\\
=&\frac{f(-q^8)}{f(-q^2)}\Big\{G(q^{24})\big(G(q^{16})+qH(-q^4)\big)+q^5H(q^{24})\big(q^3H(q^{16})+G(-q^4)\big)\Big\}\\
=&\frac{f(-q^8)}{f(-q^2)}\Big\{G(q^{16})G(q^{24})+q^8H(q^{16})H(q^{24})\\
&+q\big(H(-q^4)G(q^{24})+q^4G(-q^4)H(q^{24})\big)\Big\}. \qquad (11.4)
\end{aligned}$$

By several applications of (2.14) , we deduce from (11.4) that

$$\begin{aligned}
&G(q^{16})G(q^{24})+q^8H(q^{16})H(q^{24})+q\big(H(-q^4)G(q^{24})+q^4G(-q^4)H(q^{24})\big)\\
&=\frac{\chi(-q^3)\chi(-q^{12})f(-q^2)}{\chi(-q)\chi(-q^4)f(-q^8)}=\chi(q)\chi(-q^3)\chi(-q^{12}). \qquad (11.5)
\end{aligned}$$

Therefore, it suffices to find the even and the odd parts of $\chi(q)\chi(-q^3)$. By (2.6), (2.8), and (2.17),

$$\begin{aligned}
f(-q,-q^5) =&(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty\\
=&\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty}(q^6;q^6)_\infty\\
=&\chi(-q)\psi(q^3)=\chi(-q)\chi(q^3)f(-q^{12}). \qquad (11.6)
\end{aligned}$$

Employing (2.19) with $a=q$ and $b=q^5$, we also have

$$f(q,q^5)=f(q^8,q^{16})+qf(q^4,q^{20}). \qquad (11.7)$$

It is also easily verified that (see [3, p. 350, eq. (2.3)])

$$f(q,q^2)=\frac{\varphi(-q^3)}{\chi(-q)}. \tag{11.8}$$

Therefore, by (11.6), (11.7), and (11.8), we find that

$$\begin{aligned}\chi(q)\chi(-q^3)=\frac{f(q,q^5)}{f(-q^{12})}&=\frac{f(q^8,q^{16})}{f(-q^{12})}+q\frac{f(q^4,q^{20})}{f(-q^{12})}\\&=\frac{\varphi(-q^{24})}{\chi(-q^8)f(-q^{12})}+q\frac{\chi(q^4)\chi(-q^{12})f(-q^{48})}{f(-q^{12})}.\end{aligned} \tag{11.9}$$

Next, by several applications of (2.14), we deduce from (11.9) that

$$\chi(q)\chi(-q^3)=\frac{\chi(q^{12})}{\chi(-q^8)}+q\frac{\chi(q^4)}{\chi(-q^{24})}. \tag{11.10}$$

Therefore, by (11.5), (11.10), and (2.16),

$$\begin{aligned}&G(q^{16})G(q^{24})+q^8H(q^{16})H(q^{24})+q\big(H(-q^4)G(q^{24})+q^4G(-q^4)H(q^{24})\big)\\&=\frac{\chi(q^{12})\chi(-q^{12})}{\chi(-q^8)}+q\frac{\chi(q^4)\chi(-q^{12})}{\chi(-q^{24})}=\frac{\chi(-q^{24})}{\chi(-q^8)}+q\frac{\chi(q^4)}{\chi(q^{12})}.\end{aligned} \tag{11.11}$$

Equating the even parts on both sides of the equation (11.11), we obtain Entry 3.7 with q replaced by q^8. Similarly, equating the even parts gives Entry 3.8 with q replaced by $-q^4$. □

12. Proof of Entry 3.8

We have just given a proof of Entry 3.8 along with one of our proofs of Entry 3.7. We provide a second proof here.

If we use (2.11), we can put Entry 3.8 in the form

$$f(-q^{12},-q^{18})f(-q,-q^4)-qf(-q^2,-q^3)f(-q^6,-q^{24})=\frac{\chi(-q)}{\chi(-q^3)}f(-q)f(-q^6). \tag{12.1}$$

Using the same equalities (11.2) that we employed to prove Entry 3.7, we can rewrite (12.1) as

$$f(-q^{12},-q^{18})f(-q,-q^4)-qf(-q^2,-q^3)f(-q^6,-q^{24})=\psi(q^3)\varphi(-q). \tag{12.2}$$

In the representation,

$$2\psi(q^3)\varphi(-q)=f(1,q^3)f(-q,-q)=\sum_{m,n=-\infty}^{\infty}(-1)^nq^{(3m^2+3m+2n^2)/2}, \tag{12.3}$$

we make the change of indices

$$3m-2n=5M+a \qquad \text{and} \qquad m+n=5N+b, \tag{12.4}$$

where a and b have values selected from the integers 0, ±1, ±2. Since

$$m=M+2N+(a+2b)/5 \qquad \text{and} \qquad n=-M+3N+(3b-a)/5, \tag{12.5}$$

we see that values of a and b are associated as in the following table:

a	0	±1	±2
b	0	±2	±1

Thus, there is a one-one correspondence between all pairs of integers (m,n) and all sets of integers (M,N,a) as given above. We therefore deduce from (12.3), (2.4), and (2.5) that

$$\begin{aligned}2\psi(q^3)\varphi(-q) &= \sum_{m,n=-\infty}^{\infty}(-1)^n q^{(3m^2+3m+2n^2)/2}\\ &= -qf(-q^2,-q^3)f(-q^6,-q^{24}) - qf(-q^2,-q^3)f(-q^6,-q^{24})\\ &\quad + f(-q,-q^4)f(-q^{12},-q^{18}) - q^4f(-1,-q^5)f(-1,-q^{30})\\ &\quad - qf(-q^6,-q^{-1})f(-q^{12},-q^{18})\\ &= -2qf(-q^2,-q^3)f(-q^6,-q^{24}) + f(-q,-q^4)f(-q^{12},-q^{18})\\ &\quad + f(-q,-q^4)f(-q^{12},-q^{18})\\ &= 2f(-q,-q^4)f(-q^{12},-q^{18}) - 2qf(-q^2,-q^3)f(-q^6,-q^{24}).\end{aligned}$$

By (12.2), we see that the proof is complete.

13. Proof of Entry 3.9

By using (2.11) and (2.14), we find that Entry 3.9 is equivalent to the identity

$$\begin{aligned}f(-q^{14},-q^{21})f(-q^2,-q^8) - qf(-q^4,-q^6)f(-q^7,-q^{28}) &= f(-q^2)f(-q^7)\frac{\chi(-q)}{\chi(-q^7)}\\ &= f(-q)f(-q^{14}). \qquad (13.1)\end{aligned}$$

We invoke (4.13) with the two sets of parameters $\alpha_1 = 2, \beta_1 = 7, m_1 = 3, p_1 = 5, \lambda_1 = 5$ and $\alpha_2 = 1, \beta_2 = 14, m_2 = 1, p_2 = 3, \lambda_2 = 5$. The conditions in (4.9) are easily seen to be met. By using (4.14) and (4.15), we find that

$$f(-q^{28},-q^{42})f(-q^4,-q^{16}) - q^2f(-q^8,-q^{12})f(-q^{14},-q^{56}) = f(-q^2)f(-q^{28}).$$

Replacing q^2 by q in the last equality, we deduce (13.1) to complete the proof.

14. Proof of Entry 3.10

By using (2.11) and (2.14), we find that Entry 3.10 is equivalent to the identity

$$\begin{aligned}f(-q^2,-q^3)f(-q^{28},-q^{42}) + q^3f(-q,-q^4)f(-q^{14},-q^{56}) &= f(-q)f(-q^{14})\frac{\chi(-q^7)}{\chi(-q)}\\ &= f(-q^2)f(-q^7). \qquad (14.1)\end{aligned}$$

We now apply (4.13) with $\alpha_1 = 1, \beta_1 = 14, m_1 = 1, p_1 = 5, \lambda_1 = 3$ and $\alpha_2 = 2, \beta_2 = 7, m_2 = 1, p_2 = 3, \lambda_2 = 3$. We easily find that these sets of parameters satisfy the conditions in (4.9). Employing (4.13) and (4.15), we find that

$$f(-q^4,-q^6)f(-q^{56},-q^{84}) + q^6f(-q^2,-q^8)f(-q^{28},-q^{112}) = f(-q^4)f(-q^{14}), \qquad (14.2)$$

which is (14.1), but with q replaced by q^2, and so the proof is complete.

15. Proofs of Entry 3.11

First Proof of Entry 3.11. By employing (2.11), we see that Entry 3.11 is equivalent to the identity

$$f(-q^{16},-q^{24})f(-q^{3},-q^{12})-qf(-q^{6},-q^{9})f(-q^{8},-q^{32}) = f(-q^3)f(-q^8)\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \tag{15.1}$$

We apply (4.13) with the two sets of parameters $\alpha_1 = 3, \beta_1 = 8, m_1 = 3, p_1 = 5, \lambda_1 = 7$ and $\alpha_2 = 2, \beta_2 = 12, m_2 = 1, p_2 = 2, \lambda_2 = 7$. The conditions (4.9) are easily seen to be satisfied. Using (4.14) and (4.16), we find that

$$f(-q^{32},-q^{48})f(-q^{6},-q^{24}) - q^2 f(-q^{12},-q^{18})f(-q^{16},-q^{64}) = \psi(-q^2)\psi(-q^{12}). \tag{15.2}$$

After replacing q^2 by q in (15.2) and comparing the result with (15.1), we find that it suffices to show that

$$\frac{\psi(-q)\psi(-q^6)}{f(-q^3)f(-q^8)} = \frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}. \tag{15.3}$$

By using the product representations for $\psi(-q^a)$ and $f(-q^a)$ in (2.8) and (2.9), respectively, we find that

$$\begin{aligned}\frac{\psi(-q)\psi(-q^6)}{f(-q^3)f(-q^8)} =& \frac{(q^2;q^2)_\infty(q^{12},q^{12})_\infty}{(-q;q^2)_\infty(-q^6,q^{12})_\infty(q^3;q^3)_\infty(q^8;q^8)_\infty}\\ =&\frac{(q^2;q^2)_\infty(q;q^2)_\infty(q^{12},q^{12})_\infty(q^6;q^{12})_\infty}{(q^2;q^4)_\infty(q^{12},q^{24})_\infty(q^3;q^3)_\infty(q^8;q^8)_\infty}\\ =&\frac{\chi(-q)\chi(-q^4)}{\chi(-q^{12})}\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty}\\ =&\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}.\end{aligned}$$

Thus, the proof of (15.3) is complete, and so is that of Entry 3.11 also complete. □

Second Proof of Entry 3.11. Using (4.23) and (4.24) in (3.9), we find that

$$\begin{aligned}\frac{\chi(-q)}{\chi(-q^3)} =& G(q^6)H(q) - qG(q)H(q^6)\\ =&\frac{f(-q^8)}{f(-q^2)}\Big\{G(q^6)\big(q^3H(q^{16})+G(-q^4)\big)-qH(q^6)\big(G(q^{16})+qH(-q^4)\big)\Big\}\\ =&\frac{f(-q^8)}{f(-q^2)}\Big\{G(-q^4)G(q^6) - q^2H(-q^4)H(q^6)\\ &- q\big(H(q^6)G(q^{16}) - q^2G(q^6)H(q^{16})\big)\Big\}.\end{aligned} \tag{15.4}$$

By (2.16), and by (11.10) with q replaced by $-q$, we deduce from (15.4) that

$$\begin{aligned}&G(-q^4)G(q^6)-q^2H(-q^4)H(q^6)-q\big(H(q^6)G(q^{16})-q^2G(q^6)H(q^{16})\big)\\ =&\frac{\chi(-q)f(-q^2)}{\chi(-q^3)f(-q^8)}=\frac{f(-q^2)}{f(-q^8)\chi(-q^6)}\chi(-q)\chi(q^3)\\ =&\frac{f(-q^2)}{f(-q^8)\chi(-q^6)}\left\{\frac{\chi(q^{12})}{\chi(-q^8)}-q\frac{\chi(q^4)}{\chi(-q^{24})}\right\}=\frac{\chi(-q^2)\chi(q^{12})}{\chi(-q^6)}-q\frac{\chi(-q^2)\chi(-q^8)}{\chi(-q^6)\chi(-q^{24})}. \qquad (15.5)\end{aligned}$$

Equating the even and odd parts on both sides of the equation (15.5), we obtain Entries 3.23 and 3.11 with q replaced by $-q^2$ and q^2, respectively. □

16. Proofs of Entry 3.12

The first proof that we give is due to Bressoud [10].

First Proof of Entry 3.12. Using (2.11), we readily find that Entry 3.12 is equivalent to the identity

$$f(-q^2,-q^3)f(-q^{48},-q^{72})+q^5f(-q,-q^4)f(-q^{24},-q^{96})=f(-q)f(-q^{24})\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}. \qquad (16.1)$$

We apply (4.13) with the two sets of parameters $\alpha_1=1,\beta_1=24,m_1=1,p_1=5,\lambda_1=5$ and $\alpha_2=4,\beta_2=6,m_2=1,p_2=2,\lambda_2=5$. We find that the conditions in (4.9) are satisfied. Hence, using (4.15) and (4.18), we deduce the identity

$$f(-q^4,-q^6)f(-q^{96},-q^{144})+q^{10}f(-q^2,-q^8)f(-q^{48},-q^{192})=\psi(-q^4)\psi(-q^6). \qquad (16.2)$$

Replacing q^2 by q in (16.2) and comparing it with (16.1), we find that it suffices to prove that

$$\frac{\psi(-q^2)\psi(-q^3)}{f(-q)f(-q^{24})}=\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)}. \qquad (16.3)$$

Using the product representations of $\psi(-q^a)$ and $f(-q^a)$ from (2.8) and (2.9), respectively, we find that

$$\begin{aligned}\frac{\psi(-q^2)\psi(-q^3)}{f(-q)f(-q^{24})}=&\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{(-q^2;q^4)_\infty(-q^3;q^6)_\infty(q;q)_\infty(q^{24};q^{24})_\infty}\\ =&\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty(q^2;q^4)_\infty(q^3;q^6)_\infty}{(q^4;q^8)_\infty(q^6;q^{12})_\infty(q;q)_\infty(q^{24};q^{24})_\infty}\\ =&\frac{(q^2;q^2)_\infty(q^6;q^6)_\infty\chi(-q^3)}{\chi(-q^4)(q;q)_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}\\ =&\frac{(q^6;q^6)_\infty\chi(-q^3)}{\chi(-q)\chi(-q^4)(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}\\ =&\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)},\end{aligned}$$

which establishes (16.3), and so the proof is complete. □

Second Proof of Entry 3.12. Using (4.23) and (4.24) in (3.8) with q replaced by q^3, we arrive at

$$\begin{aligned}\frac{\chi(-q^3)}{\chi(-q)} =&G(q^2)G(q^3)+qH(q^2)H(q^3)\\ =&\frac{f(-q^{24})}{f(-q^6)}\Big\{G(q^2)\big(G(q^{48})+q^3H(-q^{12})\big)+qH(q^2)\big(q^9H(q^{48})+G(-q^{12})\big)\Big\}\\ =&\frac{f(-q^{24})}{f(-q^6)}\Big\{G(q^2)G(q^{48})+q^{10}H(q^2)H(q^{48})\\ &+q\big(H(q^2)G(-q^{12})+q^2G(q^2)H(-q^{12})\big)\Big\}. \qquad (16.4)\end{aligned}$$

That is,

$$G(q^2)G(q^{48})+q^{10}H(q^2)H(q^{48})+q\big(H(q^2)G(-q^{12})+q^2G(q^2)H(-q^{12})\big)=\frac{f(-q^6)\chi(-q^3)}{f(-q^{24})\chi(-q)}. \qquad (16.5)$$

Therefore, by (11.10), (2.15), and (2.14),

$$\frac{f(-q^6)\chi(-q^3)}{f(-q^{24})\chi(-q)}=\frac{f(-q^6)\chi(q)\chi(-q^3)}{f(-q^{24})\chi(-q^2)}=\frac{\chi(-q^6)\chi(-q^{24})}{\chi(-q^2)\chi(-q^8)}+q\frac{\chi(q^4)\chi(-q^6)}{\chi(-q^2)\chi(q^{12})}. \qquad (16.6)$$

Returning to (16.5), we use (16.6) to equate the odd parts on both sides of the equation, and, upon replacing q^2 by $-q$, we find that

$$H(-q)G(-q^6)-qG(-q)H(-q^6)=\frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)},$$

which is Entry 3.24. Similarly, equating the even parts in (16.5), employing (16.6), and replacing q^2 by q, we deduce that

$$G(q)G(q^{24})+q^5H(q)H(q^{24})=\frac{\chi(-q^3)\chi(-q^{12})}{\chi(-q)\chi(-q^4)},$$

which is Entry 3.12. □

17. Proofs of Entries 3.13 and 3.14

Throughout this section, we shall use several times without comment the elementary identity (2.5). To prove Entries 3.13 and 3.14, we also need the following lemma.

Lemma 17.1. *For* $|q|<1$, $x\neq 0$, *and* $y\neq 0$,

$$\begin{aligned}&f(-x,-x^{-1}q)f(-y,-y^{-1}q)\\ =&f(xy,(xy)^{-1}q^2)f(x^{-1}yq,xy^{-1}q)-xf(xyq,(xy)^{-1}q)f(x^{-1}y,xy^{-1}q^2). \qquad (17.1)\end{aligned}$$

In this form, Lemma 17.1 is given as Theorem 1.1 in [13, p. 649]. However, it is easily seen that Lemma 17.1 can be obtained by adding Entries 29(i) and (ii) in Chapter 16 of Ramanujan's second notebook [3, p. 45].

Lemma 17.2.

$$f^2(-q^{27},-q^{45})+q^9f^2(-q^9,-q^{63})=f(-q^3,q^6)\psi(-q^9)\chi(q^3). \qquad (17.2)$$

Proof. Replacing q, x, and y by q^{36}, $-q^{9}$, and q^{18}, respectively, in Lemma 17.1, we find that

$$f(-q^{27},-q^{45})^2+q^9f(-q^9,-q^{63})^2=f(q^9,q^{27})f(-q^{18},-q^{18})=\psi(q^9)\varphi(-q^{18}). \quad (17.3)$$

Using (2.13)–(2.15), or using (2.7)–(2.9), we can easily conclude that

$$\varphi(-q^2)\psi(q)=\varphi(q)\psi(-q). \quad (17.4)$$

Therefore, by (11.8) with q replaced by $-q^3$, and by (17.4) with q replaced by q^9, we find that

$$f(-q^3,q^6)\psi(-q^9)\chi(q^3)=\psi(-q^9)\varphi(q^9)=\psi(q^9)\varphi(-q^{18}). \quad (17.5)$$

Thus, we have proved Lemma 17.2. □

Lemma 17.3.

$$\begin{aligned}&f(-q^{21},-q^{51})f(-q^{27},-q^{45})+q^6f(-q^3,-q^{69})f(-q^{27},-q^{45})\\&+q^3f(-q^{33},-q^{39})f(-q^9,-q^{63})-q^6f(-q^{15},-q^{57})f(-q^9,-q^{63})=\psi^2(-q^9)\chi(q^3).\end{aligned} \quad (17.6)$$

Proof. Replacing q, x, and y by q^{36}, q^{6}, and $-q^{15}$, respectively, in Lemma 17.1, we find that

$$f(-q^{21},-q^{51})f(-q^{27},-q^{45})-q^6f(-q^{15},-q^{57})f(-q^9,-q^{63})=f(-q^6,-q^{30})f(q^{15},q^{21}), \quad (17.7)$$

and replacing q, x, and y by q^{36}, $-q^{3}$, and q^{6}, respectively, in Lemma 17.1, we find that

$$f(-q^9,-q^{63})f(-q^{33},-q^{39})+q^3f(-q^{27},-q^{45})f(-q^3,-q^{69})=f(q^3,q^{33})f(-q^6,-q^{30}). \quad (17.8)$$

By (2.19) with $a=q^3$, $b=q^6$ and $n=2$, we deduce that

$$f(q^3,q^6)=f(q^{15},q^{21})+q^3f(q^3,q^{33}). \quad (17.9)$$

Thus, by (17.7)–(17.9), we find that

$$\begin{aligned}&f(-q^{21},-q^{51})f(-q^{27},-q^{45})+q^6f(-q^3,-q^{69})f(-q^{27},-q^{45})\\&+q^3f(-q^{33},-q^{39})f(-q^9,-q^{63})-q^6f(-q^{15},-q^{57})f(-q^9,-q^{63})\\&=f(q^3,q^6)f(-q^6,-q^{30}).\end{aligned} \quad (17.10)$$

One can easily verify that

$$\psi^2(-q)=\psi(q^2)\varphi(-q). \quad (17.11)$$

By (11.8), (11.6), and (17.11) with q replaced by q^3, q^6, and q^9, respectively, and by (2.15), we conclude that

$$f(q^3,q^6)f(-q^6,-q^{30})=\frac{\varphi(-q^9)}{\chi(-q^3)}\chi(-q^6)\psi(q^{18})=\varphi(-q^9)\psi(q^{18})\chi(q^3)=\psi^2(-q^9)\chi(q^3). \quad (17.12)$$

Thus, we have proved Lemma 17.3. □

Theorem 17.4. *For* $|q|<1$,

$$\begin{aligned} &f(-q,-q^7)f(-q^{27},-q^{45})-q^4f(-q^3,-q^5)f(-q^9,-q^{63})\\ =&(q^4;q^4)_\infty(q^9;q^9)_\infty\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}. \end{aligned} \tag{17.13}$$

Proof. Replacing n, a, and b by 3, $-q$, and $-q^7$, respectively, in (2.19), we find that

$$f(-q,-q^7)=f(-q^{27},-q^{45})-qf(-q^{21},-q^{51})-q^7f(-q^3,-q^{69}); \tag{17.14}$$

replacing n, a, and b by 3, $-q^3$, and $-q^5$, respectively, in (2.19), we find that

$$f(-q^3,-q^5)=f(-q^{33},-q^{39})-q^3f(-q^{15},-q^{57})-q^5f(-q^9,-q^{63}); \tag{17.15}$$

and replacing n, a, and b by 3, $-q$, and $-q^3$, respectively, in (2.19), we find that (see also [3, p. 49, Cor.])

$$\begin{aligned} \psi(-q)=f(-q,-q^3)&=f(-q^{15},-q^{21})-qf(-q^9,-q^{27})-q^3f(-q^3,-q^{33})\\ &=f(-q^3,q^6)-q\psi(-q^9), \end{aligned} \tag{17.16}$$

where in the last step, we used (17.9) with q replaced by -q. Then, by (17.14) and (17.15), the left-hand side of (17.13) equals

$$\begin{aligned} &f^2(-q^{27},-q^{45})-qf(-q^{21},-q^{51})f(-q^{27},-q^{45})\\ &-q^7f(-q^3,-q^{69})f(-q^{27},-q^{45})-q^4f(-q^{33},-q^{39})f(-q^9,-q^{63})\\ &+q^7f(-q^{15},-q^{57})f(-q^9,-q^{63})+q^9f^2(-q^9,-q^{63}). \end{aligned} \tag{17.17}$$

Therefore, by (17.17), Lemma 17.2, Lemma 17.3, (17.16), (2.16) and (2.17) the left-hand side of (17.13) equals

$$\begin{aligned} &f(-q^3,q^6)\psi(-q^9)\chi(q^3)-q\psi^2(-q^9)\chi(-q^6)=\psi(-q^9)\chi(q^3)\Big\{f(-q^3,q^6)-q\psi(-q^9)\Big\}\\ &=\psi(-q)\psi(-q^9)\chi(q^3)=f(-q^4)\chi(-q)\frac{f(-q^9)}{\chi(-q^{18})}\frac{\chi(-q^6)}{\chi(-q^3)}. \end{aligned} \tag{17.18}$$

We have thus completed the proof of Theorem 17.4. □

Theorem 17.5. *For* $|q|<1$,

$$\begin{aligned} &f(-q^4,-q^{16})f(-q^{18},-q^{27})-qf(-q^8,-q^{12})f(-q^9,-q^{36})\\ =&f(-q,-q^4)f(-q^{72},-q^{108})-q^7f(-q^2,-q^3)f(-q^{36},-q^{144})\\ =&f(-q,-q^7)f(-q^{27},-q^{45})-q^4f(-q^3,-q^5)f(-q^9,-q^{63}). \end{aligned} \tag{17.19}$$

Proof. We apply the ideas of Rogers with the three sets of parameters $\alpha_1=4$, $\beta_1=9$, $m_1=3$, $p_1=5$, $\lambda_1=9$; $\alpha_2=1$, $\beta_2=36$, $m_2=3$, $p_2=5$, $\lambda_2=9$; and $\alpha_3=2$, $\beta_3=18$, $m_3=3$, $p_3=4$, $\lambda_3=9$. The requisite conditions (4.9) are satisfied. Therefore, we find that

$$\begin{aligned} &q^{9/4}f(-q^8,-q^{32})f(-q^{36},-q^{54})-q^{17/4}f(-q^{16},-q^{24})f(-q^{18},-q^{72})\\ =&q^{9/4}f(-q^2,-q^8)f(-q^{144},-q^{216})-q^{65/4}f(-q^4,-q^6)f(-q^{72},-q^{288})\\ =&q^{9/4}f(-q^2,-q^{14})f(-q^{54},-q^{90})-q^{41/4}f(-q^6,-q^{10})f(-q^{18},-q^{126}). \end{aligned} \tag{17.20}$$

Dividing each term of (17.20) by $q^{9/4}$, and replacing q^2 by q, we are able to derive (17.19) from (17.20). □

We are now going to prove Entries 3.13 and 3.14.

