AUTOMATIC PROOF OF THETA-FUNCTION IDENTITIES

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Abstract. This is a tutorial for using two new MAPLE packages, `thetaids` and `ramarobinsids`. The `thetaids` package is for proving generalized eta-product identities. It uses the valence formula for modular functions. The `thetaids` package can be used for finding theta-function identities. We show how to find and prove Ramanujan's 40 identities for his so called Rogers-Ramanujan functions $G(q)$ and $H(q)$. In his thesis Robins found similar identities for higher level generalized eta-products. The `ramarobinsids` package is for finding and proving identities for generalizations of Ramanujan's $G(q)$ and $H(q)$ and Robin's extensions. These generalizations are associated with certain real Dirichlet characters. We find a total of over 300 identities.

Contents

1. Introduction

The Rogers-Ramanujan functions are

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})},\quad H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{1.1}$$

The ratio of these two functions is the famous Rogers-Ramanujan continued fraction [1]

$$\begin{aligned}\frac{G(q)}{H(q)}&=\prod_{n=0}^{\infty}\frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})}\\&=1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\ddots}}}}\end{aligned}$$

Ramanujan also found

$$H(q)G(q)^{11}-q^2G(q)H(q)^{11}=1+11G(q)^6H(q)^6 \tag{1.2}$$

and

$$H(q)G(q^{11})-q^2G(q)H(q^{11})=1, \tag{1.3}$$

and remarked that "each of these formulae is the simplest of a large class". Here we have used the standard q-notation

$$(a;q)_n:=\prod_{j=0}^{n-1}(1-aq^j)\qquad(a;q)_\infty:=\prod_{j=0}^{\infty}(1-aq^j).$$

In 1974 B. J. Birch published a description of some manuscripts of Ramanujan including a list of forty identities for the Rogers-Ramanjan functions. Biagioli [5] show how the theory of modular forms could prove identities of this type efficiently. See [4] and [2] for recent work. It is instructive to write the Rogers-Ramanujan functions in terms of generalized eta-products.

The Dedekind eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

where $\tau \in \mathcal{H} := \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ and $q := e^{2\pi i\tau}$, and the generalized Dedekind eta function is defined to be

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2(g/\delta)} \prod_{m\equiv \pm g \pmod{\delta}} (1-q^m), \tag{1.4}$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial, $\{t\} = t - [t]$ is the fractional part of t, $g, \delta, m \in \mathbb{Z}^+$ and $0 \le g < \delta$. The function (1.4) is a modular function (modular form of weight 0) on $\mathrm{SL}_2(\mathbb{Z})$ with a multiplier system.

Ramanujan's identity (1.3) can be rewritten as

$$\frac{1}{\eta_{5,2}(\tau)\eta_{5,1}(11\tau)} - \frac{1}{\eta_{5,1}(\tau)\eta_{5,2}(11\tau)} = 1. \tag{1.5}$$

The main goal of the `thetaids` MAPLE package is to automatically verify identities for generalized eta-products using the theory of modular functions.

In Sections 3-4 we describe the `ramarobinsids` package, which uses the `thetaids` package to search for and prove theta-function identities considered by Ramanujan [2] and Robins [19]. We demonstrate how we found new identities for extensions of Ramanujan's and Robins's functions.

We note that Liangjie [12] gave an algorithm for proving relations for certain theta-functions and their derivatives using a different method. We also note that Lovejoy and Osburn [13], [14], [15], [16], have used an earlier version of the `thetaids` package to prove theta-functions identities that were needed to establish an number of results for mock-theta functions.

1.1. **Installation Instructions.** First install the `qseries` package from

`http://qseries.org/fgarvan/qmaple/qseries`

and follow the directions on that page. Before proceeding it is advisable to become familiar with the functions in the `qseries` package. See [8] for a tutorial. Then go to

`http://qseries.org/fgarvan/qmaple/thetaids`

to install the `thetaids` package. In Section 3 you will need to install the `ramarobinsids` package from

`http://qseries.org/fgarvan/qmaple/ramarobinsids`

2. Proving theta-function identities

To prove a given theta-function identity we basically to the following.

(i) We rewrite the identity in terms of generalized eta-functions.
(ii) We check that each term in the identity is a modular function on some group $\Gamma_1(N)$.
(iii) We determine the order at each cusp of $\Gamma_1(N)$ of each term in the identity.
(iv) We use the valence formula to determine up to which power of q we need to verify the identity.
(v) Then finally we prove the identity by carrying out the verification.

In this section we explain how we carry out each of these steps in MAPLE. Then we show how the whole process of proof can be automated.

2.1. **Encoding theta-functions, eta-functions and generalized eta-functions.** We recall Jacobi's triple product for theta-functions:

$$\prod_{n=1}^{\infty}(1-zq^{n-1})(1-z^{-1}q^n)(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n-1)/2}, \tag{2.1}$$

so that

$$\prod_{n=1}^{\infty}(1-q^{\delta n+\delta-g})(1-q^{\delta n+g-n})(1-q^{\delta n})=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n(\delta n-\delta+2g)}. \tag{2.2}$$

In the qseries MAPLE package the function on the left side of (2.2) is encoded symbollically as `JAC(g,`δ`,infinity)`. This is the building block of the functions in our package. In the qseries package `JAC(0,`δ`,infinity)` corresponds symbollically to

$$\prod_{n=1}^{\infty}(1-q^{\delta n})=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{\delta}{2}n(3n+1)}, \tag{2.3}$$

which is Euler's Pentagonal Number Theorem.

Function	Symbolic MAPLE form
$\prod_{n=1}^{\infty}(1-q^{\delta n+\delta-g})(1-q^{\delta n+g-n})(1-q^{\delta n})$	`JAC(g,` δ`, infinity)`
$\prod_{n=1}^{\infty}(1-q^{\delta n})$	`JAC(0,` δ`, infinity)`
$\eta_{\delta,g}(\tau)$	`GETA(`δ`,g)`
$\eta(\delta\tau)$	`EETA(`δ`)`

We will also consider generalized eta-products. Let N be a fixed positive integer. A generalized Dedekind eta-product of level N has the form

$$f(\tau) = \prod_{\substack{\delta | N \\ 0<g<\delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \tag{2.4}$$

where

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } g=0 \text{ or } g=\delta/2, \\ \mathbb{Z} & \text{otherwise.} \end{cases} \tag{2.5}$$

In MAPLE we represent the generalized eta-product

$$\eta_{N,g_1}(\tau)^{r_1}\, \eta_{N,g_1}(\tau)^{r_1} \cdots \eta_{N,g_m}(\tau)^{r_m}$$

symbollically by the list

$$[[N, g_1, r_1], [N, g_2, r_2] \ \ldots, [N, g_m, r_m]].$$

We call such a list a *geta-list*.

2.2. **Symbolic product conversion.** `jac2eprod` — Converts a quotient of theta-functions in `JAC` notation to a product of generalized eta-functions in `EETA` and `GETA` notation.

EXAMPLE:

```
> with(qseries):
> with(thetaids):
> G:=q->add(q^(n^2)/aqprod(q,q,n),n=0..10):
> H:=q->add(q^(n^2+n)/aqprod(q,q,n),n=0..10):
> JG:=jacprodmake(G(q),q,50);
```

$$\frac{JAC(0,5,\infty)}{JAC(1,5,\infty)}$$

```
> HG:=jacprodmake(H(q),q,50);
```

$$\frac{JAC(0,5,\infty