Proof of Entry 3.13. By Theorems 17.4 and 17.5, we find that

$$\begin{aligned} &f(-q^4,-q^{16})f(-q^{18},-q^{27})-qf(-q^8,-q^{12})f(-q^9,-q^{36}) \\ &=(q^4;q^4)_\infty(q^9;q^9)_\infty\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}. \end{aligned} \tag{17.21}$$

Dividing both sides of (17.21) by $(q^4;q^4)_\infty(q^9;q^9)_\infty$ and using the definitions of $G(q)$ and $H(q)$, we derive Entry 3.13. □

Proof of Entry 3.14. By Theorems 17.4 and 17.5, we find that

$$\begin{aligned} &f(-q,-q^4)f(-q^{72},-q^{108})-q^7f(-q^2,-q^3)f(-q^{36},-q^{144}) \\ =&(q^4;q^4)_\infty(q^9;q^9)_\infty\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}. \end{aligned} \tag{17.22}$$

Dividing both sides of (17.22) by $(q;q)_\infty(q^{36};q^{36})_\infty$ and using the definitions of $G(q)$ and $H(q)$, we find that the left-hand side of (17.22) equals $G(q^{36})H(q)-q^7G(q)H(q^{36})$, and the right-hand side of (17.22) equals

$$\frac{(q^4;q^4)_\infty(q^9;q^9)_\infty}{(q;q)_\infty(q^{36};q^{36})_\infty}\cdot\frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})}=\frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)},$$

which completes the proof of Entry 3.14. □

We offer now a second, completely different proof of Entry 3.13.

Second Proof of Entry 3.13. By (2.11), (2.8), (2.6), and some elementary product manipulations, Entry 3.13 is easily seen to be equivalent to

$$f(-q^4,-q^{16})f(-q^{18},-q^{27})-qf(-q^8,-q^{12})f(-q^9,-q^{36})=\psi(-q)f(q^3,q^{15}). \tag{17.23}$$

We prove (17.23).

Employing Theorem 4.5 with the set of parameters $a=q^3,\ b=q^{15},\ c=q,\ d=q^2,\ \alpha=1,\ \beta=2,\ m=5,\ \epsilon_1=0$, and $\epsilon_2=1$, we find that

$$\begin{aligned} f(q^3,q^{15})f(-q,-q^2)=&f(-q^{18},-q^{27})f(q^8,q^{22})-qf(-q^9,-q^{36})f(q^{14},q^{16}) \\ &+q^3f(-q^9,-q^{36})f(q^4,q^{26})-q^2f(-q^{18},-q^{27})f(q^2,q^{28}). \end{aligned}$$

Upon the rearrangement of terms and the use of (4.28) and (4.29) with q replaced by q^2, we obtain

$$\begin{aligned} f(q^3,q^{15})f(-q)=&f(-q^{18},-q^{27})\{f(q^8,q^{22})-q^2f(q^2,q^{28})\} \\ &-qf(-q^9,-q^{36})\{f(q^{14},q^{16})-q^2f(q^4,q^{26})\} \\ =&f(-q^{18},-q^{27})H(q^4)f(-q^2)-qf(-q^9,-q^{36})G(q^4)f(-q^2) \end{aligned}$$

$$=\frac{f(-q^2)}{f(-q^4)}\Big\{f(-q^4,-q^{16})f(-q^{18},-q^{27})-qf(-q^8,-q^{12})f(-q^9,-q^{36})\Big\}. \tag{17.24}$$

But by (2.17),

$$\frac{f(-q)f(-q^4)}{f(-q^2)}=\psi(-q).$$

Using the last equality in (17.24), we complete the proof (17.23) and also that of Entry 3.14. □

In Section 10, we promised that in the current section we would give a proof that simultaneously yields Entries 3.6 and 3.13. We show that Entry 3.11 implies both Entries 3.6 and 3.13.

Another Proof of Entries 3.6 *and* 3.13. In Entry 3.11, we employ (4.31) and (4.30) with q replaced by q^3 to find that

$$\begin{aligned}&\Big\{-q^3\frac{\chi(q^{18})}{\chi(-q^{12})}H(-q^{18})+\frac{\chi(q^6)}{\chi(-q^{36})}G(q^{72})\Big\}G(q^8)\\&-q\Big\{\frac{\chi(q^{18})}{\chi(-q^{12})}G(-q^{18})-q^{15}\frac{\chi(q^6)}{\chi(-q^{36})}H(q^{72})\Big\}H(q^8)=\chi(q^9)\chi(-q^3)\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})}.\end{aligned} \tag{17.25}$$

Upon collecting terms, we deduce from (17.25) that

$$\begin{aligned}&\frac{\chi(q^6)}{\chi(-q^{36})}\Big\{G(q^8)G(q^{72})+q^{16}H(q^8)H(q^{72})\Big\}\\&-q\frac{\chi(q^{18})}{\chi(-q^{12})}\Big\{G(-q^{18})H(q^8)+q^2H(-q^{18})G(q^8)\Big\}=\frac{\chi(-q^4)}{\chi(-q^{12})}\chi(-q)\chi(q^9).\end{aligned} \tag{17.26}$$

To equate even and odd parts on both sides of (17.26), we need the 2-dissection of $\chi(-q)\chi(q^9)$ which we obtain from Theorem 4.5. To that end, we employ Theorem 4.5 with the set of parameters $a=1,\ b=q^9,\ c=q,\ d=q^2,\ \epsilon_1=0,\ \epsilon_2=1,\ \alpha=\beta=1,$ and $m=4$ to find that

$$\begin{aligned}f(1,q^9)f(-q,-q^2)=&f(-q^2,-q^{10})f(-q^{12},-q^{24})+f(-q,-q^{11})f(-q^{15},-q^{21})\\&-qf(-q^4,-q^8)f(-q^6,-q^{30})-q^2f(-q^5,-q^7)f(-q^3,-q^{33}).\end{aligned} \tag{17.27}$$

We employ Theorem 4.5 again with the same set of parameters, except this time we take $\epsilon_1=1$, and $\epsilon_2=0$, to find that

$$\begin{aligned}f(-1,-q^9)f(q,q^2)=&f(-q^2,-q^{10})f(-q^{12},-q^{24})-f(-q,-q^{11})f(-q^{15},-q^{21})\\&-qf(-q^4,-q^8)f(-q^6,-q^{30})+q^2f(-q^5,-q^7)f(-q^3,-q^{33}).\end{aligned} \tag{17.28}$$

By (2.4), the product on the left side of (17.28) equals 0. Recalling the definitions (2.8) and (2.9), and employing (2.3), (17.27), and (17.28), we conclude that

$$\begin{aligned}\psi(q^9)f(-q) &= \frac{1}{2}f(1,q^9)f(-q,-q^2) = \frac{1}{2}\left\{f(1,q^9)f(-q,-q^2)+f(-1,-q^9)f(q,q^2)\right\}\\ &= f(-q^2,-q^{10})f(-q^{12},-q^{24}) - qf(-q^4,-q^8)f(-q^6,-q^{30})\\ &= f(-q^2,-q^{10})f(-q^{12}) - qf(-q^4)f(-q^6,-q^{30}). \qquad (17.29)\end{aligned}$$

Next, we use (2.14), (2.17), and (11.6) twice with q replaced by q^2 and q^6, respectively, to find from (17.29) that

$$\begin{aligned}&\chi(q^9)f(-q^{36})\chi(-q)f(-q^2)\\ &= f(-q^{12})\chi(-q^2)\chi(q^6)f(-q^{24}) - qf(-q^4)\chi(-q^6)\chi(q^{18})f(-q^{72}),\end{aligned}$$

which after several uses of (2.14) simplifies to

$$\chi(-q)\chi(q^9) = \frac{f(-q^{12})\psi(q^6)}{f(-q^4)f(-q^{36})} - q\frac{\chi(-q^6)}{\chi(-q^2)\chi(-q^{18})}. \qquad (17.30)$$

Returning to (17.26), we substitute the value of $\chi(-q)\chi(q^9)$ from (17.30) and equate the odd parts on both sides of the resulting equation. Hence, using (2.15), we conclude that

$$\begin{aligned}G(-q^{18})H(q^8)+q^2H(-q^{18})G(q^8) &= \frac{\chi(-q^{12})}{\chi(q^{18})}\frac{\chi(-q^4)}{\chi(-q^{12})}\frac{\chi(-q^6)}{\chi(-q^2)\chi(-q^{18})}\\ &= \frac{\chi(q^2)\chi(-q^6)}{\chi(-q^{36})},\end{aligned}$$

which is Entry 3.13 with q replaced by $-q^2$. Similarly, equating the even parts in (17.26) with the use of (17.30), using (2.14) and (2.17), and replacing q^8 by q, we deduce Entry 3.6. □

18. Proof of Entry 3.15

Let

$$M(q) := G(q^3)G(q^7) + q^2H(q^3)H(q^7) \qquad (18.1)$$

and

$$N(q) := G(q^{21})H(q) - q^4G(q)H(q^{21}). \qquad (18.2)$$

Consider the system of equations

$$N(q^2) = H(q^2)G(q^{42}) - q^8G(q^2)H(q^{42}),$$

$$\frac{\chi(-q^7)}{\chi(-q^{21})} =: R(q) = H(q^7)G(q^{42}) - q^7G(q^7)H(q^{42}), \qquad (18.3)$$

$$\frac{\chi(-q^{21})}{\chi(-q^3)} =: S(q) = G(q^3)G(q^{42}) + q^9H(q^3)H(q^{42}), \qquad (18.4)$$

arising from (18.2), Entry 3.8 with q replaced by q^7, and Entry 3.10 with q replaced by q^3, respectively. It follows that

$$\begin{vmatrix} H(q^2) & -q^8G(q^2) & N(q^2) \\ H(q^7) & -q^7G(q^7) & R(q) \\ G(q^3) & q^9H(q^3) & S(q) \end{vmatrix} = 0,$$

or, using (18.1), Entry 3.7, Entry 3.9, (18.3), and (18.4), we find that

$$\begin{aligned} 0 =& N(q^2)\left(q^9H(q^3)H(q^7)+q^7G(q^3)G(q^7)\right) - R(q)\left(q^9H(q^2)H(q^3)+q^8G(q^2)G(q^3)\right) \\ &+ S(q)\left(-q^7G(q^7)H(q^2)+q^8G(q^2)H(q^7)\right) \\ =& q^7N(q^2)M(q) - q^8\frac{\chi(-q^7)}{\chi(-q^{21})}\frac{\chi(-q^3)}{\chi(-q)} - q^7\frac{\chi(-q^{21})}{\chi(-q^3)}\frac{\chi(-q)}{\chi(-q^7)}. \end{aligned}$$

Solving the equation above for $N(q^2)M(q)$, we find that, if

$$T(q) := \frac{\chi(-q^3)\chi(-q^7)}{\chi(-q)\chi(-q^{21})}, \tag{18.5}$$

then

$$N(q^2)M(q) = qT(q) + \frac{1}{T(q)}. \tag{18.6}$$

Next, we derive a similar formula for $M(q^2)N(q)$. Using (18.1), Entry 3.10, and Entry 3.7 with q replaced by q^7, we find that

$$M(q^2) = G(q^6)G(q^{14}) + q^4H(q^6)H(q^{14}),$$

$$\frac{\chi(-q^7)}{\chi(-q)} =: R_1(q) = G(q)G(q^{14}) + q^3H(q)H(q^{14}), \tag{18.7}$$

$$\frac{\chi(-q^{21})}{\chi(-q^7)} =: S_1(q) = G(q^{21})G(q^{14}) + q^7H(q^{21})H(q^{14}), \tag{18.8}$$

which implies that

$$\begin{vmatrix} G(q^6) & q^4H(q^6) & M(q^2) \\ G(q) & q^3H(q) & R_1(q) \\ G(q^{21}) & q^7H(q^{21}) & S_1(q) \end{vmatrix} = 0.$$

Hence, by (18.2), Entry 3.9 with q replaced by q^3, Entry 3.8, (18.7), and (18.8),

$$\begin{aligned} 0 =& M(q^2)\left(q^7G(q)H(q^{21}) - q^3H(q)G(q^{21})\right) - R_1(q)\left(q^7G(q^6)H(q^{21}) - q^4H(q^6)G(q^{21})\right) \\ &+ S_1(q)\left(q^3G(q^6)H(q) - q^4H(q^6)G(q)\right) \\ =& -q^3M(q^2)N(q) + q^4\frac{\chi(-q^7)}{\chi(-q)}\frac{\chi(-q^3)}{\chi(-q^{21})} + q^3\frac{\chi(-q^{21})}{\chi(-q^7)}\frac{\chi(-q)}{\chi(-q^3)}. \end{aligned}$$

Hence, solving the equation above for $M(q^2)N(q)$, we find that

$$M(q^2)N(q) = qT(q) + \frac{1}{T(q)}, \tag{18.9}$$

where $T(q)$ is defined by (18.5). Comparing (18.6) with (18.9), we find that

$$N(q^2)M(q) = M(q^2)N(q). \tag{18.10}$$

Equation (18.10) easily implies that $M(q) = N(q)$, which is what we wanted to prove, i.e., (3.16). To see this, let

$$M(q) := \sum_{n=0}^{\infty} a_n q^n \qquad \text{and} \qquad N(q) := \sum_{n=0}^{\infty} b_n q^n.$$

From the definitions (18.1) and (18.2), we see that $a_0 = b_0$. Then by an easy inductive argument, we find that $a_n = b_n$, for every positive integer n. Hence, $M(q) = N(q)$, as we wanted to demonstrate.

As an immediate consequence of our main identity $M(q) = N(q)$ and (18.9), we derive the following curious corollary.

Corollary 18.1. *If $T(q)$ is defined by* (18.5), *then*

$$M(q)M(q^2) = N(q)N(q^2) = qT(q) + \frac{1}{T(q)}.$$

Next, we prove the second part of Entry 3.15.

Now we are ready to prove (3.17). Let $J(q)$ denote the right-hand side of (3.17), so that

$$J(q^2) = \frac{1}{2q}\left\{\chi(q)\chi(-q^3)\chi(q^7)\chi(-q^{21}) - \chi(-q)\chi(q^3)\chi(-q^7)\chi(q^{21})\right\}. \tag{18.11}$$

Recall that $M(q)$and $N(q)$ are defined by (18.1) and (18.2), respectively. Using the previously established fact, $M(q) = N(q)$, we see that it suffices to show that $M(q^2)N(q^2) = J^2(q^2)$.

Using (4.39) in (3.10) with q replaced by q^7, we obtain

$$\begin{aligned}&\left\{a(q^7)G(q^{42}) - q^7 b(q^7)H(q^{28})\right\} H(q^2)\\ &- qG(q^2)\left\{q^7 a(q^7)H(q^{42}) + b(q^7)G(q^{28})\right\} = \frac{\chi(-q)}{\chi(-q^7)}.\end{aligned}$$

Upon rearrangement and the use of (18.2) and (3.11) with q replaced by q^2, we find that

$$a(q^7)N(q^2) - qb(q^7)\frac{\chi(-q^{14})}{\chi(-q^2)} = \frac{\chi(-q)}{\chi(-q^7)},$$

from which, by (4.38), we conclude that

$$N(q^2) = \frac{1}{\chi^2(q^7)}\left\{\frac{\chi(-q)\chi(-q^{42})}{\chi(-q^7)\chi(-q^{14})} + q\frac{\chi(-q^7)\chi(-q^{14})}{\chi(-q^2)\chi(-q^{21})}\right\}. \tag{18.12}$$

Similarly, employing Lemma 4.4 in (3.11), we find that

$$\begin{aligned}&\left\{a(q)G(q^6) - qb(q)H(q^4)\right\} G(q^{14})\\ &+ q^3\left\{qa(q)H(q^6) + b(q)G(q^4)\right\} H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}.\end{aligned}$$

Upon rearrangement and the use of (18.1) and (3.10) with q replaced by q^2, we obtain

$$a(q)M(q^2) - qb(q)\frac{\chi(-q^2)}{\chi(-q^{14})} = \frac{\chi(-q^7)}{\chi(-q)},$$

from which, we similarly find that

$$M(q^2)=\frac{1}{\chi^2(q)}\left\{\frac{\chi(-q^6)\chi(-q^7)}{\chi(-q)\chi(-q^2)}+q\frac{\chi(-q)\chi(-q^2)}{\chi(-q^3)\chi(-q^{14})}\right\}. \tag{18.13}$$

Next, recall that [3, p. 124, Entries 12 (v), (vi), (vii)]

$$\chi(q)=2^{1/6}\left(\frac{q}{\alpha(1-\alpha)}\right)^{1/24} \quad\text{and}\quad \chi(-q)=2^{1/6}\left(\frac{(1-\alpha)^2q}{\alpha}\right)^{1/24}. \tag{18.14}$$

Let α, β, γ, and δ be of degrees 1, 3, 7, and 21, respectively. In (18.13), we use the representations (18.14) and (9.14) and conclude, after some algebra, that

$$\begin{aligned}M(q^2)=\frac{2^{-1/3}q^{1/3}}{\{\alpha\beta^2\gamma(1-\alpha)(1-\beta)^2(1-\gamma)\}^{1/24}}\Big\{&\alpha^{1/4}(1-\beta)^{1/8}(1-\gamma)^{1/8}\\ &+\beta^{1/8}\gamma^{1/8}(1-\alpha)^{1/4}\Big\}.\end{aligned} \tag{18.15}$$

Similarly, from (18.12), we find that

$$\begin{aligned}N(q^2)=\frac{2^{-1/3}q^{1/3}}{\{\alpha\gamma\delta^2(1-\alpha)(1-\gamma)(1-\delta)^2\}^{1/24}}\Big\{&\gamma^{1/4}(1-\alpha)^{1/8}(1-\delta)^{1/8}\\ &+\alpha^{1/8}\delta^{1/8}(1-\gamma)^{1/4}\Big\}.\end{aligned} \tag{18.16}$$

Lastly, from (18.11), we conclude that

$$\begin{aligned}J(q^2)=\frac{2^{-1/3}q^{1/3}}{\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}}\Big\{&\{(1-\beta)(1-\delta)\}^{1/8}\\ &-\{(1-\alpha)(1-\gamma)\}^{1/8}\Big\}.\end{aligned} \tag{18.17}$$

Therefore, the equation $M(q^2)N(q^2)=J^2(q^2)$ is equivalent to the modular equation

$$\begin{aligned}&\big\{\{(1-\beta)(1-\delta)\}^{1/8}-\{(1-\alpha)(1-\gamma)\}^{1/8}\big\}^2\\ &=\big\{\alpha^{1/4}(1-\beta)^{1/8}(1-\gamma)^{1/8}+\beta^{1/8}\gamma^{1/8}(1-\alpha)^{1/4}\big\}\big\{\gamma^{1/4}(1-\alpha)^{1/8}(1-\delta)^{1/8}\\ &\quad+\alpha^{1/8}\delta^{1/8}(1-\gamma)^{1/4}\big\}\\ &=\big\{\alpha^2\gamma^2(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\big\}^{1/8}+\big\{\alpha^3\delta(1-\beta)(1-\gamma)^3\big\}^{1/8}\\ &\quad+\big\{\gamma^3\beta(1-\delta)(1-\alpha)^3\big\}^{1/8}+\big\{\alpha\beta\gamma\delta(1-\alpha)^2(1-\gamma)^2\big\}^{1/8}.\end{aligned} \tag{18.18}$$

To prove (18.18), we invoke two modular equations, of degrees 3 and 7, respectively. Namely, if β has degree 3 over α, then [3, p. 230, Entry 5 (i)]

$$\{\beta(1-\alpha)^3\}^{1/8}-\{\alpha^3(1-\beta)\}^{1/8}=\{\beta(1-\beta)\}^{1/8}, \tag{18.19}$$

and, if γ has degree 7 over α, then [3, p. 314, Entry 19 (i)]

$$\{(1-\alpha)(1-\gamma)\}^{1/8}+\{\alpha\gamma\}^{1/8}=1. \tag{18.20}$$

Let

$$u := (\alpha\gamma)^{1/8}, \quad v := (\beta\delta)^{1/8}, \quad x := \{\beta(1-\alpha)^3\}^{1/8},$$
$$y := \{\alpha^3(1-\beta)\}^{1/8}, \quad \bar{x} := \{\delta(1-\gamma)^3\}^{1/8}, \quad \text{and} \quad \bar{y} := \{\gamma^3(1-\delta)\}^{1/8}.$$

Since γ has degree 7 over α and δ has degree 7 over β, by (18.20),

$$\{(1-\alpha)(1-\gamma)\}^{1/8} = 1-u \qquad \text{and} \qquad \{(1-\beta)(1-\delta)\}^{1/8} = 1-v.$$

Since β has degree 3 over α and δ has degree 3 over γ, by (18.19),

$$x-y = \{\beta(1-\beta)\}^{1/8} \qquad \text{and} \qquad \bar{x}-\bar{y} = \{\delta(1-\delta)\}^{1/8}.$$

By using the trivial identity

$$y\bar{x}+\bar{y}x = x\bar{x}+y\bar{y}-(x-y)(\bar{x}-\bar{y}),$$

we conclude that

$$y\bar{x}+\bar{y}x = v(1-u)^3+u^3(1-v)-v(1-v).$$

Returning to the equation (18.18), we see that the right-hand side of (18.18) is

$$u^2(1-u)(1-v)+y\bar{x}+\bar{y}x+uv(1-u)^2$$
$$= u^2(1-u)(1-v)+v(1-u)^3+u^3(1-v)-v(1-v)+uv(1-u)^2,$$

which, after some algebra, simplifies to

$$(u-v)^2 = \{(1-v)-(1-u)\}^2,$$

which is exactly the far left side of (18.18). Hence, the proof of (3.17) is complete.

19. Proof of Entry 3.16

We prove that both sides of (3.18) are independently equal to the right-hand side of (3.19). For brevity of exposition, we make the following definition. Assuming that S is a subset of the rational numbers and $\sum_{n\in S} a_n q^n$ is a generic q-series, we define an operator $\mathcal{L}$ acting on $\sum_{n\in S} a_n q^n$ by $\mathcal{L}\left(\sum_{n\in S} a_n q^n\right) = \sum_{n\in S'} a_n q^n$, where $S' \subseteq S$ is the set of all integers in S.

We apply Lemma 5.1 with q replaced by q^2 and q^{13} to respectively deduce that

$$f(-q^2)f(-q^{2/5}) = f^2(-q^4,-q^6) - q^{4/5}f^2(-q^2,-q^8) - q^{2/5}f(-q^2)f(-q^{10}), \tag{19.1}$$
$$f(-q^{13})f(-q^{13/5}) = f^2(-q^{26},-q^{39}) - q^{26/5}f^2(-q^{13},-q^{52}) - q^{13/5}f(-q^{13})f(-q^{65}). \tag{19.2}$$

Multiplying together (19.1) and (19.2), we obtain

$$\begin{aligned} f(-q^2)f(-q^{13})f(-q^{2/5})f(-q^{13/5}) &= f^2(-q^4,-q^6)f^2(-q^{26},-q^{39}) \\ &+ q^6 f^2(-q^2,-q^8)f^2(-q^{13},-q^{52}) + q^3 f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65}) \\ &+ q^{26/5}f^2(-q^4,-q^6)f^2(-q^{13},-q^{52}) - q^{13/5}f^2(-q^4,-q^6)f(-q^{13})f(-q^{65}) \\ &- q^{4/5}f^2(-q^2,-q^8)f^2(-q^{26},-q^{39}) + q^{17/5}f^2(-q^2,-q^8)f(-q^{13})f(-q^{65}) \\ &- q^{2/5}f^2(-q^{26},-q^{39})f(-q^2)f(-q^{10}) + q^{28/5}f^2(-q^{13},-q^{52})f(-q^2)f(-q^{10}). \end{aligned} \tag{19.3}$$

We consider terms with integral powers of q on both sides of (19.3) and observe that

$$\begin{aligned}&\mathcal{L}\left(f(-q^2)f(-q^{13})f(-q^{2/5})f(-q^{13/5})\right)\\ =&f^2(-q^4,-q^6)f^2(-q^{26},-q^{39})+q^6f^2(-q^2,-q^8)f^2(-q^{13},-q^{52})\\ &+q^3f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65}).\end{aligned} \tag{19.4}$$

We now derive an alternative expression for the left-hand side of (19.4) above. To this end, we first employ (2.19) with $a=-q$, $b=-q^2$, and $n=5$ to deduce that

$$\begin{aligned}f(-q)=&f(-q^{35},-q^{40})-qf(-q^{50},-q^{25})+q^5f(-q^{65},-q^{10})-q^{12}f(-q^{80},-q^{-5})\\ &+q^{22}f(-q^{95},-q^{-20})\\ =&f(-q^{35},-q^{40})-qf(-q^{50},-q^{25})+q^5f(-q^{65},-q^{10})+q^7f(-q^{70},-q^5)\\ &-q^2f(-q^{20},-q^{55}),\end{aligned} \tag{19.5}$$

after two applications of (2.5). We then apply (19.5) above to obtain representations for $f(-q^{2/5})$ and $f(-q^{13/5})$ by replacing q by $q^{2/5}$ and q by $q^{13/5}$, respectively. This gives us

$$\begin{aligned}f(-q^{2/5})=&f(-q^{14},-q^{16})-q^{2/5}f(-q^{20},-q^{10})+q^2f(-q^{26},-q^4)\\ &+q^{14/5}f(-q^{28},-q^2)-q^{4/5}f(-q^8,-q^{22})\end{aligned} \tag{19.6}$$

and

$$\begin{aligned}f(-q^{13/5})=&f(-q^{91},-q^{104})-q^{13/5}f(-q^{130},-q^{65})+q^{13}f(-q^{169},-q^{26})\\ &+q^{91/5}f(-q^{182},-q^{13})-q^{26/5}f(-q^{52},-q^{143}).\end{aligned} \tag{19.7}$$

Thus, multiplying (19.6) and (19.7), we see that

$$\begin{aligned}&\mathcal{L}\left(f(-q^{2/5})f(-q^{13/5})\right)=f(-q^{14},-q^{16})f(-q^{91},-q^{104})\\ &+q^3f(-q^{10},-q^{20})f(-q^{65},-q^{130})+q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169})\\ &+q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182})+q^6f(-q^8,-q^{22})f(-q^{52},-q^{143})\\ &+q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169})+q^2f(-q^4,-q^{26})f(-q^{91},-q^{104})\\ &-q^8f(-q^2,-q^{28})f(-q^{52},-q^{143})-q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182}).\end{aligned} \tag{19.8}$$

Since $f(-q^2)f(-q^{13})$ contains only integral powers of q, it follows that

$$\begin{aligned}&\mathcal{L}\left(f(-q^2)f(-q^{13})f(-q^{2/5})f(-q^{13/5})\right)=f(-q^2)f(-q^{13})\mathcal{L}\left(f(-q^{2/5})f(-q^{13/5})\right)\\ =&f(-q^2)f(-q^{13})\big\{f(-q^{14},-q^{16})f(-q^{91},-q^{104})+q^3f(-q^{10},-q^{20})f(-q^{65},-q^{130})\\ &+q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169})+q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182})\\ &+q^6f(-q^8,-q^{22})f(-q^{52},-q^{143})+q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169})\\ &+q^2f(-q^4,-q^{26})f(-q^{91},-q^{104})-q^8f(-q^2,-q^{28})f(-q^{52},-q^{143})\\ &-q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182})\big\}.\end{aligned} \tag{19.9}$$

Equating the right-hand sides of (19.4) and (19.9), we deduce that

$$\begin{aligned}
&f^2(-q^4,-q^6)f^2(-q^{26},-q^{39})+q^6f^2(-q^2,-q^8)f^2(-q^{13},-q^{52})\\
&\quad+q^3f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65})\\
&=f(-q^2)f(-q^{13})\big\{f(-q^{14},-q^{16})f(-q^{91},-q^{104})+q^3f(-q^{10},-q^{20})f(-q^{65},-q^{130})\\
&\quad+q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169})+q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182})\\
&\quad+q^6f(-q^8,-q^{22})f(-q^{52},-q^{143})+q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169})\\
&\quad+q^2f(-q^4,-q^{26})f(-q^{91},-q^{104})-q^8f(-q^2,-q^{28})f(-q^{52},-q^{143})\\
&\quad-q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182})\big\}. \qquad (19.10)
\end{aligned}$$

We seek to simplify the right-hand side of (19.10). Applying (4.13) with $\alpha=1$, $\beta=\frac{13}{2}$, $m=1$, and $p=15$, we see that

$$\begin{aligned}
&q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k)=f(-q^{14},-q^{16})f(-q^{91},-q^{104})\\
&\quad+qf(-q^{12},-q^{18})f(-q^{78},-q^{117})+q^3f(-q^{10},-q^{20})f(-q^{65},-q^{130})\\
&\quad+q^6f(-q^8,-q^{22})f(-q^{52},-q^{143})+q^{10}f(-q^6,-q^{24})f(-q^{39},-q^{156})\\
&\quad+q^{15}f(-q^4,-q^{26})f(-q^{26},-q^{169})+q^{21}f(-q^2,-q^{28})f(-q^{13},-q^{182}). \qquad (19.11)
\end{aligned}$$

We now observe that five out of the seven terms appearing on the right-hand side of (19.11) also appear on the right-hand side of (19.10). This enables us to rewrite (19.10) as

$$\begin{aligned}
&f^2(-q^4,-q^6)f^2(-q^{26},-q^{39})+q^6f^2(-q^2,-q^8)f^2(-q^{13},-q^{52})\\
&\quad+q^3f(-q^2)f(-q^{10})f(-q^{13})f(-q^{65})\\
&=f(-q^2)f(-q^{13})\Big\{q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k)-qf(-q^{12},-q^{18})f(-q^{78},-q^{117})\\
&\quad-q^{10}f(-q^6-q^{24})f(-q^{39},-q^{156})+q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169})\\
&\quad+q^2f(-q^4,-q^{26})f(-q^{91},-q^{104})-q^8f(-q^2,-q^{28})f(-q^{52},-q^{143})\\
&\quad-q^{19}f(-q^8,-q^{22})f(-q^{13},-q^{182})\Big\}. \qquad (19.12)
\end{aligned}$$

We next apply (4.13) again with $\alpha=1$, $\beta=13/2$, $m=11$, and $p=15$. This yields

$$\begin{aligned}
&q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)=q^2f(-q^{26},-q^4)f(-q^{104},-q^{91})\\
&+q^{19}f(-q^{48},-q^{-18})f(-q^{117},-q^{78})+q^{53}f(-q^{70},-q^{-40})f(-q^{130},-q^{65})\\
&+q^{104}f(-q^{92},-q^{-62})f(-q^{143},-q^{52})+q^{172}f(-q^{114},-q^{-84})f(-q^{156},-q^{39})\\
&+q^{257}f(-q^{136},-q^{-106})f(-q^{169},-q^{26})+q^{359}f(-q^{158},-q^{-128})f(-q^{182},-q^{13}). \quad (19.13)
\end{aligned}$$

After several applications of (2.5), we rewrite (19.13) as

$$\begin{aligned}
&q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)=q^{2}f(-q^{4},-q^{26})f(-q^{91},-q^{104})\\
&-qf(-q^{12},-q^{18})f(-q^{78},-q^{117})+q^{3}f(-q^{10},-q^{20})f(-q^{65},-q^{130})\\
&-q^{8}f(-q^{2},-q^{28})f(-q^{52},-q^{143})-q^{10}f(-q^{6},-q^{24})f(-q^{39},-q^{156})\\
&+q^{13}f(-q^{14},-q^{16})f(-q^{26},-q^{169})-q^{19}f(-q^{8},-q^{22})f(-q^{13},-q^{182}). \qquad (19.14)
\end{aligned}$$

We now note from (2.9) that $q^3f(-q^{10},-q^{20})f(-q^{65},-q^{130})=q^3f(-q^{10})f(-q^{65})$, and upon comparing the right-hand side of (19.14) with that of (19.12), we rewrite (19.12) as

$$\begin{aligned}
&f^{2}(-q^{4},-q^{6})f^{2}(-q^{26},-q^{39})+q^{6}f^{2}(-q^{2},-q^{8})f^{2}(-q^{13},-q^{52})\\
&+q^{3}f(-q^{2})f(-q^{10})f(-q^{13})f(-q^{65})\\
=&f(-q^{2})f(-q^{13})\Big\{q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k)+q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)\\
&-q^{3}f(-q^{10})f(-q^{65})\Big\}, \qquad (19.15)
\end{aligned}$$

or equivalently upon applying the Jacobi triple product identity (2.6) to $f(-q^4,-q^6)$, $f(-q^{26},-q^{39})$, $f(-q^2,-q^8)$, and $f(-q^{13},-q^{52})$, we deduce that

$$\begin{aligned}
&(f(-q^{4},-q^{6})f(-q^{26},-q^{39})+q^{3}f(-q^{2},-q^{8})f(-q^{13},-q^{52}))^{2}\\
=&f(-q^{2})f(-q^{13})\Big\{q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k)+q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)\Big\}. \qquad (19.16)
\end{aligned}$$

We now turn our attention to the two sums appearing on the right-hand side of (19.16). From (4.13), we see that

$$\begin{aligned}
&q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},1,15,\tfrac{1}{2},k)=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t}q^{\frac{1}{2}\left\{u+\frac{1}{2}+2t\right\}^{2}+13t^{2}}\\
=&\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t}q^{\frac{1}{2}\left\{u+\frac{1}{2}\right\}^{2}+13t^{2}}=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}q^{\frac{1}{2}\left\{u+\frac{1}{2}\right\}^{2}}\sum_{t=-\infty}^{\infty}(-1)^{t}q^{13t^{2}}\\
=&\psi(q)\varphi(-q^{13})=\frac{(q^{2};q^{2})_{\infty}}{(q;q^{2})_{\infty}}(q^{13};q^{26})_{\infty}^{2}(q^{26};q^{26})_{\infty}=\frac{(q^{2};q^{2})_{\infty}^{2}(q^{13};q^{13})_{\infty}^{2}}{(q;q)_{\infty}(q^{26};q^{26})_{\infty}}\\
=&\frac{f^{2}(-q^{2})f^{2}(-q^{13})}{f(-q)f(-q^{26})}, \qquad (19.17)
\end{aligned}$$

where we have utilized (2.7)–(2.9). Similarly, from (4.13), we find that

$$q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k)=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{17}{2}\left\{u+\frac{1}{2}+\frac{22}{17}t\right\}^2+\frac{13}{17}t^2}$$

$$=\frac{q^{-1/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{17}{2}\left\{u+\frac{1}{2}+\frac{5}{17}t\right\}^2+\frac{13}{17}t^2}$$

$$=\frac{1}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{2+\frac{3}{2}t^2+\frac{5}{2}t+5tu+\frac{17}{2}u(u+1)}$$

$$=-\frac{q}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}, \tag{19.18}$$

where in the last equality we replaced t by $t-1$.

We now claim that

$$\frac{1}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}=f(-q)f(-q^{26}). \tag{19.19}$$

To this end, we dissect the series according as $u\equiv 0,1,-1\,(\mathrm{mod}\,3)$ respectively. We consider each of the three sums separately. If we replace u by $3u$ and t by $-t-5u$, we find that

$$\sum_{\substack{u=-\infty\\ u\equiv 0\,(\mathrm{mod}\,3)}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}=\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u} q^{\frac{3}{2}t^2+\frac{1}{2}t+39u^2+13u}$$

$$=\sum_{u=-\infty}^{\infty}(-1)^u q^{39u^2+13u}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2+\frac{1}{2}t}=f(-q^{26})f(-q), \tag{19.20}$$

by (2.9). Next, if we replace u by $3u+1$ and t by $-t-5u$ in the series in (19.19), we find that

$$\sum_{\substack{u=-\infty\\ u\equiv 1\,(\mathrm{mod}\,3)}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}=\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u} q^{12+\frac{3}{2}t^2-\frac{9}{2}t+39u^2+13u}$$

$$=\sum_{u=-\infty}^{\infty}(-1)^u q^{39u^2+39u}\sum_{t=-\infty}^{\infty}(-1)^t q^{12+\frac{3}{2}t^2-\frac{9}{2}t}=0, \tag{19.21}$$

by (2.4). Finally, if we replace u by $3u-1$ and t by $-t-5u+2$ in the series in (19.19), we see that

$$\sum_{\substack{u=-\infty\\ u\equiv -1\,(\mathrm{mod}\,3)}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u}=\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u} q^{\frac{3}{2}t^2-\frac{1}{2}t+39u^2-13u}$$

$$=\sum_{u=-\infty}^{\infty}(-1)^u q^{39u^2-13u}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t}=f(-q^{26})f(-q), \tag{19.22}$$

by (2.9). Thus, in (19.20)–(19.22), we have shown that

$$\frac{1}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^t q^{\frac{3}{2}t^2-\frac{1}{2}t+5tu+\frac{17}{2}u^2+\frac{7}{2}u} = \frac{1}{2}(f(-q)f(-q^{26})+f(-q)f(-q^{26}))$$
$$=f(-q)f(-q^{26}), \tag{19.23}$$

as claimed in (19.19). Thus, from (19.18) and (19.23), we deduce that

$$q^{-1/8}\sum_{k=1}^{7}F(1,\tfrac{13}{2},11,15,\tfrac{17}{2},k) = -qf(-q)f(-q^{26}). \tag{19.24}$$

Finally, we insert (19.17) and (19.24) into (19.16) to arrive at

$$(f(-q^4,-q^6)f(-q^{26},-q^{39})+q^3f(-q^2,-q^8)f(-q^{13},-q^{52}))^2$$
$$=f(-q^2)f(-q^{13})\Big\{\frac{f^2(-q^2)f^2(-q^{13})}{f(-q)f(-q^{26})}-qf(-q)f(-q^{26})\Big\}. \tag{19.25}$$

Dividing both sides of (19.25) by $f^2(-q^2)f^2(-q^{13})$ and taking square roots, we obtain

$$\frac{f(-q^4,-q^6)f(-q^{26},-q^{39})}{f(-q^2)f(-q^{13})}+q^3\frac{f(-q^2,-q^8)f(-q^{13},-q^{52})}{f(-q^2)f(-q^{13})}$$
$$=\sqrt{\frac{f(-q^2)f(-q^{13})}{f(-q)f(-q^{26})}-q\frac{f(-q)f(-q^{26})}{f(-q^2)f(-q^{13})}}$$
$$=\sqrt{\frac{\chi(-q^{13})}{\chi(-q)}-q\frac{\chi(-q)}{\chi(-q^{13})}}, \tag{19.26}$$

by (2.14). Using (2.11), we see that we have shown that the left side in (3.18) is equal to (3.19).

We now turn to the right-hand side of (3.18) and show that it equals the expression in (3.19). Our argument is brief, since the proof is similar to the previous proof above. We apply Lemma 5.1, and then apply it a second time with q replaced by q^{26}. Then multiply the two resulting equlities together to obtain

$$f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5}) = f^2(-q^2,-q^3)f^2(-q^{52},-q^{78})$$
$$-q^{2/5}f^2(-q,-q^4)f^2(-q^{52},-q^{78})-q^{1/5}f(-q)f(-q^5)f^2(-q^{52},-q^{78})$$
$$-q^{52/5}f^2(-q^2,-q^3)f^2(-q^{26},-q^{104})+q^{54/5}f^2(-q,-q^4)f^2(-q^{26},-q^{104})$$
$$+q^{53/5}f(-q)f(-q^5)f^2(-q^{26},-q^{104})-q^{26/5}f^2(-q^2,-q^3)f(-q^{26})f(-q^{130})$$
$$+q^{28/5}f^2(-q,-q^4)f(-q^{26})f(-q^{130})+q^{27/5}f(-q)f(-q^5)f(-q^{26})f(-q^{130}). \tag{19.27}$$

Recalling the definition of $\mathcal{L}$ at the beginning of this section, we have

$$\mathcal{L}\left(q^{-2/5}f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5})\right) = -f^2(-q,-q^4)f^2(-q^{52},-q^{78})$$
$$-q^{10}f^2(-q^2,-q^3)f^2(-q^{26},-q^{104})+q^5f(-q)f(-q^5)f(-q^{26})f(-q^{130}). \tag{19.28}$$

We then apply (19.5) to obtain representations for $f(-q^{1/5})$ and $f(-q^{26/5})$ by replacing q by $q^{1/5}$ and q by $q^{26/5}$, respectively. This gives us

$$\begin{aligned} f(-q^{1/5}) =& f(-q^7,-q^8) - q^{1/5}f(-q^{10},-q^5) + qf(-q^{13},-q^2) \\ &+ q^{7/5}f(-q^{14},-q) - q^{2/5}f(-q^4,-q^{11}) \end{aligned} \tag{19.29}$$

and

$$\begin{aligned} f(-q^{26/5}) =& f(-q^{182},-q^{208}) - q^{26/5}f(-q^{260},-q^{130}) + q^{26}f(-q^{338},-q^{52}) \\ &+ q^{182/5}f(-q^{364},-q^{26}) - q^{52/5}f(-q^{104},-q^{286}). \end{aligned} \tag{19.30}$$

Thus, since $f(-q)f(-q^{26})$ contains only terms with integral powers, we find upon using (19.29) and (19.30) that

$$\begin{aligned} &\mathcal{L}\left(q^{-2/5}f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5})\right) = f(-q)f(-q^{26})\mathcal{L}\left(q^{-2/5}f(-q^{1/5})f(-q^{26/5})\right) \\ =& f(-q)f(-q^{26})\Big\{ - f(-q^4,-q^{11})f(-q^{182},-q^{208}) + q^{37}f(-q^2,-q^{13})f(-q^{26},-q^{364}) \\ &+ q^{36}f(-q^7,-q^8)f(-q^{26},-q^{364}) + q^5f(-q^5,-q^{10})f(-q^{130},-q^{260}) \\ &- q^{10}f(-q^7,-q^8)f(-q^{104},-q^{286}) - q^{11}f(-q^2,-q^{13})f(-q^{104},-q^{286}) \\ &+ qf(-q,-q^{14})f(-q^{182},-q^{208}) + q^{27}f(-q,-q^{14})f(-q^{52},-q^{338}) \\ &- q^{26}f(-q^4,-q^{11})f(-q^{52},-q^{338})\Big\}. \end{aligned} \tag{19.31}$$

Now from (4.13) and several applications of (2.5), we see that

$$\begin{aligned} & q^{-5/8}\sum_{k=1}^{7} F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k) = f(-q^4,-q^{11})f(-q^{182},-q^{208}) \\ & -q^2f(-q^3,-q^{12})f(-q^{156},-q^{234}) - q^5f(-q^5,-q^{10})f(-q^{130},-q^{260}) \\ & +q^{11}f(-q^2,-q^{13})f(-q^{104},-q^{286}) + q^{17}f(-q^{6=},-q^9)f(-q^{78},-q^{312}) \\ & -q^{27}f(-q,-q^{14})f(-q^{52},-q^{338}) - q^{36}f(-q^7,-q^{=}8)f(-q^{26},-q^{364}) \end{aligned} \tag{19.32}$$

and

$$\begin{aligned} & q^{-5/8}\sum_{k=1}^{7} F(\tfrac{1}{2},13,13,15,\tfrac{13}{2},k) = qf(-q,-q^{14})f(-q^{182},-q^{208}) \\ & -q^2f(-q^3,-q^{12})f(-q^{156},-q^{234}) + q^5f(-q^5,-q^{10})f(-q^{130},-q^{260}) \\ & -q^{10}f(-q^7,-q^8)f(-q^{104},-q^{286}) + q^{17}f(-q^6,-q^9)f(-q^{78},-q^{312}) \\ & -q^{26}f(-q^4,-q^{11})f(-q^{52},-q^{338}) + q^{37}f(-q^2,-q^{13})f(-q^{26},-q^{364}). \end{aligned} \tag{19.33}$$

Comparing the right-hand sides of (19.31), (19.32), and (19.33), we see that

$$\begin{aligned} &\mathcal{L}\left(q^{-2/5}f(-q)f(-q^{26})f(-q^{1/5})f(-q^{26/5})\right) \\ =& f(-q)f(-q^{26})\Big\{ - q^{-5/8}\sum_{k=1}^{7} F(\tfrac{1}{2},13,7,15,\tfrac{5}{2},k) + q^{-5/8}\sum_{k=1}^{7} F(\tfrac{1}{2},13,13,15,\tfrac{13}{2},k) \\ &- q^5f(-q^5)f(-q^{130})\Big\}. \end{aligned} \tag{19.34}$$

Now, combining the right-hand sides of (19.28) and (19.34), applying the Jacobi triple product identity (2.6) to $f(-q,-q^4)$, $f(-q^{52},-q^{78})$, $f(-q^2,-q^3)$, and $f(-q^{26},-q^{104})$, and simplifying, we deduce that

$$\begin{aligned}&\big(f(-q,-q^4)f(-q^{52},-q^{78})-q^5f(-q^2,-q^3)f(-q^{26},-q^{104})\big)^2\\=&-f(-q)f(-q^{26})\Big\{-q^{-5/8}\sum_{k=1}^{7}F(\tfrac12,13,7,15,\tfrac52,k)+q^{-5/8}\sum_{k=1}^{7}F(\tfrac12,13,13,15,\tfrac{13}{2},k)\Big\}.\end{aligned}\tag{19.35}$$

We now concentrate on the two sums arising on the right-hand side of (19.35) above. From (4.13), we have

$$\begin{aligned}q^{-5/8}&\sum_{k=1}^{7}F(\tfrac12,13,13,15,\tfrac{13}{2},k)=\frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac{13}{2}\left\{u+\frac12+t\right\}^2+t^2}\\=&\frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac{13}{2}\left\{u+\frac12\right\}^2+t^2}=\frac{q}{2}\sum_{u=-\infty}^{\infty}q^{\frac{13}{2}u(u+1)}\sum_{t=-\infty}^{\infty}(-1)^tq^{t^2}\\=&q\psi(q^{13})\varphi(-q)=q\frac{(q^{26};q^{26})_\infty}{(q^{13};q^{26})_\infty}(q;q^2)_\infty^2(q^2;q^2)_\infty=q\frac{(q^{26};q^{26})_\infty^2(q;q)_\infty^2}{(q^{13};q^{13})_\infty(q^2;q^2)_\infty}\\=&q\frac{f^2(-q^{26})f^2(-q)}{f(-q^{13})f(-q^2)},\end{aligned}\tag{19.36}$$

where we have used (2.7)–(2.9). Similarly, from (4.13), we find that

$$\begin{aligned}q^{-5/8}&\sum_{k=1}^{7}F(\tfrac12,13,7,15,\tfrac52,k)=\frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac52\left\{u+\frac12+\frac75t\right\}^2+\frac{13}{5}t^2}\\=&\frac{q^{-5/8}}{2}\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{\frac52\left\{u+\frac12+\frac25t\right\}^2+\frac{13}{5}t^2}=\frac12\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{3t^2+t+2tu+\frac52u(u+1)}.\end{aligned}\tag{19.37}$$

As in the previous proof, we dissect the series above according as $u\equiv 0,1,-1 \pmod 3$. Assuming that $u\equiv 0 \pmod 3$, we replace u by $3u$ and t by $-t-u$ to find that

$$\begin{aligned}&\sum_{\substack{u=-\infty\\u\equiv 0\,(\mathrm{mod}\,3)}}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^tq^{3t^2+t+2tu+\frac52u(u+1)}=\sum_{u=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}(-1)^{t+u}q^{3t^2-t+\frac{39}{2}u^2+\frac{13}{2}u}\\=&\sum_{u=-\infty}^{\infty}(-1)^uq^{\frac{13}{2}u(3u+1)}=\sum_{t=-\infty}^{\infty}(-1)^tq^{t(3t-1)}=f(-q^{13})f(-q^2),\end{aligned}\tag{19.38}$$

by (2.9). Next, if we replace u by $3u+1$ and t by $t-u$ in the series in (19.19), we find that

$$\sum_{\substack{u=-\infty \\ u\equiv 1 \pmod 3}}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{3t^2+t+2tu+\frac{5}{2}u(u+1)} = \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^{t+u} q^{3t^2+3t+\frac{39}{2}u^2+\frac{39}{2}u+2}$$

$$= \sum_{u=-\infty}^{\infty} (-1)^u q^{\frac{39}{2}u^2+\frac{39}{2}u+2} \sum_{t=-\infty}^{\infty} (-1)^t q^{3t^2+3t} = 0, \tag{19.39}$$

by (2.4). Finally, if we replace u by $3u-1$ and t by $t-u$, we obtain

$$\sum_{\substack{u=-\infty \\ u\equiv -1 \pmod 3}}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^t q^{3t^2+t+2tu+\frac{5}{2}u(u+1)} = \sum_{u=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (-1)^{t+u} q^{3t^2-t+\frac{39}{2}u^2-\frac{13}{2}u}$$

$$= f(-q^{13})f(-q^2), \tag{19.40}$$

by (2.9). Thus, from (19.37)–(19.40), we have shown that

$$q^{-5/8}\sum_{k=1}^{7} F(\tfrac{1}{2}, 13, 7, 15, \tfrac{5}{2}, k) = \frac{1}{2}\left(f(-q^{13})f(-q^2) + f(-q^{13})f(-q^2)\right)$$

$$= f(-q^{13})f(-q^2). \tag{19.41}$$

Finally, we insert (19.36) and (19.41) into (19.35) and conclude that

$$\left(f(-q,-q^4)f(-q^{52},-q^{78}) - q^5 f(-q^2,-q^3)f(-q^{26},-q^{104})\right)^2$$

$$= -f(-q)f(-q^{26})\left\{-f(-q^{13})f(-q^2) + q\frac{f^2(-q^{26})f^2(-q)}{f(-q^{13})f(-q^2)}\right\}. \tag{19.42}$$

Dividing both sides of (19.42) by $f^2(-q)f^2(-q^{26})$ and taking square roots, we arrive at

$$\frac{f(-q,-q^4)f(-q^{52},-q^{78}) - q^5 f(-q^2,-q^3)f(-q^{26},-q^{104})}{f(-q^2)f(-q^{26})}$$

$$= \sqrt{\frac{f(-q^2)f(-q^{13})}{f(-q)f(-q^{26})} - q\frac{f(-q)f(-q^{26})}{f(-q^2)f(-q^{13})}}$$

$$= \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}, \tag{19.43}$$

which completes the proof of the second part of Entry 3.16.

20. Proof of Entry 3.17

Using the product representations of $\chi(q)$ and $f(-q)$, given in (2.9) and (2.10), respectively, together with (2.7) and (2.8), we find that

$$\varphi(q) = f(q,q) = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \chi^2(q) f(-q^2) \tag{20.1}$$

and

$$\psi(q) = f(q,q^3) = \frac{(q^2:q^2)_\infty}{(q;q^2)_\infty} = \frac{(q;q)_\infty}{(q;q^2)_\infty}\frac{1}{(q;q^2)_\infty} = \frac{f(-q)}{\chi^2(-q)}. \tag{20.2}$$

By (2.11), (20.2), and (20.1) with q replaced by $q^{1/2}$ and $-q^{1/2}$, respectively, we find that Entry 3.17 is equivalent to the identity

$$\begin{aligned}
&f(-q^2,-q^3)f(-q^{38},-q^{57}) + q^4 f(-q,-q^4)f(-q^{19},-q^{76}) \\
&= f(-q)f(-q^{19})\Big\{\frac{\chi^2(q^{1/2})\chi^2(q^{19/2})}{4\sqrt{q}} - \frac{\chi^2(-q^{1/2})\chi^2(-q^{19/2})}{4\sqrt{q}} - \frac{q^2}{\chi^2(-q)\chi^2(-q^{19})}\Big\} \\
&= \frac{1}{4\sqrt{q}}\Big(\varphi(q^{1/2})\varphi(q^{19/2}) - \varphi(-q^{1/2})\varphi(-q^{19/2})\Big) - q^2\psi(q)\psi(q^{19}).
\end{aligned} \tag{20.3}$$

We now apply Theorem 4.5 with the parameters $\epsilon_1 = \epsilon_2 = 0,\ a = b = q,\ c = d = q^{19},\ \alpha = 1,\ \beta = 19,$ and $m = 20$. Accordingly, we deduce that

$$\begin{aligned}
\varphi(q)\varphi(q^{19}) &= f(q^{20},q^{20})f(q^{380},q^{380}) + q^{19}f(q^{-18},q^{58})f(q^{342},q^{418}) \\
&+ q^{76}f(q^{-56},q^{96})f(q^{304},q^{456}) + q^{171}f(q^{-94},q^{134})f(q^{266},q^{494}) \\
&+ q^{304}f(q^{-132},q^{172})f(q^{228},q^{532}) + q^{475}f(q^{-170},q^{210})f(q^{190},q^{570}) \\
&+ q^{684}f(q^{-208},q^{248})f(q^{152},q^{608}) + q^{931}f(q^{-246},q^{286})f(q^{114},q^{646}) \\
&+ q^{1216}f(q^{-284},q^{324})f(q^{76},q^{684}) + q^{1539}f(q^{-322},q^{362})f(q^{38},q^{722}) \\
&+ q^{1900}f(q^{-360},q^{400})f(1,q^{760}) + q^{2299}f(q^{-398},q^{438})f(q^{-38},q^{798}) \\
&+ q^{2736}f(q^{-436},q^{476})f(q^{-76},q^{836}) + q^{3211}f(q^{-474},q^{514})f(q^{-114},q^{874}) \\
&+ q^{3724}f(q^{-512},q^{552})f(q^{-152},q^{912}) + q^{4275}f(q^{-550},q^{590})f(q^{-190},q^{950}) \\
&+ q^{4864}f(q^{-588},q^{628})f(q^{-228},q^{988}) + q^{5491}f(q^{-626},q^{666})f(q^{-266},q^{1026}) \\
&+ q^{6156}f(q^{-664},q^{704})f(q^{-304},q^{1064}) + q^{6859}f(q^{-702},q^{742})f(q^{-342},q^{1102}) \\
&= f(q^{20},q^{20})f(q^{380},q^{380}) + 2\,qf(q^{18},q^{22})f(q^{342},q^{418}) \\
&+ 2\,q^4f(q^{16},q^{24})f(q^{304},q^{456}) + 2\,q^9f(q^{14},q^{26})f(q^{266},q^{494}) + 2\,q^{16}f(q^{12},q^{28})f(q^{228},q^{532}) \\
&+ 2\,q^{25}f(q^{10},q^{30})f(q^{190},q^{570}) + 2\,q^{36}f(q^{8},q^{32})f(q^{152},q^{608}) + 2\,q^{49}f(q^{6},q^{34})f(q^{114},q^{646}) \\
&+ 2\,q^{64}f(q^{4},q^{36})f(q^{76},q^{684}) + 2\,q^{81}f(q^{2},q^{38})f(q^{38},q^{722}) + q^{100}f(1,q^{40})f(1,q^{760}),
\end{aligned} \tag{20.4}$$

after several applications of (2.5). Upon replacing q by $-q$ in (20.4), we conclude that

$$\begin{aligned}
&\frac{1}{4q}\Big(\varphi(q)\varphi(q^{19}) - \varphi(-q)\varphi(-q^{19})\Big) \\
&= f(q^{18},q^{22})f(q^{342},q^{418}) + q^8f(q^{14},q^{26})f(q^{266},q^{494}) + q^{24}f(q^{10},q^{30})f(q^{190},q^{570}) \\
&\quad + q^{48}f(q^{6},q^{34})f(q^{114},q^{646}) + q^{80}f(q^{2},q^{38})f(q^{38},q^{722}).
\end{aligned} \tag{20.5}$$

Next, we employ (4.14) with the two sets of parameters $\alpha_1 = 1,\ \beta_1 = 19,\ m_1 = 1,\ p_1 = 2,\ \lambda_1 = 10$ and $\alpha_2 = 1,\ \beta_2 = 19,\ m_2 = 9,\ p_2 = 10,\ \lambda_2 = 10$. We find that the conditions in (4.9) are satisfied. Hence, using (4.18) and (4.14), we find that

$$q^{5/2}\psi(-q)\psi(-q^{19}) = q^{5/2}f(-q^{19},-q)f(-q^{209},-q^{171}) + q^{45/2}f(-q^{37},-q^{-17})f(-q^{247},-q^{133})$$
$$+q^{125/2}f(-q^{55},-q^{-35})f(-q^{285},-q^{95}) + q^{245/2}f(-q^{73},-q^{-53})f(-q^{323},-q^{57})$$
$$+q^{405/2}f(-q^{91},-q^{-71})f(-q^{361},-q^{19}).$$

After several applications of (2.5), we obtain the identity

$$\begin{aligned}\psi(-q)\psi(-q^{19}) =& f(-q,-q^{19})f(-q^{171},-q^{209}) - q^3f(-q^3,-q^{17})f(-q^{133},-q^{247})\\ &+ q^{10}f(-q^5,-q^{15})f(-q^{95},-q^{285}) - q^{21}f(-q^7,-q^{13})f(-q^{57},-q^{323})\\ &+ q^{36}f(-q^9,-q^{11})f(-q^{19},-q^{361}).\end{aligned} \tag{20.6}$$

By using (20.5) with q replaced by $q^{1/2}$ and (20.6) with q replaced by $-q$, we conclude that

$$\begin{aligned}&\frac{1}{4\sqrt{q}}\Big(\varphi(q^{1/2})\varphi(q^{19/2}) - \varphi(-q^{1/2})\varphi(-q^{19/2})\Big) - q^2\psi(q)\psi(q^{19})\\
&= f(q^9,q^{11})f(q^{171},q^{209}) + q^4f(q^7,q^{13})f(q^{133},q^{247}) + q^{24}f(q^3,q^{17})f(q^{57},q^{323})\\
&\quad + q^{40}f(q,q^{19})f(q^{19},q^{361}) - q^2f(q,q^{19})f(q^{171},q^{209}) - q^5f(q^3,q^{17})f(q^{133},q^{247})\\
&\quad - q^{23}f(q^7,q^{13})f(q^{57},q^{323}) - q^{38}f(q^9,q^{11})f(q^{19},q^{361})\\
&= \left(f(q^9,q^{11}) - q^2f(q,q^{19})\right)\left(f(q^{171},q^{209}) - q^{38}f(q^{19},q^{361})\right)\\
&\quad + q^4\left(f(q^7,q^{13}) - qf(q^3,q^{17})\right)\left(f(q^{133},q^{247}) - q^{19}f(q^{57},q^{323})\right)\\
&= f(-q^2,-q^3)f(-q^{38},-q^{57}) + q^4f(-q,-q^4)f(-q^{19},-q^{76}),\end{aligned}$$

where in the last step we used (26.4) and (26.5) with q replaced by $-q$ and $-q^{19}$, respectively. This completes the proof of Entry 3.17.

21. Proof of Entry 3.18

The following proof of Entry 3.18 is due to Bressoud [10].

By (2.11), (2.7), (2.8), and (2.14), it is easy to see that (3.21) is equivalent to the identity

$$\begin{aligned}&f(-q,-q^4)f(-q^{62},-q^{93}) - q^6f(-q^2,-q^3)f(-q^{31},-q^{124})\\ =&\frac{1}{2q}\varphi(-q^2)\varphi(-q^{62}) - \frac{1}{2q}\varphi(-q)\varphi(-q^{31}) + q^3\psi(-q)\psi(-q^{31}).\end{aligned} \tag{21.1}$$

By (4.16), and (4.13), with the set of parameters, $\alpha = 1/2$, $\beta = 31/2$, $m = 3$, $p = 5$, and $\lambda = 4$, we find that

$$\begin{aligned}&qf(-q,-q^4)f(-q^{62},-q^{93}) - q^7f(-q^2,-q^3)f(-q^{31},-q^{124})\\ &= \sum_{k=1}^{2} F(\tfrac{1}{2},\tfrac{31}{2},3,5,4,k) = \frac{1}{2}\sum_{u,\,t=-\infty}^{\infty}(-1)^tq^I =: \frac{1}{2}R,\end{aligned} \tag{21.2}$$

where, by (4.7), I is given by

$$I = 4\left\{u + \frac{1}{2} + \frac{3t}{8}\right\}^2 + \frac{31}{16}t^2 = (2u+1)^2 + \frac{3}{2}(2u+1)t + \frac{5}{2}t^2. \tag{21.3}$$

Therefore, by (21.1)–(21.3), it suffices to prove that

$$\begin{aligned} R &= \sum_{u,\,t=-\infty}^{\infty} (-1)^t q^{(2u+1)^2 + \frac{3}{2}(2u+1)t + \frac{5}{2}t^2} \\ &= \varphi(-q^2)\varphi(-q^{62}) - \varphi(-q)\varphi(-q^{31}) + 2q^4\psi(-q)\psi(-q^{31}). \end{aligned} \tag{21.4}$$

We establish (21.4) by a series of changes of the indices of summation. To that end

$$\begin{aligned} R &= \sum_{u,\,t=-\infty}^{\infty} (-1)^t q^{(2u+1)^2 + \frac{3}{2}(2u+1)t + \frac{5}{2}t^2} \\ &= \sum_{j=0}^{1} \sum_{u,\,r=-\infty}^{\infty} (-1)^{2r+j} q^{(2u+1)^2 + \frac{3}{2}(2u+1)(2r+j) + \frac{5}{2}(2r+j)^2} \\ &= \sum_{u,\,r=-\infty}^{\infty} q^{4u^2+10r^2+6ru+4u+3r+1} - \sum_{u,\,r=-\infty}^{\infty} q^{4u^2+10r^2+6ru+7u+13r+5} \\ &= \sum_{s,\,r=-\infty}^{\infty} q^{4(s-r)^2+10r^2+6r(s-r)+4(s-r)+3r+1} \\ &\quad - \sum_{s,\,r=-\infty}^{\infty} q^{4(s-r-1)^2+10r^2+6r(s-r-1)+7(s-r-1)+13r+5} \\ &= \sum_{s,\,r=-\infty}^{\infty} q^{(2s+1)^2-(2s+1)r+8r^2} - \sum_{s,\,r=-\infty}^{\infty} q^{4s^2-s(2r+1)+2(2r+1)^2} \\ &= \sum_{\substack{s,\,r=-\infty \\ s \text{ odd}}}^{\infty} q^{s^2-sr+8r^2} - \sum_{\substack{s,\,r=-\infty \\ r \text{ odd}}}^{\infty} q^{4s^2-sr+2r^2}. \end{aligned} \tag{21.5}$$

But, trivially,

$$\sum_{\substack{s,\,r=-\infty \\ r \text{ even}}}^{\infty} q^{4s^2-sr+2r^2} - \sum_{\substack{s,\,r=-\infty \\ s \text{ even}}}^{\infty} q^{s^2-sr+8r^2} = 0. \tag{21.6}$$

Therefore, returning to (21.5), we find that

$$\begin{aligned} R &= \sum_{\substack{s,\,r=-\infty \\ s \text{ odd}}}^{\infty} q^{s^2-sr+8r^2} - \sum_{\substack{s,\,r=-\infty \\ r \text{ odd}}}^{\infty} q^{4s^2-sr+2r^2} + \sum_{\substack{s,\,r=-\infty \\ r \text{ even}}}^{\infty} q^{4s^2-sr+2r^2} - \sum_{\substack{s,\,r=-\infty \\ s \text{ even}}}^{\infty} q^{s^2-sr+8r^2} \\ &= \sum_{s,\,r=-\infty}^{\infty} (-1)^r q^{4s^2-sr+2r^2} - \sum_{s,\,r=-\infty}^{\infty} (-1)^s q^{s^2-sr+8r^2} =: R_1 - R_2. \end{aligned} \tag{21.7}$$

Next, we evaluate R_1 and R_2 separately. First,

$$\begin{aligned}
R_1 &= \sum_{s,\,r=-\infty}^{\infty} (-1)^r q^{4s^2-sr+2r^2} = \sum_{j=0}^{3} \sum_{t,\,r=-\infty}^{\infty} (-1)^r q^{4(4t+j)^2-(4t+j)r+2r^2} \\
&= \sum_{j=0}^{3} \sum_{t,\,r=-\infty}^{\infty} (-1)^r q^{64t^2+32tj+4j^2-4rt-jr+2r^2} = \sum_{j=0}^{3} \sum_{t,\,r=-\infty}^{\infty} (-1)^r q^{62(t+\frac{j}{4})^2+2(t-r+\frac{j}{4})^2} \\
&= \sum_{j=0}^{3} \sum_{t,\,s=-\infty}^{\infty} (-1)^{t-s} q^{62(t+\frac{j}{4})^2+2(s+\frac{j}{4})^2} = \sum_{j=0}^{3} \Big\{ \sum_{t=-\infty}^{\infty} (-1)^t q^{62(t+\frac{j}{4})^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{2(s+\frac{j}{4})^2} \Big\}.
\end{aligned} \tag{21.8}$$

Observe that

$$\begin{aligned}
\sum_{s=-\infty}^{\infty} (-1)^s q^{(s+\frac{1}{4})^2} &= \sum_{s=-\infty}^{\infty} (-1)^s q^{(s+1-\frac{3}{4})^2} = \sum_{s=-\infty}^{\infty} (-1)^{s-1} q^{(s-\frac{3}{4})^2} \\
&= -\sum_{s=-\infty}^{\infty} (-1)^{-s} q^{(-s-\frac{3}{4})^2} = -\sum_{s=-\infty}^{\infty} (-1)^s q^{(s+\frac{3}{4})^2}.
\end{aligned} \tag{21.9}$$

Thus, in (21.8), the contributions from the terms when $j = 1$ and $j = 3$ are the same. By (2.4), the contribution from the term $j = 2$ is 0. Therefore, by (2.1), (2.7) and (2.8), we conclude that

$$\begin{aligned}
R_1 &= \sum_{t=-\infty}^{\infty} (-1)^t q^{62t^2} \sum_{s=-\infty}^{\infty} (-1)^s q^{2s^2} + 2q^4 \sum_{t=-\infty}^{\infty} (-1)^t q^{62t^2+31t} \sum_{s=-\infty}^{\infty} (-1)^s q^{2s^2+s} \\
&= \varphi(-q^2)\varphi(-q^{62}) + 2q^4 f(-q,-q^3) f(-q^{31},-q^{93}) \\
&= \varphi(-q^2)\varphi(-q^{62}) + 2q^4 \psi(-q)\psi(-q^{31}).
\end{aligned} \tag{21.10}$$

Similarly,

$$\begin{aligned}
R_2 &= \sum_{s,\,r=-\infty}^{\infty} (-1)^s q^{s^2-sr+8r^2} = \sum_{j=0}^{1} \sum_{s,\,t=-\infty}^{\infty} (-1)^s q^{s^2-s(2t+j)+8(2t+j)^2} \\
&= \sum_{j=0}^{1} \sum_{s,\,t=-\infty}^{\infty} (-1)^s q^{s^2-sj-2st+32t^2+32tj+8j^2} = \sum_{j=0}^{1} \sum_{s,\,t=-\infty}^{\infty} (-1)^s q^{31(t+\frac{j}{2})^2+(t-s+\frac{j}{2})^2} \\
&= \sum_{j=0}^{1} \sum_{t,\,r=-\infty}^{\infty} (-1)^{t-r} q^{31(t+\frac{j}{2})^2+(r+\frac{j}{2})^2} = \sum_{j=0}^{1} \Big\{ \sum_{t=-\infty}^{\infty} (-1)^t q^{31(t+\frac{j}{2})^2} \sum_{r=-\infty}^{\infty} (-1)^r q^{(r+\frac{j}{2})^2} \Big\} \\
&= \sum_{t=-\infty}^{\infty} (-1)^t q^{31t^2} \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2} = \varphi(-q)\varphi(-q^{31}),
\end{aligned} \tag{21.11}$$

where we used (2.4) again. By (21.7), (21.10), and (21.11), we conclude that

$$R = \varphi(-q^2)\varphi(-q^{62}) + 2q^4\psi(-q)\psi(-q^{31}) - \varphi(-q)\varphi(-q^{31}),$$

which is (21.4). Hence, the proof of Entry 3.18 is complete.

22. Proof of Entry 3.19

In the proof below, we actually provide two variations. Like Bressoud [10], we begin with an application of Rogers's ideas, but then the proofs diverge.

Proof. By (2.11), the first part of Entry 3.19 can be put in the form

$$\begin{aligned} &f(-q^2,-q^3)f(-q^{78},-q^{117})+q^8f(-q,-q^4)f(-q^{39},-q^{156})\\ =&f(-q^{26},-q^{39})f(-q^3,-q^{12})-q^2f(-q^6,-q^9)f(-q^{13},-q^{52}). \end{aligned} \tag{22.1}$$

We apply Rogers's method first with $\alpha_1=\frac{1}{2}$, $\beta_1=\frac{39}{2}$, $p_1=5$, $m_1=1$, and $\lambda_1=4$, and secondly with $\alpha_2=\frac{3}{2}$, $\beta_2=\frac{13}{2}$, $p_2=5$, $m_2=3$, and $\lambda_2=4$. Then both sets of parameters satisfy (4.9). By (4.15) and (4.16), respectively, we find that

$$\sum_{k=1}^{2}F\left(\tfrac{1}{2},\tfrac{39}{2},1,5,4,k\right)=qf(-q^2,-q^3)f(-q^{78},-q^{117})+q^9f(-q,-q^4)f(-q^{39},-q^{156}) \tag{22.2}$$

and

$$\sum_{k=1}^{2}F\left(\tfrac{3}{2},\tfrac{13}{2},3,5,4,k\right)=qf(-q^3,-q^{12})f(-q^{26},-q^{39})-q^3f(-q^6,-q^9)f(-q^{13},-q^{52}). \tag{22.3}$$

Combining (22.2) and (22.3) together, we deduce (22.1) to complete the proof.

Next, we prove (3.23). Let us define, by (2.11),

$$g(q):=f(-q)G(q)=f(-q^2,-q^3)\qquad\text{and}\qquad h(q):=f(-q)H(q)=f(-q,-q^4). \tag{22.4}$$

Therefore, by (22.1), we can define $N(q)$ by

$$N(q):=g(q)g(q^{39})+q^8h(q)h(q^{39})=g(q^{13})h(q^3)-q^2g(q^3)h(q^{13}). \tag{22.5}$$

Let us also define

$$M(q):=g(q^2)g(q^{13})+q^3h(q^2)h(q^{13}), \tag{22.6}$$

$$L(q):=g(q^{26})h(q)-q^5g(q)h(q^{26}). \tag{22.7}$$

Lemma 22.1.

$$\begin{aligned} g(q)&=\frac{1}{\varphi(-q^9)}\Big\{-q^2\varphi(-q)h(q^9)+\varphi(-q^3)\chi(-q^3)g(q^6)\Big\},\\ h(q)&=\frac{1}{\varphi(-q^9)}\Big\{\varphi(-q)g(q^9)+q\varphi(-q^3)\chi(-q^3)h(q^6)\Big\}. \end{aligned} \tag{22.8}$$

Proof. To prove (22.8) we employ (2.19) with $a=-q^2$, $b=-q^3$, and $n=3$, we find that

$$\begin{aligned} f(-q^2,-q^3)&=f(-q^{21},-q^{24})-q^2f(-q^{36},-q^9)+q^9f(-q^{51},-q^{-6})\\ &=f(-q^{21},-q^{24})-q^2f(-q^9,-q^{36})-q^3f(-q^6,-q^{39}), \end{aligned} \tag{22.9}$$

where in the last step (2.5) is used. Similarly, with the choice of parameters $a = -q,\ b = -q^4$, and $n = 3$ one obtains

$$f(-q, -q^4) = f(-q^{18}, -q^{27}) - qf(-q^{12}, -q^{33}) - q^4 f(-q^3, -q^{42}). \tag{22.10}$$

Therefore, in the notation of (22.4), we obtain

$$g(q) = A(q^3) - q^2 h(q^9) \ \text{ and } \ h(q) = g(q^9) - qB(q^3), \tag{22.11}$$

where

$$A(q) = f(-q^7, -q^8) - qf(-q^2, -q^{13}) \ \text{ and } \ B(q) = f(-q^4, -q^{11}) + qf(-q, -q^{14}). \tag{22.12}$$

Next, we use Entries 3.6–3.8. By (2.14), we can rewrite them in their equivalent forms

$$g(q)g(q^9) + q^2 h(q)h(q^9) = f^2(-q^3), \tag{22.13}$$

$$g(q^2)g(q^3) + qh(q^2)h(q^3) = \psi(q)\varphi(-q^3), \tag{22.14}$$

and

$$h(q)g(q^6) - qg(q)h(q^6) = \psi(q^3)\varphi(-q). \tag{22.15}$$

Starting from (22.13), we have, by (22.11),

$$\begin{aligned} f^2(-q^3) &= g(q)g(q^9) + q^2 h(q)h(q^9) \\ &= \big(A(q^3) - q^2 h(q^9)\big)g(q^9) + q^2\big(g(q^9) - qB(q^3)\big)h(q^9) \\ &= A(q^3)g(q^9) - q^3 B(q^3)h(q^9). \end{aligned} \tag{22.16}$$

Similarly, starting from (22.15) and using (22.11) and (22.14) with q replaced by q^3, we deduce that

$$\begin{aligned} &\psi(q^3)\varphi(-q) \\ &= h(q)g(q^6) - qg(q)h(q^6) \\ &= g(q^6)g(q^9) + q^3 h(q^6)h(q^9) - q\big(A(q^3)h(q^6) + B(q^3)g(q^6)\big) \\ &= \psi(q^3)\varphi(-q^9) - q\big(A(q^3)h(q^6) + B(q^3)g(q^6)\big). \end{aligned} \tag{22.17}$$

Solving for $A(q)$ from the last two equations, we find that

$$\begin{aligned} &\big(g(q^6)g(q^9) + q^3 h(q^6)h(q^9)\big)A(q^3) \\ =&f^2(-q^3)g(q^6) + q^2\psi(q^3)\big(\varphi(-q^9) - \varphi(-q)\big)h(q^9). \end{aligned}$$

Using (22.14) again with q replaced by q^3, we conclude that

$$\varphi(-q^9)A(q^3) = \frac{f^2(-q^3)}{\psi(q^3)}g(q^6) + q^2\big(\varphi(-q^9) - \varphi(-q)\big)h(q^9). \tag{22.18}$$

Substituting this value of $A(q)$ in (22.11) yields the first identity of (22.8) after observing that

$$\frac{f^2(-q^3)}{\psi(q^3)} = \varphi(-q^3)\chi(-q^3).$$

Similarly one solves for $B(q)$ and obtains the analogous identity for $h(q)$. □

Using (22.8) in the first equation of (22.5), we find that

$$\begin{aligned}N(q) =&\frac{1}{\varphi(-q^9)}\Big\{ -q^2\varphi(-q)h(q^9)+\varphi(-q^3)\chi(-q^3)g(q^6)\Big\}g(q^{39})\\&+\frac{q^8}{\varphi(-q^9)}\Big\{\varphi(-q)g(q^9)+q\varphi(-q^3)\chi(-q^3)h(q^6)\Big\}h(q^{39})\\=&-q^2\frac{\varphi(-q)}{\varphi(-q^9)}\Big\{g(q^{39})h(q^9)-q^6g(q^9)h(q^{39})\Big\}\\&+\frac{\varphi(-q^3)\chi(-q^3)}{\varphi(-q^9)}\Big\{g(q^6)g(q^{39})+q^9h(q^6)h(q^{39})\Big\}\\=&-q^2\frac{\varphi(-q)}{\varphi(-q^9)}N(q^3)+\frac{\varphi(-q^3)\chi(-q^3)}{\varphi(-q^9)}M(q^3).\end{aligned}\tag{22.19}$$

Employing (22.8) again, this time with q replaced by q^{13} in the second equation of (22.5), we find that

$$\begin{aligned}N(q) =&\frac{1}{\varphi(-q^{117})}\Big\{ -q^{26}\varphi(-q^{13})h(q^{117})+\varphi(-q^{39})\chi(-q^{39})g(q^{78})\Big\}h(q^3)\\&-\frac{q^2}{\varphi(-q^{117})}\Big\{\varphi(-q^{13})g(q^{117})+q^{13}\varphi(-q^{39})\chi(-q^{39})h(q^{78})\Big\}g(q^3)\\=&-q^2\frac{\varphi(-q^{13})}{\varphi(-q^{117})}\Big\{g(q^3)g(q^{117})+q^{24}h(q^3)h(q^{117})\Big\}\\&+\frac{\varphi(-q^{39})\chi(-q^{39})}{\varphi(-q^{117})}\Big\{h(q^3)g(q^{78})-q^{15}g(q^3)h(q^{78})\Big\}\\=&-q^2\frac{\varphi(-q^{13})}{\varphi(-q^{117})}N(q^3)+\frac{\varphi(-q^{39})\chi(-q^{39})}{\varphi(-q^{117})}L(q^3).\end{aligned}\tag{22.20}$$

From (3.18), (2.11), and (2.14), we deduce that

$$\frac{L(q)}{M(q)}=\frac{f(-q)f(-q^{26})}{f(-q^2)f(-q^{13})}=\frac{\chi(-q)}{\chi(-q^{13})}.\tag{22.21}$$

Thus, by (22.19)–(22.21), we conclude that

$$q^2\left\{\frac{\varphi(-q^{13})}{\varphi(-q^{117})}-\frac{\varphi(-q)}{\varphi(-q^9)}\right\}N(q^3)=\left\{\frac{\varphi(-q^{39})\chi(-q^3)}{\varphi(-q^{117})}-\frac{\varphi(-q^3)\chi(-q^3)}{\varphi(-q^9)}\right\}M(q^3).\tag{22.22}$$

By (32.23), with q replaced by q^{13} and q, respectively, we find that

$$\begin{aligned}&\varphi(-q^9)\varphi(-q^{13})-\varphi(-q)\varphi(-q^{117})\\&\quad=\varphi(-q^9)\left\{\varphi(-q^{117})-2q^{13}f(-q^{39},-q^{195})\right\}-\left\{\varphi(-q^9)-2qf(-q^3,-q^{15})\right\}\varphi(-q^{117})\\&\quad=2q\left\{f(-q^3,-q^{15})\varphi(-q^{117})-q^{12}\varphi(-q^9)f(-q^{39},-q^{195})\right\}.\end{aligned}\tag{22.23}$$

Using (22.23) in (22.22) and replacing by q^3 by q, we arrive at

$$2q\left\{f(-q,-q^5)\varphi(-q^{39})-q^4\varphi(-q^3)f(-q^{13},-q^{65})\right\}N(q)$$
$$=\chi(-q)\left\{\varphi(-q^3)\varphi(-q^{13})-\varphi(-q)\varphi(-q^{39})\right\}M(q). \tag{22.24}$$

Comparing (22.24) to (3.23), we see that it suffices to prove that

$$M(q)=\frac{1}{\chi(-q)}\Big\{f(-q,-q^5)\varphi(-q^{39})-q^4\varphi(-q^3)f(-q^{13},-q^{65})\Big\}$$
$$=\psi(q^3)\varphi(-q^{39})-q^4f(q,q^2)f(-q^{13},-q^{65}), \tag{22.25}$$

where in the last step we used (11.6) and (11.8). To verify (22.25), we employ Theorem 4.5 with the parameters $a=b=q^{39}$, $c=1$, $d=q^3$, $\epsilon_1=1$, $\epsilon_2=0$, $\alpha=2$, $\beta=1$, and $m=15$, to find that

$$f(1,q^3)\varphi(-q^{39})=2\,f(-q^{42},-q^{48})f(-q^{273},-q^{312})+2\,q^3f(-q^{36},-q^{54})f(-q^{234},-q^{351})$$
$$+2\,q^9f(-q^{30},-q^{60})f(-q^{195},-q^{390})+2\,q^{18}f(-q^{24},-q^{66})f(-q^{156},-q^{429})$$
$$+2\,q^{30}f(-q^{18},-q^{72})f(-q^{117},-q^{468})+2\,q^{45}f(-q^{12},-q^{78})f(-q^{78},-q^{507})$$
$$+2\,q^{63}f(-q^6,-q^{84})f(-q^{39},-q^{546}). \tag{22.26}$$

Employing Theorem 4.5 again, this time with the parameters $a=q^{13}$, $b=q^{65}$, $c=q$, $d=q^2$, $\epsilon_1=1$, $\epsilon_2=0$, $\alpha=13$, $\beta=1$, and $m=15$, we find that

$$f(-q^{13},-q^{65})f(q,q^2)=f(-q^{273},-q^{312})f(-q^{18},-q^{72})+qf(-q^{234},-q^{351})f(-q^{24},-q^{66})$$
$$+q^5f(-q^{195},-q^{390})f(-q^{30},-q^{60})+q^{12}f(-q^{156},-q^{429})f(-q^{36},-q^{54})$$
$$+q^{22}f(-q^{117},-q^{468})f(-q^{42},-q^{48})+q^{35}f(-q^{78},-q^{507})f(-q^{42},-q^{48})$$
$$+q^{51}f(-q^{39},-q^{546})f(-q^{36},-q^{54})-q^{53}f(-q^{39},-q^{546})f(-q^{24},-q^{66})$$
$$-q^{39}f(-q^{78},-q^{507})f(-q^{18},-q^{72})-q^{28}f(-q^{117},-q^{468})f(-q^{12},-q^{78})$$
$$-q^{20}f(-q^{156},-q^{429})f(-q^6,-q^{84})+q^7f(-q^{234},-q^{351})f(-q^6,-q^{84})$$
$$+q^2f(-q^{273},-q^{312})f(-q^{12},-q^{78}). \tag{22.27}$$

Now by (2.3), (22.26), and (22.27), we conclude that

$$\psi(q^3)\varphi(-q^{39})-q^4f(q,q^2)f(-q^{13},-q^{65})$$
$$=\left\{f(-q^{42},-q^{48})-q^4f(-q^{18},-q^{72})-q^6f(-q^{12},-q^{78})\right\}\left\{f(-q^{273},-q^{312})\right.$$
$$\left.-q^{26}f(-q^{117},-q^{468})-q^{39}f(-q^{78},-q^{507})\right\}$$
$$+q^3\left\{f(-q^{36},-q^{54})-q^2f(-q^{24},-q^{66})-q^8f(-q^6,-q^{84})\right\}\left\{f(-q^{234},-q^{351})\right.$$
$$\left.-q^{13}f(-q^{156},-q^{429})-q^{52}f(-q^{39},-q^{546})\right\}. \tag{22.28}$$

But, from (2.19) with $n=3$, we know that

$$g(q)=f(-q^2,-q^3)=f(-q^{21},-q^{24})-q^2f(-q^9,-q^{36})-q^3f(-q^6,-q^{39}), \tag{22.29}$$
$$h(q)=f(-q,-q^4)=f(-q^{18},-q^{27})-qf(-q^{12},-q^{33})-q^4f(-q^3,-q^{42}), \tag{22.30}$$

where we used (2.5). Replacing q by q^2 and q^{13} in each of (22.29) and (22.30), we see that (22.25) holds, since the right-hand side of (22.28) is exactly

$$g(q^2)g(q^{13}) + q^3h(q^2)h(q^{13}) = M(q).$$

Hence, the proof of Entry 19 is complete. □

Next, we sketch a different prove for Entry 3.19 which, by (22.21), is equivalent to showing that

$$\frac{L(q)}{M(q)} = \frac{\chi(-q)}{\chi(-q^{13})}. \tag{22.31}$$

Therefore, by (22.25), one needs to prove that

$$\begin{aligned} L(q) &= \frac{\chi(-q)}{\chi(-q^{13})}\Big\{\psi(q^3)\varphi(-q^{39}) - q^4 f(q,q^2)f(-q^{13},-q^{65})\Big\} \\ &= f(-q,-q^5)f(q^{13},q^{26}) - q^4\varphi(-q^3)\psi(q^{39}), \end{aligned} \tag{22.32}$$

where in the last step we used (11.6) and (11.8). The equality (22.32) is proved in the same way that we proved (22.25), and so we omit the details.

23. Proofs of Entry 3.20

First Proof of Entry 3.20. Using (4.23) and (4.24) in (3.3), we arrive at

$$\begin{aligned} \chi^2(q) &= G(q)G(q^4) + qH(q)H(q^4) \\ &= \frac{f(-q^8)}{f(-q^2)}\left\{G(q^4)\Big(G(q^{16}) + qH(-q^4)\Big) + qH(q^4)\left(q^3H(q^{16}) + G(-q^4)\right)\right\} \\ &= \frac{f(-q^8)}{f(-q^2)}\Big\{G(q^4)G(q^{16}) + q^4H(q^4)H(q^{16}) + q\left(H(-q^4)G(q^4) + H(q^4)G(-q^4)\right)\Big\}. \end{aligned} \tag{23.1}$$

Separating out the even and odd indexed terms, we easily show that

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{23.2}$$

Using (2.7), and (23.2) we conclude from (23.1) that

$$\begin{aligned} &G(q^4)G(q^{16}) + q^4H(q^4)H(q^{16}) + q\Big(H(-q^4)G(q^4) + H(q^4)G(-q^4)\Big) \\ &\quad = \frac{\chi^2(q)f(-q^2)}{f(-q^8)} = \frac{\varphi(q)}{f(-q^8)} = \frac{\varphi(q^4) + 2q\psi(q^8)}{f(-q^8)}. \end{aligned} \tag{23.3}$$

Equating the odd parts of both sides of the equation (23.3), we obtain

$$H(-q^4)G(q^4) + H(q^4)G(-q^4) = 2\frac{\psi(q^8)}{f(-q^8)}, \tag{23.4}$$

which is Entry 3.20 with q replaced by q^4. □

Second Proof of Entry 3.20. This proof of Entry 3.20 is due to Watson [25] and uses Entry 3.22 below. We need an analogue of Lemma 7.1.

Lemma 23.1. *We have*

$$\psi^2(q) - q\psi^2(q^5) = f^2(-q^5)\frac{\chi(-q^5)}{\chi(-q)}. \tag{23.5}$$

By Entry 10(iv) in Chapter 19 of Ramanujan's second notebook [3, p. 262] and the Jacobi triple product identity (2.6), we find that

$$\begin{aligned}\psi^2(q) - q\psi^2(q^5) =& f(q,q^4)f(q^2,q^3)\\ =&(-q;q^5)_\infty(-q^4;q^5)_\infty(-q^2;q^5)_\infty(-q^3;q^5)_\infty(q^5;q^5)_\infty^2\\ =&\frac{f(-q^2)f^3(-q^5)}{f(-q^{10})f(-q)}\\ =&f^2(-q^5)\frac{\chi(-q^5)}{\chi(-q)},\end{aligned}$$

by (2.14), and the proof is complete.

Identity (23.5) is an analogue of

$$\psi^2(q) - 5q\psi^2(q^5) = f^2(-q)\frac{\chi(-q)}{\chi(-q^5)}, \tag{23.6}$$

which is found in Ramanujan's lost notebook [21] and was first proved by Kang [14, Thm. 2.2(ii)].

By Entry 3.21, (2.11) along with (5.4), and Lemma 23.1,

$$\begin{aligned}\{G(q)H(-q)+G(-q)H(q)\}^2 =& \{G(q)H(-q)-G(-q)H(q)\}^2\\ &+ 4G(q)H(q)G(-q)H(-q)\\ =&4\frac{q^2\psi^2(q^{10})}{f^2(-q^2)} + 4\frac{f(-q^5)f(q^5)}{f(-q)f(q)}\\ =&4\frac{q^2\psi^2(q^{10})}{f^2(-q^2)} + 4\frac{f^2(-q^{10})\chi(-q^{10})}{f^2(-q^2)\chi(-q^2)}\\ =&4\frac{\psi^2(q^2)}{f^2(-q^2)}.\end{aligned}$$

Taking the square root of both sides yields the desired result. ☐

24. Proof of Entry 3.21

We shall see that Entry 3.21 follows by combining Entries 3.1 and 3.2 with some elementary identities for theta functions.

From Entries 3.2 and 3.3, we easily deduce that

$$G(q) = \frac{\varphi(q)+\varphi(q^5)}{2G(q^4)f(-q^2)} \qquad \text{and} \qquad qH(q) = \frac{\varphi(q)-\varphi(q^5)}{2H(q^4)f(-q^2)}. \tag{24.1}$$

Applying each of the equalities in (24.1) twice, but with q replaced by $-q$ in two instances, using (2.16), using Lemma 7.1, and invoking (2.11) along with (5.4), we find that

$$\begin{aligned}
qG(q)H(-q)-qG(-q)H(q) =& \frac{\varphi(q^5)\varphi(-q^5)-\varphi(q)\varphi(-q)}{2G(q^4)H(q^4)f^2(-q^2)} \\
=& \frac{\varphi^2(-q^{10})-\varphi^2(-q^2)}{2G(q^4)H(q^4)f^2(-q^2)} \\
=& \frac{2q^2\chi(-q^2)f^2(-q^{20})}{G(q^4)H(q^4)\chi(-q^{10})f^2(-q^2)} \\
=& \frac{2q^2f(-q^{20})}{\chi(-q^{10})}\cdot\frac{f(-q^4)\chi(-q^2)}{f^2(-q^2)} \\
=& \frac{2q^2\psi(q^{10})}{f(-q^2)},
\end{aligned} \tag{24.2}$$

where we applied the elementary identities

$$f(-q^2)=\psi(q)\chi(-q)=\frac{f(-q)}{\chi(-q)}, \tag{24.3}$$

with q replaced by q^{10} and q^2, respectively. The identities in (24.3) both follow from (2.14). The truth of (3.21) is readily apparent from (24.2).

25. Proof of Entry 3.22

First Proof of Entry 3.22. Using (4.23) and (4.24) in (3.5), we find that

$$\begin{aligned}
\chi(q^2) &= G(q^{16})H(q)-q^3G(q)H(q^{16}) \\
&= \left\{\frac{f(-q^2)}{f(-q^8)}G(q)-qH(-q^4)\right\}H(q)-\left\{\frac{f(-q^2)}{f(-q^8)}H(q)-G(-q^4)\right\}G(q) \\
&= G(q)G(-q^4)-qH(q)H(-q^4),
\end{aligned}$$

which is Entry 3.22 with q replaced by $-q$. □

Second Proof of Entry 3.22. Consider the system of three equations,

$$G(-q)G(-q^4)+qH(-q)H(-q^4)=:T(q), \tag{25.1}$$

$$H(q^4)G(-q^4)+G(q^4)H(-q^4)=\frac{2\psi(q^8)}{f(-q^8)}, \tag{25.2}$$

$$-H(q^4)G(-q^4)+G(q^4)H(-q^4)=\frac{2q^4\psi(q^{40})}{f(-q^8)}. \tag{25.3}$$

Note that (25.1) merely gives the definition of $T(q)$, and that our goal is to show that $T(q)=\chi(q^2)$. The equality (25.2) is (3.24) with q replaced by q^4, and (25.3) is (3.25) with q replaced by q^4. We regard (25.1)–(25.3) as a system of three equations in the "variables" $G(-q^4)$, $H(-q^4)$, and -1. Thus, we have

$$\begin{vmatrix} G(-q) & qH(-q) & T(q) \\ H(q^4) & G(q^4) & \dfrac{2\psi(q^8)}{f(-q^8)} \\ -H(q^4) & G(q^4) & \dfrac{2q^4\psi(q^{40})}{f(-q^8)} \end{vmatrix} = 0. \tag{25.4}$$

Expanding (25.4) by the last column, we find that

$$2T(q)G(q^4)H(q^4) - \frac{2\psi(q^8)}{f(-q^8)}\left\{G(-q)G(q^4) + qH(-q)H(q^4)\right\} + \frac{2q^4\psi(q^{40})}{f(-q^8)}\left\{G(-q)G(q^4) - qH(-q)H(q^4)\right\} = 0. \tag{25.5}$$

Using (2.12), (3.4) with q replaced by $-q$, and (3.3) with q replaced by $-q$, we rewrite (25.5) in the form

$$\frac{T(q)f(-q^{20})}{f(-q^4)} - \frac{\psi(q^8)\varphi(-q^5)}{f(-q^8)f(-q^2)} + \frac{q^4\psi(q^{40})\varphi(-q)}{f(-q^8)f(-q^2)} = 0, \tag{25.6}$$

or, upon rearrangement,

$$T(q) = \frac{f(-q^4)}{f(-q^2)f(-q^8)f(-q^{20})}\left\{\varphi(-q^5)\psi(q^8) - q^4\varphi(-q)\psi(q^{40})\right\}. \tag{25.7}$$

By (2.15), (25.7) can be rewritten as

$$T(q) = \frac{\chi(-q^2)\chi(q^2)}{f(-q^2)f(-q^{20})}\left\{\varphi(-q^5)\psi(q^8) - q^4\varphi(-q)\psi(q^{40})\right\}. \tag{25.8}$$

From (9.12), we find, after simplification, that

$$\begin{aligned} &\varphi(-q^5)\psi(q^8) - q^4\varphi(-q)\psi(q^{40}) \\ =&\sqrt{z_5}(1-\beta)^{1/4}\frac{1}{4q}\sqrt{z_1}\left\{1-(1-\alpha)^{1/4}\right\} - q^4\sqrt{z_1}(1-\alpha)^{1/4}\frac{1}{4q^5}\sqrt{z_5}\left\{1-(1-\beta)^{1/4}\right\} \\ =&\frac{\sqrt{z_1z_5}}{4q}\left\{(1-\beta)^{1/4}-(1-\alpha)^{1/4}\right\}. \end{aligned} \tag{25.9}$$

Putting (9.15) and (25.9) in (25.8), we arrive at

$$T(q) = \frac{\chi(q^2)\left\{(1-\beta)^{1/4}-(1-\alpha)^{1/4}\right\}}{2^{2/3}(\alpha\beta)^{1/6}\left\{(1-\alpha)(1-\beta)\right\}^{1/24}}. \tag{25.10}$$

In comparing (25.10) with (3.26), we see that it remains to show that

$$\frac{(1-\beta)^{1/4}-(1-\alpha)^{1/4}}{2^{2/3}(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/24}} = 1. \tag{25.11}$$

But (25.11) is equivalent to (9.1), and so the proof is complete. □

26. Proof of Entry 3.23

We first remark that we have already given one proof of Entry 3.23 along with one of our proofs of Entry 3.11. We provide a second proof here.

Using (2.11) and (2.14), we see that Entry 3.23 is equivalent to the identity

$$\begin{aligned} f(-q^4,q^6)f(-q^6,q^9)+qf(q^2,-q^8)f(q^3,-q^{12}) =&\frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)}f(q^2)f(q^3)\\ =&f(-q^4)f(-q^6)\chi(q)\chi(q^6). \qquad (26.1)\end{aligned}$$

by (2.14). Using the product representations of $\chi(q)$ and $f(-q)$ given in (2.9) and (2.10), respectively, together with (2.6), we find that

$$\begin{aligned} f(-q^4)\chi(q) =&(q^4;q^4)_\infty(-q;q^2)_\infty = (q^4;q^4)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty\\ =&f(q,q^3)=\psi(q)\end{aligned}$$

and

$$\begin{aligned} f(-q^6)\chi(q^6) =&(q^6;q^6)_\infty(-q^6;q^{12})_\infty = (q^6;q^{12})_\infty(q^{12};q^{12})_\infty(-q^6;q^{12})_\infty\\ =&(q^{12};q^{24})_\infty(q^{12};q^{12})_\infty\\ =&(q^{12};q^{24})_\infty(q^{12};q^{24})_\infty(q^{24};q^{24})_\infty\\ =&f(-q^{12},-q^{12})=\varphi(-q^{12}),\end{aligned}$$

by (2.7). It thus suffices to prove that

$$f(-q^4,q^6)f(-q^6,q^9)+qf(q^2,-q^8)f(q^3,-q^{12})=\varphi(-q^{12})\psi(q). \qquad (26.2)$$

We now apply Theorem 4.5 with the parameters $\epsilon_1=1$, $\epsilon_2=0$, $a=b=q^{12}$ $c=q$, $d=q^3$, $\alpha=2$, $\beta=1$, and $m=5$. We consequently find that

$$\begin{aligned} \varphi(-q^{12})\psi(q) =&f(-q^{22},-q^{18})f(-q^{33},-q^{27})+qf(-q^{14},-q^{26})f(-q^{21},-q^{39})\\ &+q^6f(-q^6,-q^{34})f(-q^9,-q^{51})+q^{15}f(-q^{-2},-q^{42})f(-q^{-3},-q^{63})\\ &+q^{28}f(-q^{-10},-q^{50})f(-q^{-15},-q^{75})\\ =&f(-q^{18},-q^{22})f(-q^{33},-q^{27})+qf(-q^{14},-q^{26})f(-q^{21},-q^{39})\\ &+q^6f(-q^6,-q^{34})f(-q^9,-q^{51})+q^{10}f(-q^2,-q^{38})f(-q^3,-q^{57})\\ &+q^3f(-q^{10},-q^{30})f(-q^{15},-q^{45}), \qquad (26.3)\end{aligned}$$

where we applied (2.5) four times in the last equality. By (2.19), with $a=-q^2, b=q^3$, and $n=2$, and with $a=q, b=-q^4$, and $n=2$, respectively,

$$f(-q^2,q^3)=f(-q^9,-q^{11})-q^2f(-q^{19},-q), \qquad (26.4)$$

$$f(q,-q^4)=f(-q^7,-q^{13})+qf(-q^{17},-q^3). \qquad (26.5)$$

Replacing q by q^2 and q^3 in each of (26.4) and (26.5), respectively, we obtain

$$\begin{aligned} f(-q^4,q^6) =& f(-q^{18},-q^{22}) - q^4 f(-q^{38},-q^2),\\ f(-q^6,q^9) =& f(-q^{27},-q^{33}) - q^6 f(-q^{57},-q^3),\\ f(q^2,-q^8) =& f(-q^{14},-q^{26}) + q^2 f(-q^{34},-q^6),\\ f(q^3,-q^{12}) =& f(-q^{21},-q^{39}) + q^3 f(-q^{51},-q^9). \end{aligned}$$

Return to (26.2) and substitute each of the equalities above to deduce that

$$\begin{aligned} &\varphi(-q^{12})\psi(q) - \{f(-q^4,q^6)f(-q^6,q^9) + qf(q^2,-q^8)f(q^3,-q^{12})\}\\ =& q^3 f(-q^{10},-q^{30})f(-q^{15},-q^{45}) + q^4 f(-q^2,-q^{38})f(-q^{27},-q^{33})\\ &- q^3 f(-q^6,-q^{34})f(-q^{21},-q^{39}) - q^4 f(-q^{14},-q^{26})f(-q^9,-q^{51})\\ &+ q^6 f(-q^{18},-q^{22})f(-q^3,-q^{57}). \end{aligned} \tag{26.6}$$

We now use Theorem 4.5 again, but now with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = 1$, $b = q^{24}$, $c = q$, $d = q^3$, $\alpha = 2$, $\beta = 1$, and $m = 5$. Hence, using (2.4), we find that

$$\begin{aligned} q^3 f(-1,-q^{24})\psi(q) =& q^3 f(-q^{10},-q^{30})f(-q^{15},-q^{45}) + q^4 f(-q^2,-q^{38})f(-q^{27},-q^{33})\\ &+ q^9 f(-q^{-6},-q^{46})f(-q^{39},-q^{21}) + q^{18} f(-q^{-14},-q^{54})f(-q^{51},-q^9)\\ &+ q^{31} f(-q^{-22},-q^{62})f(-q^{63},-q^{-3})\\ =& q^3 f(-q^{10},-q^{30})f(-q^{15},-q^{45}) + q^4 f(-q^2,-q^{38})f(-q^{27},-q^{33})\\ &- q^3 f(-q^6,-q^{34})f(-q^{21},-q^{39}) - q^4 f(-q^{14},-q^{26})f(-q^9,-q^{51})\\ &+ q^6 f(-q^{18},-q^{22})f(-q^3,-q^{57}), \end{aligned} \tag{26.7}$$

after four applications of (2.5). The product on the far left side of (26.7) equals 0, by (2.4), Hence, using (26.7) in (26.6), we complete the proof of Entry 3.23.

27. Proof of Entry 3.24

We first remark that we have already given one proof of Entry 3.24 along with one of our proofs of Entry 3.12. We provide a second proof here.

The proof of Entry 3.24 is very similar to that of Entry 3.23. Using (2.11) and (2.14), we see that Entry 3.24 is equivalent to the identity

$$\begin{aligned} f(-q^{12},q^{18})f(q,-q^4) - qf(q^6,-q^{24})f(-q^2,q^3) =& \frac{\chi(q^2)\chi(q^3)}{\chi(q)\chi(q^6)} f(q)f(q^6)\\ =& f(-q^2)f(-q^{12})\chi(q^2)\chi(q^3). \end{aligned} \tag{27.1}$$

Using the product representations of $\chi(q)$ and $f(-q)$ from (2.10) and (2.9), respectively, together with (2.6), we obtain

$$\begin{aligned}
&f(-q^2)f(-q^{12})\chi(q^2)\chi(q^3) = (q^2;q^2)_\infty(q^{12};q^{12})_\infty(-q^2;q^4)_\infty(-q^3;q^6)_\infty \\
&= \frac{(q^4;q^8)_\infty}{(q^2;q^4)_\infty}(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty(q^{12};q^{12})_\infty \\
&= f(q^3,q^9)(q^4;q^8)_\infty(q^4;q^4)_\infty \\
&= f(q^3,q^9)f(-q^4,-q^4) = \psi(q^3)\varphi(-q^4).
\end{aligned}$$

It therefore remains to prove that

$$f(-q^{12},q^{18})f(q,-q^4) - qf(q^6,-q^{24})f(-q^2,q^3) = \varphi(-q^4)\psi(q^3). \tag{27.2}$$

We now apply Theorem 4.5 with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = b = q^4$, $c = q^3$, $d = q^9$, $\alpha = 1$, $\beta = 3$, and $m = 5$. Accordingly, we find that

$$\begin{aligned}
\varphi(-q^4)\psi(q^3) =&f(-q^{13},-q^7)f(-q^{66},-q^{54}) + q^3f(-q,-q^{19})f(-q^{42},-q^{78}) \\
&+ q^{18}f(-q^{-11},-q^{31})f(-q^{18},-q^{102}) + q^{45}f(-q^{-23},-q^{43}) \\
&\times f(-q^{-6},-q^{126}) + q^{84}f(-q^{-35},-q^{55})f(-q^{-30},-q^{150}) \\
=&f(-q^7,-q^{13})f(-q^{54},-q^{66}) + q^3f(-q,-q^{19})f(-q^{42},-q^{78}) \\
&- q^7f(-q^9,-q^{11})f(-q^{18},-q^{102}) - q^{13}f(-q^3,-q^{17})f(-q^6,-q^{114}) \\
&- q^4f(-q^5,-q^{15})f(-q^{30},-q^{90}),
\end{aligned} \tag{27.3}$$

where we applied (2.5) five times in the last equality. Recording again (26.4) and (26.5) as well as their analogues with q replaced by q^6, we find that

$$\begin{aligned}
f(-q^2,q^3) =&f(-q^9,-q^{11}) - q^2f(-q^{19},-q), \\
f(-q^{12},q^{18}) =&f(-q^{54},-q^{66}) - q^{12}f(-q^{114},-q^6). \\
f(q,-q^4) =&f(-q^7,-q^{13}) + qf(-q^{17},-q^3), \\
f(q^6,-q^{24}) =&f(-q^{42},-q^{78}) + q^6f(-q^{102},-q^{18}).
\end{aligned}$$

Using these identities in (27.3), we find that, after some elementary algebra,

$$\begin{aligned}
&\varphi(-q^4)\psi(q^3) - \left\{f(-q^{12},q^{18})f(q,-q^4) - qf(q^6,-q^{24})f(-q^2,q^3)\right\} \\
=&qf(-q^9,-q^{11})f(-q^{42},-q^{78}) - qf(-q^3,-q^{17})f(-q^{54},-q^{66}) \\
&- q^4f(-q^5,-q^{15})f(-q^{30},-q^{90}) + q^{12}f(-q^7,-q^{13})f(-q^6,-q^{114}) \\
&- q^9f(-q,-q^{19})f(-q^{18},-q^{102}).
\end{aligned} \tag{27.4}$$

Next, we apply Theorem 4.5 again, but now with the parameters $\epsilon_1 = 1$, $\epsilon_2 = 0$, $a = 1$, $b = q^8$, $c = q^3$, $d = q^9$, $\alpha = 1$, $\beta = 3$, and $m = 5$. Accordingly, we find that

$$\begin{aligned} qf(-1,-q^8)\psi(q^3) =& qf(-q^9,-q^{11})f(-q^{78},-q^{42}) + q^4f(-q^{-3},-q^{23})f(-q^{54},-q^{66}) \\ &+ q^{19}f(-q^{-15},-q^{35})f(-q^{30},-q^{90}) + q^{46}f(-q^{-27},-q^{47})f(-q^6,-q^{114}) \\ &+ q^{85}f(-q^{-39},-q^{59})f(-q^{138},-q^{-18}) \\ =& qf(-q^9,-q^{11})f(-q^{42},-q^{78}) - qf(-q^3,-q^{17})f(-q^{54},-q^{66}) \\ &- q^4f(-q^5,-q^{15})f(-q^{30},-q^{90}) + q^{12}f(-q^7,-q^{13})f(-q^6,-q^{114}) \\ &- q^9f(-q,-q^{19})f(-q^{18},-q^{102}), \end{aligned}$$

after several applications of (2.5). Using the result above in (27.4), we complete the proof of Entry 3.24.

28. Proofs of Entry 3.25

First Proof of Entry 3.25. Using (4.23) and (4.24) in (3.14) with q replaced by q^9, we arrive at

$$\begin{aligned} \frac{\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})} =& G(q^9)H(q^4) - qH(q^9)G(q^4) \\ =& \frac{f(-q^{72})}{f(-q^{18})}\Big\{H(q^4)\big(G(q^{144}) + q^9H(-q^{36})\big) - qG(q^4)\big(q^{27}H(q^{144}) + G(-q^{36})\big)\Big\} \\ =& \frac{f(-q^{72})}{f(-q^{18})}\Big\{H(q^4)G(q^{144}) - q^{28}G(q^4)H(q^{144}) \\ &- q\big(G(q^4)G(-q^{36}) - q^8H(q^4)H(-q^{36})\big)\Big\} \end{aligned} \tag{28.1}$$

Using (2.14), (2.16) and (11.10) with q replaced by $-q$, we obtain from (28.1) that

$$\begin{aligned} &H(q^4)G(q^{144}) - q^{28}G(q^4)H(q^{144}) - q\big(G(q^4)G(-q^{36}) - q^8H(q^4)H(-q^{36})\big) \\ &= \frac{f(-q^{18})\chi(-q)\chi(-q^6)}{\chi(-q^3)\chi(-q^{18})f(-q^{72})} = \chi(-q)\chi(q^3)\chi(-q^{36}) = \chi(-q^{36})\Big\{\frac{\chi(q^{12})}{\chi(-q^8)} - q\frac{\chi(q^4)}{\chi(-q^{24})}\Big\}. \end{aligned} \tag{28.2}$$

Equating the even and odd parts in both sides of the equation (28.2), we readily obtain Entries 3.14 and 3.25 with q replaced by q^4 and $-q$, respectively. □

Second Proof of Entry 3.25. Employing Theorem 4.5 with the set of parameters $a = q^6$, $b = q^{12}$, $c = q$, $d = q^2$, $\alpha = 2$, $\beta = 1$, $m = 5$, $\epsilon_1 = 0$, and $\epsilon_2 = 1$, we find that

$$\begin{aligned} f(q^6,q^{12})f(-q) =& f(q^{13},q^{17})f(-q^{18},-q^{27}) - qf(q^7,q^{23})f(-q^{18},-q^{27}) \\ &+ q^5f(q,q^{29})f(-q^9,-q^{36}) - q^2f(q^{11},q^{19})f(-q^9,-q^{36}). \end{aligned}$$

Upon the rearrangement of terms and use of (2.11), (4.26), and (4.27) with q replaced by $-q$, we obtain

$$\begin{aligned} f(q^6,q^{12})f(-q) =& f(-q^{18},-q^{27})\{f(q^{13},q^{17})-qf(q^7,q^{23})\} \qquad (28.3)\\ &-q^2f(-q^9,-q^{36})\{f(q^{11},q^{19})-q^3f(q,q^{29})\}\\ =& f(-q^{18},-q^{27})G(-q)f(-q^2)-q^2f(-q^9,-q^{36})H(-q)f(-q^2)\\ =& f(-q^2)f(-q^9)\left\{G(q^9)G(-q)-q^2H(q^9)H(-q)\right\}. \qquad (28.4)\end{aligned}$$

By (11.8) with q replaced by q^6, (2.14), and (2.17) in the form $\chi(q)f(-q)=\varphi(-q^2)$, we find that

$$\frac{f(-q)f(q^6,q^{12})}{f(-q^2)f(-q^9)}=\chi(-q)\frac{\varphi(-q^{18})}{\chi(-q^6)f(-q^9)}=\frac{\chi(-q)\chi(q^9)}{\chi(-q^6)}. \qquad (28.5)$$

Hence, by (28.3) and (28.5), the proof of Entry 3.26 is complete. □

Third Proof of Entry 3.25. To prove Entry 3.25, we need the identity [3, p. 349, Entry 2(i)]

$$\varphi(q)\varphi(q^9)-\varphi^2(q^3)=2q\varphi(-q^2)\psi(q^9)\chi(q^3). \qquad (28.6)$$

For convenience, define, by (2.11),

$$g(q):=f(-q)G(q)=f(-q^2,-q^3)$$

and

$$h(q):=f(-q)H(q)=f(-q,-q^4).$$

Using (2.11), (2.7), (2.6), and some elementary product manipulations, we see that Entry 3.25 is equivalent to the identity

$$g(-q)g(q^9)-q^2h(-q)h(q^9)=\varphi(-q^2)f(q^6,q^{12}). \qquad (28.7)$$

Replacing q by $-q$ in (28.7) gives

$$g(q)g(-q^9)-q^2h(q)h(-q^9)=\varphi(-q^2)f(q^6,q^{12}). \qquad (28.8)$$

We prove (28.8).

Using (2.11), (2.8), (2.6), and some elementary product manipulations we can express Entry 3.13 as

$$g(q^9)h(q^4)-qg(q^4)h(q^9)=\psi(-q)f(q^3,q^{15}). \qquad (28.9)$$

It is also easily verified that Entry 3.20 is equivalent to the identity

$$g(q)h(-q)+g(-q)h(q)=2\psi(q)\psi(-q). \qquad (28.10)$$

Consider the system of three equations,

$$g(q)g(-q^9)-q^2h(q)h(-q^9)=:T(q), \qquad (28.11)$$

$$g(q)g(q^9)+q^2h(q)h(q^9)=f^2(-q^3), \qquad (28.12)$$

$$g(q)g(q^4)+qh(q)h(q^4)=\psi(q)\varphi(-q^2). \qquad (28.13)$$

Equation (28.12) above is equation (10.1), while equation (28.13) is equation (6.2). We wish to show that $T(q) = \varphi(-q^2)f(q^6, q^{12})$. Regarding this system in the variables $g(q)$, $qh(q)$, and -1, we find that

$$\begin{vmatrix} g(-q^9) & -qh(-q^9) & T(q) \\ g(q^9) & qh(q^9) & f^2(-q^3) \\ g(q^4) & h(q^4) & \varphi(-q^2)\psi(q) \end{vmatrix} = 0. \tag{28.14}$$

Expanding the determinant in (28.14) along the last column, we find that

$$\begin{aligned} &T(q)\Big\{g(q^9)h(q^4) - qg(q^4)h(q^9)\Big\} \\ &- f^2(-q^3)\Big\{g(-q^9)h(q^4) + qg(q^4)h(-q^9)\Big\} \\ &+ \varphi(-q^2)\psi(q)\Big\{qg(-q^9)h(q^9) + qg(q^9)h(-q^9)\Big\} = 0. \end{aligned} \tag{28.15}$$

Using (28.9), (28.9) with q replaced by $-q$, and (28.10) with q replaced by q^9 in (28.15), we find that

$$T(q)\psi(-q)f(q^3, q^{15}) = f^2(-q^3)\psi(q)f(-q^3, -q^{15}) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9).$$

It suffices then to prove that

$$\begin{aligned} &\varphi(-q^2)f(q^6, q^{12})\psi(-q)f(q^3, q^{15}) \\ &= f^2(-q^3)\psi(q)f(-q^3, -q^{15}) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9). \end{aligned} \tag{28.16}$$

By (2.6), (2.7), and (2.8), we find that

$$\begin{aligned} &f(q^2, q^4)f(-q, -q^5)f(-q^2) \\ &= (-q^2; q^6)_\infty(-q^4; q^6)_\infty(q^6; q^6)_\infty(q; q^6)_\infty(q^5; q^6)_\infty(q^6; q^6)_\infty(q^2; q^2)_\infty \\ &= \frac{(-q^2; q^2)_\infty}{(-q^6; q^6)_\infty}(q^6; q^6)_\infty\frac{(q; q^2)_\infty}{(q^3; q^6)_\infty}(q^6; q^6)_\infty(q^2; q^2)_\infty \\ &= \frac{(q^6; q^6)_\infty}{(-q^6; q^6)_\infty}(q; q)_\infty(-q^2; q^2)_\infty\frac{(q^6; q^6)_\infty}{(q^3; q^6)_\infty} \\ &= \varphi(-q^6)\psi(-q)\psi(q^3). \end{aligned} \tag{28.17}$$

We return to (28.16) and use (28.17) with q replaced by $-q^3$ to deduce that

$$\begin{aligned} &\varphi(-q^2)\psi(-q)\varphi(-q^{18})\psi(q^3)\psi(-q^9) \\ &= f^2(-q^3)\psi(q)f(-q^3, -q^{15})f(-q^6) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9)f(-q^6). \end{aligned} \tag{28.18}$$

From (11.6) and (2.14), we find that

$$f(-q, -q^5)f(-q^2) = f(-q)\psi(q^3). \tag{28.19}$$

Using (28.19) with q replaced by q^3 in (28.18), we find that

$$\begin{aligned} &\varphi(-q^2)\psi(-q)\varphi(-q^{18})\psi(q^3)\psi(-q^9) \\ &= f^3(-q^3)\psi(q)\psi(q^9) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9)f(-q^6). \end{aligned} \tag{28.20}$$

Using (2.13)–(2.15), or using (2.7)–(2.9), we can easily verify that

$$f^3(-q) = \psi(q)\varphi^2(-q). \tag{28.21}$$

Using (17.4) twice with q replaced by $-q$ and $-q^9$, respectively, and (28.21) with q replaced by q^3, we deduce from (28.20) that

$$\begin{aligned}&\varphi(-q)\psi(q)\varphi(-q^9)\psi(q^3)\psi(q^9)\\ =&\psi(q^3)\varphi^2(-q^3)\psi(q)\psi(q^9) - 2q\varphi(-q^2)\psi(q)\psi(-q^9)\psi(q^9)f(-q^6).\end{aligned} \tag{28.22}$$

Divide both sides of (28.22) with $\psi(q)\psi(q^3)\psi(q^9)$ to conclude that

$$\begin{aligned}\varphi(-q)\varphi(-q^9) &= \varphi^2(-q^3) - \frac{2q}{\psi(q^3)}\varphi(-q^2)\psi(-q^9)f(-q^6)\\ &= \varphi^2(-q^3) - 2q\varphi(-q^2)\psi(-q^9)\chi(-q^3),\end{aligned} \tag{28.23}$$

where in the last step we used the extremal equality in (2.14) with q replaced by $-q^3$. If we replace q by $-q$, then (28.23) reduces to (28.6). Hence, the proof of Entry 3.25 is complete. □

29. Proofs of Entries 3.26 and 3.27

The proofs in this section are due to Watson [25].

Recall that we have by (23.2),

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{29.1}$$

Returning to (24.1), we use (29.1) twice. Then we apply Entries 3.2, 3.3, and 3.20, with q replaced by q^4, and Entry 3.21, with q replaced by q^2. In these resulting equalities, we solve for $\varphi(q^4)$, $\varphi(q^{20})$, $\psi(q^8)$, and $\psi(q^{20})$, respectively, and substitute them in the second equality below. Accordingly, we find that

$$\begin{aligned}G(q) =&\frac{\varphi(q)+\varphi(q^5)}{2G(q^4)f(-q^2)}\\ =&\frac{\varphi(q^4)+\varphi(q^{20})}{2G(q^4)f(-q^2)} + \frac{q\psi(q^8)+q^5\psi(q^{20})}{G(q^4)f(-q^2)}\\ =&\frac{f(-q^8)}{f(-q^2)}\left(G(q^{16})+qH(-q^4)\right).\end{aligned} \tag{29.2}$$

Performing exactly the same steps on the second equality of (24.1), we find that

$$\begin{aligned}qH(q) =&\frac{\varphi(q)-\varphi(q^5)}{2H(q^4)f(-q^2)}\\ =&\frac{\varphi(q^4)-\varphi(q^{20})}{2H(q^4)f(-q^2)} + \frac{q\psi(q^8)-q^5\psi(q^{20})}{H(q^4)f(-q^2)}\\ =&\frac{f(-q^8)}{f(-q^2)}\left(q^4H(q^{16})+qG(-q^4)\right).\end{aligned} \tag{29.3}$$

For brevity, set

$$T(q) := G(q^{11})H(-q) + q^2G(-q)H(q^{11}). \tag{29.4}$$

Next, in the definition (29.4), we substitute for each of the functions G and H their respective representations from (29.2) and (29.3). We therefore deduce that

$$\begin{aligned}\frac{f(-q^2)}{f(-q^8)}\cdot\frac{f(-q^{22})}{f(-q^{88})}T(q) &= \left\{G(q^{176})+q^{11}H(-q^{44})\right\}\left\{G(-q^4)-q^3H(q^{16})\right\}\\ &\quad + q^2\left\{G(q^{16})-qH(-q^4)\right\}\left\{G(-q^{44})+q^{33}H(q^{176})\right\}\\ &= \left\{G(-q^4)G(q^{176})-q^{36}H(-q^4)H(q^{176})\right\}\\ &\quad + q^2\left\{G(q^{16})G(-q^{44})-q^{12}H(q^{16})H(-q^{44})\right\}\\ &\quad - q^3\left\{G(q^{176})H(q^{16})-q^{32}G(q^{16})H(q^{176})\right\}\\ &\quad - q^3\left\{G(-q^{44})H(-q^4)-q^8G(-q^4)H(-q^{44})\right\}. \qquad (29.5)\end{aligned}$$

Recalling the definitions of U and V in (3.31) and (3.32), respectively, recalling the definition (29.4), and using Entry 3.4, we find that (29.5) can be written in the form

$$\chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44})T(q) = U(-q^4)+q^2V(-q^4)-2q^3. \qquad (29.6)$$

Now replace q by $-q$ in (29.6) and subtract the two equalities to deduce that

$$\chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44})\left\{T(-q)-T(q)\right\} = 4q^3. \qquad (29.7)$$

We can obtain a second equation connecting $T(q)$ and $T(-q)$ in the following manner. We record (3.5), (29.4), and (3.24), with q replaced by q^{11}. Accordingly,

$$\begin{aligned}G(q^{11})H(q)-q^2G(q)H(q^{11}) &= 1,\\ G(q^{11})H(-q)+q^2G(-q)H(q^{11}) &= T(q),\\ G(q^{11})H(-q^{11})+G(-q^{11})H(q^{11}) &= \frac{2}{\chi^2(-q^{22})}.\end{aligned}$$

Regarding $G(q^{11})$, $H(q^{11})$, and 1 as the "variables," we conclude from this triple of equations that

$$\begin{vmatrix} H(q) & -q^2G(q) & 1\\ H(-q) & q^2G(-q) & T(q)\\ H(-q^{11}) & G(-q^{11}) & \dfrac{2}{\chi^2(-q^{22})}\end{vmatrix} = 0. \qquad (29.8)$$

Expanding this determinant (29.8) by the last column, using Entry 3.4, recalling the definition (29.4), and using Entry 3.20, we find that

$$1-T(q)T(-q)+\frac{4q^2}{\chi^2(-q^2)\chi^2(-q^{22})} = 0. \qquad (29.9)$$

We now use the theory of modular equations. In Ramanujan's notation for the moduli k and ℓ, let $\alpha = k^2$ and $\beta = \ell^2$, where β is of degree 11 over α. The standard modular equation of degree 11, first found by H. Schröter and rediscovered by Ramanujan [3, p. 363, Entry 7(i)], is given by

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \qquad (29.10)$$

We also need the representations [3, p. 124, Entries 12(v), (vii)]

$$\chi(q) = 2^{1/6}\left(\frac{q}{\alpha(1-\alpha)}\right)^{1/24} \qquad \text{and} \qquad \chi(-q^2) = 2^{1/3}\left(\frac{(1-\alpha)q^2}{\alpha^2}\right)^{1/24}. \tag{29.11}$$

Lastly, we set $-q^2 = Q$. Thus, using (29.7) and (29.9), the modular equation (29.10), and lastly (29.11), we deduce that

$$\begin{aligned}
&\chi^2(-q^2)\chi^2(-q^4)\chi^2(-q^{22})\chi^2(-q^{44})\left\{T(q)+T(-q)\right\}^2\\
&\quad=4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22}) - 16Q\chi^2(-Q^2)\chi^2(-Q^{22}) - 16Q^3\\
&\quad=4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})\\
&\qquad\times\left(1-16\frac{Q}{\chi^2(Q)\chi^2(Q^{11})} - 16\frac{Q^3}{\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})}\right)\\
&\quad=4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})\left(1-2^{4/3}\left\{\alpha\beta(1-\alpha)(1-\beta)\right\}^{1/12} - (\alpha\beta)^{1/4}\right)\\
&\quad=4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})\left\{(1-\alpha)(1-\beta)\right\}^{1/4}\\
&\quad=4\chi^2(Q)\chi^2(-Q^2)\chi^2(Q^{11})\chi^2(-Q^{22})\frac{\chi^2(-Q^2)\chi^2(-Q^{22})}{\chi^4(Q)\chi^4(Q^{11})}.
\end{aligned} \tag{29.12}$$

Changing back to the original variable q, we take the square root of both sides of (29.12) to deduce that

$$T(q)+T(-q) = 2\frac{\chi(-q^4)\chi(-q^{44})}{\chi^2(-q^2)\chi^2(-q^{22})} = 2\frac{\chi(q^2)\chi(q^{22})}{\chi(-q^2)\chi(-q^{22})}, \tag{29.13}$$

by (2.15). Now combine (29.13) with (29.7) to derive the desired formula (3.30).

It remains to prove (3.33) and (3.34). Return to (29.6) and insert the just proved formula for $T(q)$ in Entry 3.26. We thus find that

$$U(-q^4) + q^2V(-q^4) = \chi(q^2)\chi(-q^4)\chi(q^{22})\chi(-q^{44}). \tag{29.14}$$

Changing the sign of q^2 in (29.14), we find that

$$U(-q^4) - q^2V(-q^4) = \chi(-q^2)\chi(-q^4)\chi(-q^{22})\chi(-q^{44}). \tag{29.15}$$

Multiplying (29.14) and (29.15) together, we arrive at

$$\begin{aligned}
U^2(-q^4) - q^4V^2(-q^4) &= \chi(q^2)\chi(-q^2)\chi(q^{22})\chi(-q^{22})\chi^2(-q^4)\chi^2(-q^{44})\\
&= \chi^3(-q^4)\chi^3(-q^{44}).
\end{aligned} \tag{29.16}$$

If we replace $-q^4$ by q in (29.16), we obtain (3.33).

Finally, we prove (3.34). We record the definition (3.31) of $U(q)$, (3.5), with q replaced by q^4, and (3.3), with q replaced by q^{11}, in the array

$$\begin{aligned}
G(q)G(q^{44}) + q^9H(q)H(q^{44}) &= U(q),\\
H(q^4)G(q^{44}) - q^8G(q^4)H(q^{44}) &= 1,\\
G(q^{11})G(q^{44}) + q^{11}H(q^{11})H(q^{44}) &= \chi^2(q^{11}).
\end{aligned}$$

Regard this system of equations as three equations in the unknowns $G(q^{44})$, $q^8H(q^{44})$, and -1. It follows that

$$\begin{vmatrix} G(q) & qH(q) & U(q) \\ H(q^4) & -qG(q^4) & 1 \\ G(q^{11}) & q^3H(q^{11}) & \chi^2(q^{11}) \end{vmatrix} = 0. \tag{29.17}$$

Expanding the determinant (29.17) by the last column, and then using the definition (3.32) of V, (3.5), and (3.3), we find that

$$U(q)V(q) + q - \chi^2(q^{11})\chi^2(q) = 0,$$

which is precisely (3.34).

30. Proof of the First Part of Entry 3.28

Our argument below is the same as that of Bressoud [10].

To prove (3.35), we use (2.11) to rewrite the identity as

$$\frac{f(-q^{34},-q^{51})f(-q^2,-q^8) - q^3f(-q^4,-q^6)f(-q^{17},-q^{68})}{f(-q^2,-q^3)f(-q^{68},-q^{102}) + q^7f(-q,-q^4)f(-q^{34},-q^{136})} = \frac{\chi(-q)f(-q^2)f(-q^{17})}{\chi(-q^{17})f(-q)f(-q^{34})}. \tag{30.1}$$

From (2.9) and (2.10), it is easy to see that

$$\begin{aligned} \chi(-q)f(-q^2)f(-q^{17}) &= (q;q^2)_\infty(q^2;q^2)_\infty(q^{17};q^{17})_\infty \\ &= (q;q)_\infty(q^{17};q^{34})_\infty(q^{34};q^{34})_\infty \\ &= f(-q)\chi(-q^{17})f(-q^{34}). \end{aligned}$$

Thus, the right-hand side of (30.1) equals 1, and so (3.35) is equivalent to

$$\begin{aligned} &f(-q^{34},-q^{51})f(-q^2,-q^8) - q^3f(-q^4,-q^6)f(-q^{17},-q^{68}) \\ =&f(-q^2,-q^3)f(-q^{68},-q^{102}) + q^7f(-q,-q^4)f(-q^{34},-q^{136}). \end{aligned} \tag{30.2}$$

We now apply (4.16) with $\alpha = 1$ and $\beta = \frac{17}{2}$ to obtain

$$\sum_{k=1}^{2} F(1,\tfrac{17}{2},3,5,\tfrac{7}{2},k) = q^{\frac{7}{8}}(f(-q^{34},-q^{51})f(-q^2,-q^8) - q^3f(-q^4,-q^6)f(-q^{17},-q^{68})). \tag{30.3}$$

Similarly, letting $\alpha = \frac{1}{2}$ and $\beta = 17$ in (4.15), we deduce that

$$\sum_{k=1}^{2} F(\tfrac{1}{2},17,1,5,\tfrac{7}{2},k) = q^{\frac{7}{8}}(f(-q^2,-q^3)f(-q^{68},-q^{102}) + q^7f(-q,-q^4)f(-q^{34},-q^{136})). \tag{30.4}$$

The two sets of parameters $\{1,\frac{17}{2},3,5,\frac{7}{2}\}$ and $\{\frac{1}{2},17,1,5,\frac{7}{2}\}$ give rise to the same series on the right-hand side of (4.8), since the parameters satisfy the conditions in (4.9). Hence, the right-hand sides of (30.3) and (30.4) are equal. This completes the proof of the first part of Entry 3.28.

31. Proof of the Equivalence of Entries 3.31 and 3.32

For brevity, define

$$\begin{aligned} M(q) &:= G(q^2)G(q^{33}) + q^7 H(q^2)H(q^{33}), && (31.1)\\ N(q) &:= G(q^{66})H(q) - q^{13}H(q^{66})G(q), && (31.2)\\ T(q) &:= G(q^3)G(q^{22}) + q^5 H(q^3)H(q^{22}), && (31.3)\\ U(q) &:= G(q^{11})H(q^6) - qH(q^{11})G(q^6). && (31.4)\end{aligned}$$

Using (31.2), Entry 3.4 with q replaced by q^6, and Entry 3.8 with q replaced by q^{11}, we consider the system of three equations,

$$\begin{aligned} N(q) &:= G(q^{66})H(q) - q^{13}H(q^{66})G(q),\\ 1 &= G(q^{66})H(q^6) - q^{12}H(q^{66})G(q^6),\\ \frac{\chi(-q^{11})}{\chi(-q^{33})} =: R(q) &= G(q^{66})H(q^{11}) - q^{11}H(q^{66})G(q^{11}). && (31.5)\end{aligned}$$

It follows that

$$\begin{vmatrix} H(q) & -q^{13}G(q) & N(q)\\ H(q^6) & -q^{12}G(q^6) & 1\\ H(q^{11}) & -q^{11}G(q^{11}) & R(q)\end{vmatrix} = 0,$$

or, upon using (31.4), Entry 3.4, (31.5), and Entry 3.8, we deduce that

$$\begin{aligned} 0 &= N(q)\left(-q^{11}H(q^6)G(q^{11}) + q^{12}G(q^6)H(q^{11})\right) + q^{11}H(q)G(q^{11}) - q^{13}G(q)H(q^{11})\\ &\quad + R(q)\left(-q^{12}G(q^6)H(q) + q^{13}G(q)H(q^6)\right)\\ &= -N(q)q^{11}U(q) + q^{11} - q^{12}\frac{\chi(-q^{11})}{\chi(-q^{33})}\frac{\chi(-q)}{\chi(-q^3)}. && (31.6)\end{aligned}$$

Hence, if we define

$$W(q) := \frac{\chi(-q)\chi(-q^{11})}{\chi(-q^3)\chi(-q^{33})}, \tag{31.7}$$

then, from (31.6), we deduce that

$$N(q)U(q) = 1 - qW(q). \tag{31.8}$$

Next, using (31.1), Entry 3.4 with q replaced by q^{11}, and Entry 3.7 with q replaced by q^{11}, we consider the system of equations

$$\begin{aligned} M(q) &:= G(q^2)G(q^{33}) + q^7H(q^2)H(q^{33}),\\ 1 &:= H(q^3)G(q^{33}) - q^6G(q^3)H(q^{33}),\\ \frac{\chi(-q^{33})}{\chi(-q^{11})} =: S(q) &= G(q^{22})G(q^{33}) + q^{11}H(q^{22})H(q^{33}). && (31.9)\end{aligned}$$

It follows that

$$\begin{vmatrix} G(q^2) & q^7H(q^2) & M(q)\\ H(q^3) & -q^6G(q^3) & 1\\ G(q^{22}) & q^{11}H(q^{22}) & S(q)\end{vmatrix} = 0.$$

Hence, employing (31.3), Entry 3.4 with q replaced by q^2, (31.9), and Entry 3.7, we find that

$$\begin{aligned}0 =&M(q)\left(q^{11}H(q^3)H(q^{22})+q^6G(q^3)G(q^{22})\right)-q^{11}G(q^2)H(q^{22})+q^7H(q^2)G(q^{22})\\&+S(q)\left(-q^6G(q^2)G(q^3)-q^7H(q^2)H(q^3)\right)\\=&M(q)q^6T(q)+q^7-q^6\frac{\chi(-q^{33})}{\chi(-q^{11})}\frac{\chi(-q^3)}{\chi(-q)}.\end{aligned}\tag{31.10}$$

Hence, using the definition of $W(q)$ in (31.7), we deduce from (31.10) that

$$M(q)T(q)=-q+\frac{1}{W(q)}.\tag{31.11}$$

Hence, dividing (31.11) by (31.8), we conclude that

$$\frac{M(q)T(q)}{N(q)U(q)}=\frac{1}{W(q)}.\tag{31.12}$$

Examining Entries 3.31 and 3.32, we see that it suffices to prove just one of them, for then the other one would follow immediately from (31.12).

32. Proof of Entry 3.33

We provide two proofs.

First Proof of Entry 3.33. Let us define $K(q)$ and $L(q)$ by

$$K(q):=G(q)G(q^{54})+q^{11}H(q)H(q^{54}),\tag{32.1}$$

$$L(q):=H(q^2)G(q^{27})-q^5G(q^2)H(q^{27}),\tag{32.2}$$

so that Entry 3.33 reads

$$\frac{K(q)}{L(q)}=\frac{\chi(-q^3)\chi(-q^{27})}{\chi(-q)\chi(-q^9)}.\tag{32.3}$$

Starting from (3.15) and arguing as in (4.32), we find that

$$\frac{\chi(-q^6)\chi(-q^9)}{\chi(-q^2)\chi(-q^3)}G(-q)+\frac{\chi(-q^6)\chi(q^9)}{\chi(-q^2)\chi(q^3)}G(q)=2\frac{G(q^{36})}{\chi^2(-q^2)}.\tag{32.4}$$

By (2.15), we see that (32.4) simplifies to

$$\chi(q^3)\chi(-q^9)G(-q)+\chi(-q^3)\chi(q^9)G(q)=2\frac{G(q^{36})}{\chi(-q^2)}.\tag{32.5}$$

Similarly, we find that

$$\chi(-q^3)\chi(q^9)H(q)-\chi(q^3)\chi(-q^9)H(-q)=2q^7\frac{H(q^{36})}{\chi(-q^2)}.\tag{32.6}$$

In (32.1), we replace q by q^2 and employ (32.5) and (32.6) with q replaced by q^3 to find that

$$\begin{aligned}2\frac{K(q^2)}{\chi(-q^6)} =&\frac{2}{\chi(-q^6)}\left\{G(q^2)G(q^{108})+q^{22}H(q^2)H(q^{108})\right\}\\ =&G(q^2)\left\{\chi(q^9)\chi(-q^{27})G(-q^3)+\chi(-q^9)\chi(q^{27})G(q^3)\right\}\\ &+qH(q^2)\left\{\chi(-q^9)\chi(q^{27})H(q^3)-\chi(q^9)\chi(-q^{27})H(-q^3)\right\}\\ =&\chi(q^9)\chi(-q^{27})\left\{G(q^2)G(-q^3)-qH(q^2)H(-q^3)\right\}\\ &+\chi(-q^9)\chi(q^{27})\left\{G(q^2)G(q^3)+qH(q^2)H(q^3)\right\}. \end{aligned} \tag{32.7}$$

Using (3.8) twice, once with q replaced by $-q$, we see that (32.7) can be put in the form

$$2\frac{K(q^2)}{\chi(-q^6)}=\chi(q^9)\chi(-q^{27})\frac{\chi(q^3)}{\chi(q)}+\chi(-q^9)\chi(q^{27})\frac{\chi(-q^3)}{\chi(-q)}.$$

Using (2.15), we conclude that

$$2K(q^2)=\frac{\chi(-q^6)}{\chi(-q^2)}\Big\{\chi(-q)\chi(q^3)\chi(q^9)\chi(-q^{27})+\chi(q)\chi(-q^3)\chi(-q^9)\chi(q^{27})\Big\}. \tag{32.8}$$

To obtain the desired expression for $L(q^2)$, we use Lemma 4.3. Then, in (32.2), replacing q by q^2, employing (4.36) and (4.37), and arguing as in (32.7), we find that

$$\begin{aligned}2\frac{L(q^2)}{\chi(-q^{18})} =&\chi(q)\chi(-q^3)\left\{G(q^{54})H(q^9)-q^9H(q^{54})G(q^9)\right\}\\ &+\chi(-q)\chi(q^3)\left\{G(q^{54})H(-q^9)+q^9H(q^{54})G(-q^9)\right\}. \end{aligned} \tag{32.9}$$

Using (3.9), with q replaced by q^9 and $-q^9$, we find from (32.9) that

$$2\frac{L(q^2)}{\chi(-q^{18})}=\chi(q)\chi(-q^3)\frac{\chi(-q^9)}{\chi(-q^{27})}+\chi(-q)\chi(q^3)\frac{\chi(q^9)}{\chi(q^{27})},$$

which, by (2.15), implies that

$$2L(q^2)=\frac{\chi(-q^{18})}{\chi(-q^{54})}\Big\{\chi(-q)\chi(q^3)\chi(q^9)\chi(-q^{27})+\chi(q)\chi(-q^3)\chi(-q^9)\chi(q^{27})\Big\}. \tag{32.10}$$

Dividing (32.8) by (32.10), we obtain (32.3) with q replaced by q^2. Hence, the proof of Entry 3.33 is complete. □

Second Proof of Entry 3.33. For convenience, define, by (2.11),

$$g(q)=f(-q)G(q)=f(-q^2,-q^3)$$

and

$$h(q)=f(-q)H(q)=f(-q,-q^4).$$

Our proof of Entry 3.33 uses Entries 3.6, 3.7, and 3.8, which we write in their equivalent forms (see (10.1), (11.3), and (12.2)),

$$g(q)g(q^9)+q^2h(q)h(q^9)=f^2(-q^3), \tag{32.11}$$
$$g(q^2)g(q^3)+qh(q^2)h(q^3)=\psi(q)\varphi(-q^3), \tag{32.12}$$
$$g(q^6)h(q)-qg(q)h(q^6)=\psi(q^3)\varphi(-q). \tag{32.13}$$

Let us define $M(q)$ and $N(q)$ by

$$M(q):=h(q^2)g(q^{27})-q^5g(q^2)h(q^{27}), \tag{32.14}$$
$$N(q):=g(q)g(q^{54})+q^{11}h(q)h(q^{54}). \tag{32.15}$$

By (2.11) and (2.14), Entry 3.33 is equivalent to the identity

$$\frac{N(q)}{M(q)}=\frac{f(-q)f(-q^{54})\chi(-q^3)\chi(-q^{27})}{f(-q^2)f(-q^{27})\chi(-q)\chi(-q^9)}=\frac{\chi(-q^3)}{\chi(-q^9)}. \tag{32.16}$$

By (32.15), (32.11), and (32.13) with q replaced by q^6 and q^9, respectively, in the latter two cases, we deduce the following system of three equations:

$$\begin{aligned} g(q)g(q^{54})+q^{11}h(q)h(q^{54}) &=N(q),\\ g(q^6)g(q^{54})+q^{12}h(q^6)h(q^{54}) &=f^2(-q^{18}),\\ h(q^9)g(q^{54})-q^9g(q^9)h(q^{54}) &=\psi(q^{27})\varphi(-q^9). \end{aligned}$$

Regarding this system in the variables $G(q^{54}),\ q^9h(q^{54}),\ \text{and}\ -1$, we find that

$$\begin{vmatrix} g(q) & q^2h(q) & N(q)\\ g(q^6) & q^3h(q^6) & f^2(-q^{18})\\ h(q^9) & -g(q^9) & \psi(q^{27})\varphi(-q^9) \end{vmatrix}=0. \tag{32.17}$$

Expanding the determinant in (32.17) along the last column, we find that

$$\begin{aligned} &-N(q)\left\{g(q^6)g(q^9)+q^3h(q^6)h(q^9)\right\}\\ &+f^2(-q^{18})\left\{g(q)g(q^9)+q^2h(q)h(q^9)\right\}\\ &-q^2\varphi(-q^9)\psi(q^{27})\left\{h(q)g(q^6)-qg(q)h(q^6)\right\}=0. \end{aligned} \tag{32.18}$$

Using (32.12) with q replaced by q^3 , (32.11), and (32.13) in (32.18), we find that

$$-N(q)\psi(q^3)\varphi(-q^9)+f^2(-q^{18})f^2(-q^3)-q^2\psi(q^{27})\varphi(-q^9)\psi(q^3)\varphi(-q)=0.$$

Solving for $N(q)$, we deduce that

$$N(q)=\frac{f^2(-q^{18})f^2(-q^3)}{\psi(q^3)\varphi(-q^9)}-q^2\psi(q^{27})\varphi(-q). \tag{32.19}$$

Next, we determine $M(q)$. By (32.14), (32.12), and (32.11) with q replaced by q^9 and q^3, respectively, in the latter two equalities, we find that

$$\begin{aligned} h(q^2)g(q^{27})-q^5g(q^2)h(q^{27}) &=M(q),\\ g(q^{18})g(q^{27})+q^9h(q^{18})h(q^{27}) &=\psi(q^9)\varphi(-q^{27}),\\ g(q^3)g(q^{27})+q^6h(q^3)h(q^{27}) &=f^2(-q^9). \end{aligned}$$

Regarding this system in the variables $g(q^{27}),\ q^5h(q^{27}),$ and -1 and arguing similarly, we find that

$$qM(q)\psi(q^9)\varphi(-q^3)-\psi(q^9)\varphi(-q^{27})\psi(q)\varphi(-q^3)+f^2(-q^9)f^2(-q^6)=0.$$

Solving for $qM(q)$, we arrive at

$$qM(q)=\psi(q)\varphi(-q^{27})-\frac{f^2(-q^9)f^2(-q^6)}{\psi(q^9)\varphi(-q^3)}. \tag{32.20}$$

Recall that, by (11.6) and (11.8),

$$f(-q,-q^5)=\chi(-q)\psi(q^3) \tag{32.21}$$

and

$$f(q,q^2)=\frac{\varphi(-q^3)}{\chi(-q)}. \tag{32.22}$$

By (2.19) with $n=3$ (see also [3, p. 49, Cor.]), we find that

$$\varphi(-q)=\varphi(-q^9)-2qf(-q^3,-q^{15}), \tag{32.23}$$

$$\psi(q)=f(q^3,q^6)+q\psi(q^9). \tag{32.24}$$

Using (32.21) and (32.22) in (32.23) and (32.24), we obtain, respectively,

$$\varphi(-q)=\varphi(-q^9)-2q\chi(-q^3)\psi(q^9), \tag{32.25}$$

$$\psi(q)=\frac{\varphi(-q^9)}{\chi(-q^3)}+q\psi(q^9). \tag{32.26}$$

We deduce from (32.25) and (32.26) that

$$\chi(-q^3)\psi(q^9)=\frac{1}{2q}\Big\{\varphi(-q^9)-\varphi(-q)\Big\}, \tag{32.27}$$

$$\psi(q)\chi(-q^3)=\frac{1}{2}\Big\{3\varphi(-q^9)-\varphi(-q)\Big\}. \tag{32.28}$$

By (2.14), we easily conclude that

$$\frac{f^2(-q^2)\chi(-q)}{\varphi(-q)}=\psi(q), \tag{32.29}$$

$$\frac{f^2(-q)}{\psi(q)}=\varphi(-q)\chi(-q). \tag{32.30}$$

By (32.19), (32.29), and (32.30) with q replaced by q^9 and q^3, respectively, in the latter two equations, we find that

$$\chi(-q^9)N(q)=\chi(-q^3)\varphi(-q^3)\psi(q^9)-q^2\psi(q^{27})\chi(-q^9)\varphi(-q). \tag{32.31}$$

Using (32.27) twice in (32.31) with q replaced by q and q^3, respectively, we find that

$$\begin{aligned}\chi(-q^9)N(q)&=\frac{1}{2q}\varphi(-q^3)\Big\{\varphi(-q^9)-\varphi(-q)\Big\}-q^2\frac{1}{2q^3}\Big\{\varphi(-q^{27})-\varphi(-q^3)\Big\}\varphi(-q)\\&=\frac{1}{2q}\Big\{\varphi(-q^3)\varphi(-q^9)-\varphi(-q)\varphi(-q^{27})\Big\}.\end{aligned} \tag{32.32}$$

Similarly, by (32.20), (32.29), and (32.30) with q replaced by q^3 and q^9, respectively, we find that

$$q\chi(-q^3)M(q) = \chi(-q^3)\varphi(-q^{27})\psi(q) - \psi(q^3)\chi(-q^9)\varphi(-q^9). \qquad (32.33)$$

Using (32.28) twice with q replaced by q and q^3, respectively, we find that

$$\begin{aligned} q\chi(-q^3)M(q) &= \frac{1}{2}\varphi(-q^{27})\Big\{3\varphi(-q^9) - \varphi(-q)\Big\} - \frac{1}{2}\Big\{3\varphi(-q^{27}) - \varphi(-q^3)\Big\}\varphi(-q^9) \\ &= \frac{1}{2}\Big\{\varphi(-q^3)\varphi(-q^9) - \varphi(-q)\varphi(-q^{27})\Big\}. \qquad (32.34) \end{aligned}$$

Dividing (32.34) by (32.32), we see that (32.16) is verified. Hence, the second proof of Entry 3.33 is complete. ☐

33. Proof of Entry 3.34

Our proof is a moderate modification of the proof given by Bressoud [10].

Using (1.2), (2.6), and some elementary product manipulations, we can show that

$$G(q)G(-q) = \frac{f(q^4, q^6)}{f(-q^2)} \qquad \text{and} \qquad H(q)H(-q) = \frac{f(q^2, q^8)}{f(-q^2)}. \qquad (33.1)$$

Adding Entries 3.20 and 3.21, we find that

$$G(q)H(-q) = \frac{1}{f(-q^2)}\left\{\psi(q^2) + q\psi(q^{10})\right\}. \qquad (33.2)$$

Next, we recall Entry 3.34:

$$\begin{aligned} &\left\{G(q)G(-q^{19}) - q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19}) - q^4H(-q)H(q^{19})\right\} \\ =&G(q^2)G(q^{38}) + q^8H(q^2)H(q^{38}). \qquad (33.3) \end{aligned}$$

Expanding the product on the left side of (33.3) and then using (33.1) and (33.2), we find that

$$\begin{aligned} &\left\{G(q)G(-q^{19}) - q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19}) - q^4H(-q)H(q^{19})\right\} \\ =&G(q)G(-q)G(q^{19})G(-q^{19}) + q^8H(q)H(-q)H(q^{19})H(-q^{19}) \\ &- q^4G(-q)H(q)G(q^{19})H(-q^{19}) - q^4G(q)H(-q)G(-q^{19})H(q^{19}) \\ =&\frac{1}{f(-q^2)f(-q^{38})}f(q^4, q^6)f(q^{76}, q^{114}) + q^8\frac{1}{f(-q^2)f(-q^{38})}f(q^2, q^8)f(q^{38}, q^{152}) \\ &- q^4\frac{1}{f(-q^2)f(-q^{38})}\left\{\psi(q^2) - q\psi(q^{10})\right\}\left\{\psi(q^{38}) + q^{19}\psi(q^{190})\right\} \\ &- q^4\frac{1}{f(-q^2)f(-q^{38})}\left\{\psi(q^2) + q\psi(q^{10})\right\}\left\{\psi(q^{38}) - q^{19}\psi(q^{190})\right\} \\ =&\frac{1}{f(-q^2)f(-q^{38})}\left\{f(q^4, q^6)f(q^{76}, q^{114}) + q^8f(q^2, q^8)f(q^{38}, q^{152})\right. \\ &\left.-2q^4\psi(q^2)\psi(q^{38}) + 2q^{24}\psi(q^{10})\psi(q^{190})\right\}. \qquad (33.4) \end{aligned}$$

Returning to (33.3), we see that it suffices to prove that

$$\begin{aligned}&G(q^2)G(q^{38})+q^8H(q^2)H(q^{38})\\=&\frac{1}{f(-q^2)f(-q^{38})}\Big\{f(q^4,q^6)f(q^{76},q^{114})+q^8f(q^2,q^8)f(q^{38},q^{152})\\&-2q^4\psi(q^2)\psi(q^{38})+2q^{24}\psi(q^{10})\psi(q^{190})\Big\}.\end{aligned}\tag{33.5}$$

Multiplying both sides of (33.5) by $f(-q^2)f(-q^{38})$ and using (2.11), we can rewrite (33.5) in its equivalent form

$$\begin{aligned}&f(-q^4,-q^6)f(-q^{76},-q^{114})+q^8f(-q^2,-q^8)f(-q^{38},-q^{152})\\=&f(q^4,q^6)f(q^{76},q^{114})+q^8f(q^2,q^8)f(q^{38},q^{152})-2q^4\psi(q^2)\psi(q^{38})+2q^{24}\psi(q^{10})\psi(q^{190}).\end{aligned}\tag{33.6}$$

By (20.3), with q replaced by q^2, the left-hand side of (33.6) is

$$\frac{1}{4q}\Big(\varphi(q)\varphi(q^{19})-\varphi(-q)\varphi(-q^{19})\Big)-q^4\psi(q^2)\psi(q^{38}).\tag{33.7}$$

Therefore, it remains to show that

$$\begin{aligned}&\frac{1}{4q}\Big(\varphi(q)\varphi(q^{19})-\varphi(-q)\varphi(-q^{19})\Big)+q^4\psi(q^2)\psi(q^{38})\\=&f(q^4,q^6)f(q^{76},q^{114})+q^8f(q^2,q^8)f(q^{38},q^{152})+2q^{24}\psi(q^{10})\psi(q^{190}).\end{aligned}\tag{33.8}$$

By (20.5) and (20.6) with q replaced by $-q^2$, we deduce that

$$\begin{aligned}&\frac{1}{4q}\Big(\varphi(q)\varphi(q^{19})-\varphi(-q)\varphi(-q^{19})\Big)+q^4\psi(q^2)\psi(q^{38})\\=&f(q^{342},q^{418})f(q^{18},q^{22})+q^8f(q^{266},q^{494})f(q^{14},q^{26})+2\,q^{24}f(q^{190},q^{570})f(q^{10},q^{30})\\&+q^{48}f(q^{114},q^{646})f(q^6,q^{34})+q^{80}f(q^{38},q^{722})f(q^2,q^{38})+q^4f(q^2,q^{38})f(q^{342},q^{418})\\&+q^{10}f(q^6,q^{34})f(q^{266},q^{494})+q^{46}f(q^{14},q^{26})f(q^{114},q^{646})+q^{76}f(q^{18},q^{22})f(q^{38},q^{722})\\=&\left(f(q^{18},q^{22})+q^4f(q^2,q^{38})\right)\left(f(q^{342},q^{418})+q^{76}f(q^{38},q^{722})\right)\\&+q^8\left(f(q^{14},q^{26})+q^2f(q^6,q^{34})\right)\left(f(q^{266},q^{494})+q^{38}f(q^{114},q^{646})\right)+2q^{24}\psi(q^{10})\psi(q^{190}).\end{aligned}\tag{33.9}$$

But by (2.19) with $n=2$, first with $a=q$ and $b=q^4$, and secondly with $a=q^2$ and $b=q^3$, we have

$$f(q,q^4)=f(q^7,q^{13})+qf(q^3,q^{17})\ \text{ and }\ f(q^2,q^3)=f(q^9,q^{11})+q^2f(q,q^{19}).\tag{33.10}$$

Using both parts of (33.10) in (33.9), each with q replaced by q^2 and q^{38}, we see that (33.8) holds. Hence, the proof of Entry 3.37 is complete.

34. New Identities for $G(q)$ and $H(q)$

We first demonstrate that Entry 3.30 generates several further new identities involving $G(q)$ and $H(q)$.

Theorem 34.1. *We have*

$$\frac{G(-q^2)G(q^{38})-q^8H(-q^2)H(q^{38})}{G(q^{152})H(q^2)-q^{30}G(q^2)H(q^{152})}=\frac{G(q^{38})H(q^8)-q^6G(q^8)H(q^{38})}{G(q^2)G(-q^{38})-q^8H(q^2)H(-q^{38})}$$
$$=\frac{G(q^{19})H(q^4)-q^3G(q^4)H(q^{19})}{G(q^{76})H(-q)+q^{15}G(-q)H(q^{76})}=\frac{\chi(-q^2)}{\chi(-q^{38})} \tag{34.1}$$

and

$$\begin{aligned}&\left\{G(q)G(-q^{19})-q^4H(q)H(-q^{19})\right\}\left\{G(-q)G(q^{19})-q^4H(-q)H(q^{19})\right\}\\ =&\left\{G(q^{19})H(q^4)-q^3H(q^{19})G(q^4)\right\}\left\{G(q^{76})H(q)-q^{15}H(q^{76})G(q)\right\}\\ =&G(q^2)G(q^{38})+q^8H(q^2)H(q^{38}).\end{aligned} \tag{34.2}$$

Proof. Observe that (34.1) together with Entry 3.34 implies (34.2), and so we prove (34.1). For simplicity, let us define

$$R(q):=G(q^{19})H(q^4)-q^3H(q^{19})G(q^4), \tag{34.3}$$
$$S(q):=G(q^{76})H(-q)+q^{15}H(q^{76})G(-q), \tag{34.4}$$
$$T(q):=G(q)G(-q^{19})-q^4H(q)H(-q^{19}). \tag{34.5}$$
(34.6)

Our proof assumes Entry 3.30 which we can now restate in the form

$$\frac{R(q)}{S(q)}=\frac{\chi(-q^2)}{\chi(-q^{38})}. \tag{34.7}$$

In (34.3), we employ Lemma 4.1 with q replaced by q^{19} to find that

$$\begin{aligned}\frac{f(-q^{38})}{f(-q^{152})}R(q)=&\left(G(q^{304})+q^{19}H(-q^{76})\right)H(q^4)-q^3\left(q^{57}H(q^{304})+G(-q^{76})\right)G(q^4)\\ =&H(q^4)G(q^{304})-q^{60}G(q^4)H(q^{304})\\ &-q^3\left(G(q^4)G(-q^{76})-q^{16}H(q^4)H(-q^{76})\right)=S(-q^4)-q^3T(q^4).\end{aligned} \tag{34.8}$$

Similarly, we obtain

$$\begin{aligned}\frac{f(-q^2)}{f(-q^8)}S(q)&=\left(-q^3H(q^{16})+G(-q^4)\right)G(q^{76})+q^{15}\left(G(q^{16})-qH(-q^4)\right)H(q^{76})\\ &=T(-q^4)-q^3R(q^4).\end{aligned} \tag{34.9}$$

Combining (34.7), (34.8), and (34.9), we find that

$$\begin{aligned}\frac{S(-q^4)-q^3T(q^4)}{T(-q^4)-q^3R(q^4)}&=\frac{f(-q^{38})f(-q^8)R(q)}{f(-q^{152})f(-q^2)S(q)}\\ &=\frac{f(-q^{38})f(-q^8)\chi(-q^2)}{f(-q^{152})f(-q^2)\chi(-q^{38})}=\frac{\chi(-q^{76})}{\chi(-q^4)}=\frac{S(q^2)}{R(q^2)},\end{aligned} \tag{34.10}$$

where (2.14) was used four times. We conclude from (34.10) that

$$\frac{S(-q^4)}{T(-q^4)} = \frac{T(q^4)}{R(q^4)} = \frac{S(q^2)}{R(q^2)} = \frac{\chi(-q^{76})}{\chi(-q^4)}, \tag{34.11}$$

which is (34.1) with q replaced by q^2. □

Theorem 34.2. *Let*

$$B(q) := G(q^{12})H(-q^7) + qG(-q^7)H(q^{12}), \tag{34.12}$$
$$C(q) := G(q)G(q^{84}) + q^{17}H(q)H(q^{84}), \tag{34.13}$$
$$V(q) := H(-q)G(q^{21}) + q^4G(-q)H(q^{21}), \tag{34.14}$$
$$W(q) := G(q^4)G(q^{21}) + q^5H(q^4)H(q^{21}), \tag{34.15}$$
$$Z(q) := H(q^3)G(q^{28}) - q^5G(q^3)H(q^{28}), \tag{34.16}$$
$$Y(q) := G(q^3)G(-q^7) - q^2H(q^3)H(-q^7). \tag{34.17}$$

Then,

$$\frac{C(q^2)}{Y(-q^2)} = \frac{V(-q^2)}{B(-q^2)} = \frac{C(q)}{B(q)} = \frac{f(-q^{12})f(-q^{14})}{f(-q^2)f(-q^{84})} \tag{34.18}$$

and

$$\frac{Z(-q)}{W(q)} = \frac{Z(q)}{W(-q)} = \frac{Y(q^2)}{W(q^2)} = \frac{Z(q^2)}{V(q^2)} = \frac{f(-q^4)f(-q^{42})}{f(-q^6)f(-q^{28})}. \tag{34.19}$$

Proof. Using (4.30) and (4.31) in Entry 3.10, we find that

$$\begin{aligned}&\left\{\frac{\chi(q^6)}{\chi(-q^4)}G(-q^6) - q^5\frac{\chi(q^2)}{\chi(-q^{12})}H(q^{24})\right\}G(q^{14})\\ &+ q^3\left\{-q\frac{\chi(q^6)}{\chi(-q^4)}H(-q^6) + \frac{\chi(q^2)}{\chi(-q^{12})}G(q^{24})\right\}H(q^{14})\\ =&\frac{\chi(-q^7)}{\chi(-q)}\chi(-q)\chi(q^3) = \chi(q^3)\chi(-q^7).\end{aligned} \tag{34.20}$$

Collecting terms and equating odd parts on both sides of (34.20), we conclude that

$$2q^3\left\{G(q^{24})H(q^{14}) - q^2G(q^{14})H(q^{24})\right\} = \frac{\chi(-q^{12})}{\chi(q^2)}\left\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\right\}. \tag{34.21}$$

At the end of this section, we prove that

$$\begin{aligned}&\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\\ =&2q^3\frac{f(q^2)f(-q^{168})}{f(-q^{12})f(-q^{28})}\left\{G(-q^2)G(q^{168}) - q^{34}H(-q^2)H(q^{168})\right\}.\end{aligned} \tag{34.22}$$

Assuming (34.22) for the time being, we conclude from (34.21) and (34.22) that

$$\begin{aligned}\frac{G(q^{24})H(q^{14})-q^2G(q^{14})H(q^{24})}{G(-q^2)G(q^{168})-q^{34}H(-q^2)H(q^{168})}=&\frac{\chi(-q^{12})}{\chi(q^2)}\frac{f(q^2)f(-q^{168})}{f(-q^{12})f(-q^{28})}\\=&\frac{f(-q^4)f(-q^{168})}{f(-q^{24})f(-q^{28})},\end{aligned}\tag{34.23}$$

by (2.14).

Next, we show that (34.23) yields two more identities of the same type. By (4.23) and (4.24),

$$\begin{aligned}&\frac{f(-q^2)}{f(-q^8)}\Big\{G(q)G(q^{84})+q^{17}H(q)H(q^{84})\Big\}\\&=\Big\{G(q^{16})+qH(-q^4)\Big\}G(q^{84})+q^{17}\Big\{G(-q^4)+q^3H(q^{16})\Big\}H(q^{84})\\&=G(q^{16})G(q^{84})+q^{20}H(q^{16})H(q^{84})+q\Big\{H(-q^4)G(q^{84})+q^{16}G(-q^4)H(q^{84})\Big\}.\end{aligned}\tag{34.24}$$

Arguing in the same way, we deduce that

$$\begin{aligned}&\frac{f(-q^{14})}{f(-q^{56})}\Big\{H(-q^7)G(q^{12})+qG(-q^7)H(q^{12})\Big\}\\&=G(q^{12})G(-q^{28})-q^8H(q^{12})H(-q^{28})+q\Big\{H(q^{12})G(q^{112})-q^{20}G(q^{12})H(q^{112})\Big\}.\end{aligned}\tag{34.25}$$

By (34.14)–(34.17), (34.23), (34.24), and (34.25), we deduce that

$$\frac{Y(q^4)+qZ(q^4)}{W(q^4)+qV(q^4)}=\frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})}\frac{f(-q^8)f(-q^{14})}{f(-q^2)f(-q^{56})}=\frac{f(-q^8)f(-q^{84})}{f(-q^{12})f(-q^{56})}.\tag{34.26}$$

Hence, we deduce that

$$\frac{Y(q)}{W(q)}=\frac{Z(q)}{V(q)}=\frac{f(-q^2)f(-q^{21})}{f(-q^3)f(-q^{14})}.\tag{34.27}$$

As promised above, we now verify (34.22). For convenience, let us define, by (2.11),

$$g(q)=f(-q^2,-q^3)=f(-q)G(q)\quad\text{and}\quad h(q)=f(-q,-q^4)=f(-q)H(q).\tag{34.28}$$

By (2.17) and (34.28), (34.22) is clearly equivalent to

$$\psi(q^3)\psi(-q^7)-\psi(-q^3)\psi(q^7)=2q^3\left\{g(-q^2)g(q^{168})-q^{34}h(-q^2)h(q^{168})\right\},\tag{34.29}$$

which we now prove.

We employ Theorem 4.5 with the set of parameters $a=q^{21},\ b=q^{63},\ c=q,\ d=q^3,\ \alpha=3,\ \beta=1,\ m=10,\ \epsilon_1=1$, and $\epsilon_2=0$ to deduce that

$$\begin{aligned}\psi(-q^{21})\psi(q)=&f(-q^{42},-q^{78})f(-q^{112},-q^{168})+qf(-q^{30},-q^{90})f(-q^{140},-q^{140})\\&+q^6f(-q^{18},-q^{102})f(-q^{112},-q^{168})+q^{15}f(-q^6,-q^{114})f(-q^{84},-q^{196})\\&-q^{22}f(-q^6,-q^{114})f(-q^{56},-q^{224})-q^{27}f(-q^{18},-q^{102})f(-q^{28},-q^{252})\\&+q^{21}f(-q^{42},-q^{78})f(-q^{28},-q^{252})+q^{10}f(-q^{54},-q^{66})f(-q^{56},-q^{224})\\&+q^3f(-q^{54},-q^{66})f(-q^{84},-q^{196}).\end{aligned}$$

Replacing q by $-q$ and adding the resulting equality to that above, we find that

$$\begin{aligned}\psi(q)\psi(-q^{21})+\psi(-q)\psi(q^{21}) &= 2f(-q^{112},-q^{168})\{f(-q^{42},-q^{78})+q^6f(-q^{18},-q^{102})\}\\ &\quad+2q^{10}f(-q^{56},-q^{224})\{f(-q^{54},-q^{66})-q^{12}f(-q^6,-q^{114})\}. \qquad (34.30)\end{aligned}$$

But, by (34.28) and (2.19), with $n=2$ and $a=-q^2, b=q^3$ and $a=q, b=-q^4$, respectively,

$$g(-q)=f(-q^2,q^3)=f(-q^9,-q^{11})-q^2f(-q,-q^{19})$$

and

$$h(-q)=f(q,-q^4)=f(-q^7,-q^{13})+qf(-q^3,-q^{17}).$$

Return to (34.30) and substitute each of the equalities above with q replaced by q^6 to deduce that

$$\psi(q)\psi(-q^{21})+\psi(-q)\psi(q^{21})=2\left\{g(q^{56})h(-q^6)+q^{10}h(q^{56})g(-q^6)\right\}. \qquad (34.31)$$

In what follows, $J(q)$ will denote an arbitrary power series, usually not the same with each appearance. By (2.19) with $n=3$ in each instance,

$$\begin{aligned}g(q) &= f(-q^2,-q^3)=f(-q^{21},-q^{24})-q^2f(-q^9,-q^{36})-q^3f(-q^6,-q^{39})\\ &= J(q^3)-q^2h(q^9), \qquad (34.32)\\ h(q) &= f(-q,-q^4)=f(-q^{18},-q^{27})-qf(-q^{12},-q^{33})-q^4f(-q^3,-q^{42})\\ &= g(q^9)-qJ(q^3), \qquad (34.33)\\ \psi(q) &= f(q^3,q^6)+q\psi(q^9)=J(q^3)+q\psi(q^9), \qquad (34.34)\end{aligned}$$

where, in the last application of (2.19), we set $a=1$ and $b=q$ and used (2.3) and (2.8). By (34.34),

$$\begin{aligned}\psi(q)\psi(-q^{21})+\psi(-q)\psi(q^{21}) &= \left\{J(q^3)+q\psi(q^9)\right\}\psi(-q^{21})+\left\{J(q^3)-q\psi(-q^9)\right\}\psi(q^{21})\\ &= J(q^3)+q\left\{\psi(q^9)\psi(-q^{21})-\psi(-q^9)\psi(q^{21})\right\}. \qquad (34.35)\end{aligned}$$

Similarly, by (34.32) and (34.33) with q replaced by q^{56}, we find that

$$\begin{aligned}&2\left\{g(q^{56})h(-q^6)+q^{10}h(q^{56})g(-q^6)\right\}\\ &\quad= J(q^3)+2q^{10}\left\{g(-q^6)g(q^{504})-q^{102}h(-q^6)h(q^{504})\right\}. \qquad (34.36)\end{aligned}$$

From these last two equalities and (34.31), we conclude that

$$\psi(q^9)\psi(-q^{21})-\psi(-q^9)\psi(q^{21})=2q^9\left\{g(-q^6)g(q^{504})-q^{102}h(-q^6)h(q^{504})\right\}, \qquad (34.37)$$

which is (34.29) with q replaced by q^3.

By Theorem 4.5, we can also verify that

$$\psi(q^3)\psi(-q^7)+\psi(-q^3)\psi(q^7)=2\left\{g(q^8)g(-q^{42})-q^{10}h(q^8)h(-q^{42})\right\}. \qquad (34.38)$$

Considering the 3-dissection of both sides of (34.38), we similarly deduce that

$$\psi(q)\psi(-q^{21})-\psi(-q)\psi(q^{21})=2q\left\{h(-q^{14})g(q^{24})+q^2g(-q^{14})h(q^{24})\right\}. \qquad (34.39)$$

Further applications of Theorem 4.5 give the identities

$$\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})=4q\left\{g(q^{12})g(q^{28})+q^8h(q^{12})h(q^{28})\right\} \qquad (34.40)$$

and

$$\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)=4q^3\left\{g(q^{84})h(q^4)-q^{16}h(q^{84})g(q^4)\right\}. \tag{34.41}$$

These two identities similarly imply each other, and so they are not independent. However, combining (3.16), (34.40), and (34.41), we obtain the interesting theta function identity

$$\frac{\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)}{\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})}=q^2\frac{f(-q^4)f(-q^{84})}{f(-q^{12})f(-q^{28})}. \tag{34.42}$$

Starting from (3.11), and arguing as in (4.32), we find that

$$\frac{\chi(-q^7)}{\chi(-q)}G(-q)-\frac{\chi(q^7)}{\chi(q)}G(q)=2q^3\frac{H(q^{14})}{\chi^2(-q^2)}. \tag{34.43}$$

By (2.15), we see that (34.43) simplifies to

$$\chi(q)\chi(-q^7)G(-q)-\chi(-q)\chi(q^7)G(q)=2q^3\frac{H(q^{14})}{\chi(-q^2)}. \tag{34.44}$$

Similarly, we can find that

$$\chi(q)\chi(-q^7)H(-q)+\chi(-q)\chi(q^7)H(q)=2\frac{G(q^{14})}{\chi(-q^2)}. \tag{34.45}$$

In (34.15), we replace q by q^2 and employ (34.44) and (34.45) with q replaced by q^3 to find that

$$\begin{aligned}2\frac{W(q^2)}{\chi(-q^6)}=&G(q^8)\left\{\chi(q^3)\chi(-q^{21})H(-q^3)+\chi(-q^3)\chi(q^{21})H(q^3)\right\}\\&+qH(q^8)\left\{\chi(q^3)\chi(-q^{21})G(-q^3)-\chi(-q^3)\chi(q^{21})G(q^3)\right\}\\=&\chi(q^3)\chi(-q^{21})\left\{H(-q^3)G(q^8)+qG(-q^3)H(q^8)\right\}\\&+\chi(-q^3)\chi(q^{21})\left\{H(q^3)G(q^8)-qG(q^3)H(q^8)\right\}\\=&\chi(q^3)\chi(-q^{21})\frac{\chi(q)\chi(-q^4)}{\chi(q^3)\chi(-q^{12})}+\chi(-q^3)\chi(q^{21})\frac{\chi(-q)\chi(-q^4)}{\chi(-q^3)\chi(-q^{12})},\end{aligned} \tag{34.46}$$

where in the last step we used (3.12) twice, once with q replaced by $-q$. By (2.15), we conclude from (34.46) that

$$2W(q^2)=\frac{\chi(-q^4)}{\chi(q^6)}\Big\{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})\Big\}. \tag{34.47}$$

Similarly, in (34.14), we replace q by q^2 and employ (34.44) and (34.45) with q replaced by q^3, and arguing as in (34.46), we find that

$$\begin{aligned}2q\frac{V(q^2)}{\chi(-q^6)}=&\chi(q^3)\chi(-q^{21})\left\{G(-q^2)G(-q^3)+qH(-q^2)H(-q^3)\right\}\\&+\chi(-q^3)\chi(q^{21})\left\{G(-q^2)G(q^3)-qH(-q^2)H(q^3)\right\}.\end{aligned} \tag{34.48}$$

Using (3.27) twice, once with q replaced by $-q$, we find from (34.48) that

$$2q\frac{V(q^2)}{\chi(-q^6)}=\chi(q^3)\chi(-q^{21})\frac{\chi(q)\chi(q^6)}{\chi(q^2)\chi(q^3)}-\chi(-q^3)\chi(q^{21})\frac{\chi(-q)\chi(q^6)}{\chi(q^2)\chi(-q^3)}, \tag{34.49}$$

which, by (2.15), implies that

$$2qV(q^2) = \frac{\chi(-q^{12})}{\chi(q^2)}\Big\{\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21})\Big\}. \tag{34.50}$$

Starting from (3.10), and arguing as in (4.32), we find that

$$\frac{\chi(q)}{\chi(q^7)}G(q^7) - \frac{\chi(-q)}{\chi(-q^7)}G(-q^7) = 2q\frac{G(q^2)}{\chi^2(-q^{14})}. \tag{34.51}$$

By (2.15), we see that (34.51) simplifies to

$$\chi(q)\chi(-q^7)G(q^7) - \chi(-q)\chi(q^7)G(-q^7) = 2q\frac{G(q^2)}{\chi(-q^{14})}. \tag{34.52}$$

Similarly, we can find that

$$\chi(q)\chi(-q^7)H(q^7) + \chi(-q)\chi(q^7)H(-q^7) = 2\frac{H(q^2)}{\chi(-q^{14})}. \tag{34.53}$$

In (34.17), we replace q by q^2 and employ (34.52) and (34.53) with q replaced by q^3 to find that

$$\begin{aligned}
2q^3\frac{Y(q^2)}{\chi(-q^{42})} =& G(-q^{14})\left\{\chi(q^3)\chi(-q^{21})G(q^{21}) - \chi(-q^3)\chi(q^{21})G(-q^{21})\right\}\\
&- q^7H(-q^{14})\left\{\chi(q^3)\chi(-q^{21})H(q^{21}) + \chi(-q^3)\chi(q^{21})H(-q^{21})\right\}\\
=&\chi(q^3)\chi(-q^{21})\left\{G(-q^{14})G(q^{21}) - q^7H(-q^{14})H(q^{21})\right\}\\
&- \chi(-q^3)\chi(q^{21})\left\{G(-q^{14})G(-q^{21}) + q^7H(-q^{14})H(-q^{21})\right\}\\
=&\chi(q^3)\chi(-q^{21})\frac{\chi(-q^7)\chi(q^{42})}{\chi(q^{14})\chi(-q^{21})} - \chi(-q^3)\chi(q^{21})\frac{\chi(q^7)\chi(q^{42})}{\chi(q^{14})\chi(-q^{21})},
\end{aligned} \tag{34.54}$$

where in the last step we used (3.27) twice, with q replaced by q^7 and $-q^7$. By (2.15), we conclude from (34.54) that

$$2q^3Y(q^2) = \frac{\chi(-q^{84})}{\chi(q^{14})}\Big\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\Big\}. \tag{34.55}$$

Similarly, in (34.16), we replace q by q^2 and employ (34.52) and (34.53) with q replaced by q^3, and arguing as in (34.54), we find that

$$\begin{aligned}
2\frac{Z(q^2)}{\chi(-q^{42})} =&\chi(q^3)\chi(-q^{21})\left\{H(q^{21})G(q^{56}) - q^7G(q^{21})H(q^{56})\right\}\\
&+ \chi(-q^3)\chi(q^{21})\left\{H(-q^{21})G(q^{56}) + q^7G(-q^{21})H(q^{56})\right\}.
\end{aligned} \tag{34.56}$$

Using (3.12) twice, with q replaced by q^7 and $-q^7$, we find from (34.56) that

$$2\frac{Z(q^2)}{\chi(-q^{42})} = \chi(q^3)\chi(-q^{21})\frac{\chi(-q^7)\chi(-q^{28})}{\chi(-q^{21})\chi(-q^{84})} + \chi(-q^3)\chi(q^{21})\frac{\chi(-q^7)\chi(-q^{28})}{\chi(-q^{21})\chi(-q^{84})}, \tag{34.57}$$

which, by (2.15), implies that

$$2Z(q^2) = \frac{\chi(-q^{28})}{\chi(q^{42})}\Big\{\chi(q^3)\chi(-q^7) + \chi(-q^3)\chi(q^7)\Big\}. \tag{34.58}$$

Recall that $B(q)$ and $C(q)$ are defined by (34.12) and (34.13), respectively. By (34.23) with q^2 replaced by $-q$, we have

$$\frac{B(q)}{C(q)} = \frac{f(-q^2)f(-q^{84})}{f(-q^{12})f(-q^{14})}. \tag{34.59}$$

Using (2.17) and (34.28), we can easily express (34.22) and (34.39) in their equivalent forms

$$2q^3C(-q^2) = \frac{f(-q^{12})f(-q^{28})}{f(q^2)f(-q^{168})}\Big\{\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7)\Big\}, \tag{34.60}$$

$$2qB(q^2) = \frac{f(-q^4)f(-q^{84})}{f(q^{14})f(-q^{24})}\Big\{\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21})\Big\}. \tag{34.61}$$

By (34.60), (34.61), (34.50), and (34.55), we conclude that

$$\frac{C(-q^2)}{Y(q^2)} = \frac{V(q^2)}{B(q^2)} = \frac{f(q^{14})f(-q^{12})}{f(q^2)f(-q^{84})}. \tag{34.62}$$

Hence, combining (34.62) with (34.59), we see that (34.18) is proved

Recall that $W(q)$ and $Z(q)$ are defined by (34.15) and (34.16), respectively. Using (2.17) and (34.28), we have, by (34.38) and (34.31), respectively,

$$2W(-q^2) = \frac{f(-q^{12})f(-q^{28})}{f(-q^8)f(q^{42})}\Big\{\chi(q^3)\chi(-q^7) + \chi(-q^3)\chi(q^7)\Big\}, \tag{34.63}$$

$$2Z(-q^2) = \frac{f(-q^4)f(-q^{84})}{f(q^6)f(-q^{56})}\Big\{\chi(q)\chi(-q^{21}) + \chi(-q)\chi(q^{21})\Big\}. \tag{34.64}$$

By (34.47), (34.58), (34.63), and (34.64), we find that

$$\frac{Z(-q^2)}{W(q^2)} = \frac{Z(q^2)}{W(-q^2)} = \frac{f(-q^8)f(-q^{84})}{f(-q^{12})f(-q^{56})}. \tag{34.65}$$

Hence, combining (34.27) and (34.65), we see that (34.19) is proved.

By (34.58), (34.55), (2.15), and the the trivial identity,

$$\chi^2(q) = \frac{\varphi(q)}{f(-q^2)},$$

we find that

$$\begin{aligned}4q^3Z(q^2)Y(q^2) =& \frac{\chi(-q^{28})\chi(-q^{84})}{\chi(q^{42})\chi(q^{14})}\Big\{\chi^2(q^3)\chi^2(-q^7) - \chi^2(-q^3)\chi^2(q^7)\Big\}\\ =& \frac{\chi(-q^{14})\chi(-q^{42})}{f(-q^6)f(-q^{14})}\Big\{\varphi(q^3)\varphi(-q^7) - \varphi(-q^3)\varphi(q^7)\Big\}.\end{aligned} \tag{34.66}$$

Similarly, by (34.50) and (34.47), we deduce that

$$\begin{aligned}4qV(q^2)W(q^2) =& \frac{\chi(-q^4)\chi(-q^{12})}{\chi(q^2)\chi(q^6)}\Big\{\chi^2(q)\chi^2(-q^{21}) - \chi^2(-q)\chi^2(q^{21})\Big\}\\ =& \frac{\chi(-q^2)\chi(-q^6)}{f(-q^2)f(-q^{42})}\Big\{\varphi(q)\varphi(-q^{21}) - \varphi(-q)\varphi(q^{21})\Big\}.\end{aligned} \tag{34.67}$$

By (34.27), we conclude from (34.66), (34.67), and (2.14), that

$$\begin{aligned}\frac{\varphi(q^3)\varphi(-q^7)-\varphi(-q^3)\varphi(q^7)}{\varphi(q)\varphi(-q^{21})-\varphi(-q)\varphi(q^{21})}&=q^2\frac{\chi(-q^{14})\chi(-q^{42})}{f(-q^6)f(-q^{14})}\frac{\chi(-q^2)\chi(-q^6)}{f(-q^2)f(-q^{42})}\frac{f^2(-q^4)f^2(-q^{42})}{f^2(-q^6)f^2(-q^{28})}\\&=q^2\frac{f(-q^4)f(-q^{84})}{f(-q^{12})f(-q^{28})},\end{aligned}\tag{34.68}$$

which is (34.42).

Alternatively, we can derive two apparently new theta function identities which yield a factorization of (34.42). Namely,

$$\frac{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})}=q^3\frac{\varphi(-q^4)\psi(-q^{42})}{\varphi(-q^{12})\psi(-q^{14})},\tag{34.69}$$

$$q\frac{\chi(q^3)\chi(-q^7)+\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})-\chi(-q)\chi(q^{21})}=\frac{\varphi(-q^{28})\psi(-q^6)}{\varphi(-q^{84})\psi(-q^2)}.\tag{34.70}$$

To prove (34.69) and (34.70), we employ (34.27) with q replaced by q^2 and use the representations for $Z(q^2)$, $Y(q^2)$, $V(q^2)$, and $W(q^2)$ obtained in (34.58), (34.55), (34.50), and (34.47), respectively. Thus, by (34.55), (34.47), and (34.27) with q replaced by q^2,

$$\begin{aligned}&\frac{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})}=q^3\frac{\chi(q^{14})\chi(-q^4)Y(q^2)}{\chi(-q^{84})\chi(q^6)W(q^2)}\\&=q^3\frac{\chi(q^{14})\chi(-q^4)f(-q^4)f(-q^{42})}{\chi(-q^{84})\chi(q^6)f(-q^6)f(-q^{28})}=q^3\frac{f(-q^{42})}{\chi(-q^{84})}\frac{1}{\chi(q^6)f(-q^6)}f(-q^4)\chi(-q^4)\frac{\chi(q^{14})}{f(-q^{28})}\\&=q^3\psi(-q^{42})\frac{1}{\varphi(-q^{12})}\varphi(-q^4)\frac{1}{\psi(-q^{14})},\end{aligned}\tag{34.71}$$

where in the last step two applications of (2.17) and (2.14) are used. Therefore, (34.69) has been established. The identity (34.70) is proved in a similar way, and so we omit the details. □

Theorem 34.3.

$$\frac{G(q)G(-q^{14})-q^3H(q)H(-q^{14})}{G(q^7)H(-q^2)+qH(q^7)G(-q^2)}=\frac{G(q^{56})H(q)-q^{11}H(q^{56})G(q)}{G(q^7)G(q^8)+q^3H(q^7)H(q^8)}\tag{34.72}$$

$$=\frac{\chi(-q^{14})}{\chi(-q^2)}=\frac{G(q)G(q^{14})+q^3H(q)H(q^{14})}{G(-q^7)H(q^2)+qH(-q^7)G(q^2)}.\tag{34.73}$$

Proof. Let $N(q)$ and $M(q)$ be defined by Entries 3.9 and 3.10, i.e.,

$$N(q):=G(q^7)H(q^2)-qH(q^7)G(q^2)=\frac{\chi(-q)}{\chi(-q^7)}\tag{34.74}$$

and

$$M(q):=G(q)G(q^{14})+q^3H(q)H(q^{14})=\frac{\chi(-q^7)}{\chi(-q)}.\tag{34.75}$$

Starting from (34.75) and employing (4.23) and (4.24) as in (16.4), we find that

$$\begin{aligned}\frac{\chi(-q^7)}{\chi(-q)} =& G(q)G(q^{14}) + q^3H(q)H(q^{14})\\ =& \frac{f(-q^8)}{f(-q^2)}\Big\{G(q^{14})G(q^{16}) + q^6H(q^{14})H(q^{16})\\ &+ q\big(G(q^{14})H(-q^4) + q^2H(q^{14})G(-q^4)\big)\Big\}. \qquad (34.76)\end{aligned}$$

Upon equating the even parts in (34.76) and using (2.17) and (2.15), we find that

$$\begin{aligned}G(q^{14})G(q^{16}) + q^6H(q^{14})H(q^{16}) &= \frac{1}{2}\frac{f(-q^2)}{f(-q^8)}\left\{\frac{\chi(q^7)}{\chi(q)} + \frac{\chi(-q^7)}{\chi(-q)}\right\}\\ &= \frac{1}{2}\chi(-q^4)\Big\{\chi(q)\chi(-q^7) + \chi(-q)\chi(q^7)\Big\}. \qquad (34.77)\end{aligned}$$

Similarly, equating odd parts in (34.76), we deduce that

$$\begin{aligned}&G(q^{14})H(-q^4) + q^2H(q^{14})G(-q^4)\\ =&\frac{1}{2q}\chi(-q^4)\Big\{\chi(q)\chi(-q^7) - \chi(-q)\chi(q^7)\Big\}. \qquad (34.78)\end{aligned}$$

Analogously, starting from (34.74), using (4.23) and (4.24) with q replaced by q^7, and equating even and odd parts on both sides of the resulting equality, we can deduce that

$$G(q^{112})H(q^2) - q^{22}H(q^{112})G(q^2) = \frac{1}{2}\chi(-q^{28})\left\{\chi(q)\chi(-q^7) - \chi(-q)\chi(q^7)\right\} \qquad (34.79)$$

and

$$G(q^2)G(-q^{28}) - q^6H(q^2)H(-q^{28}) = \frac{1}{2q}\chi(-q^{28})\Big\{\chi(q)\chi(-q^7) - \chi(-q)\chi(q^7)\Big\}. \qquad (34.80)$$

Hence, by (34.74), (34.75), (34.77)–(34.80) with q^2 replaced by q, and (2.15), we conclude that

$$\begin{aligned}\frac{G(q)G(-q^{14}) - q^3H(q)H(-q^{14})}{G(q^7)H(-q^2) + qH(q^7)G(-q^2)} &= \frac{G(q^{56})H(q) - q^{11}H(q^{56})G(q)}{G(q^7)G(q^8) + q^3H(q^7)H(q^8)}\\ &= \frac{\chi(-q^{14})}{\chi(-q^2)} = \frac{M(q)}{N(-q)},\end{aligned}$$

and so the proof of Theorem 34.3 is complete. □

The only other new identities for $G(q)$ and $H(q)$ in the literature that are in the spirit of Ramanujan's identities (of which we are aware) were found by M. Koike [15]. He discovered them using Thompson series and a computer, but he did not prove them.

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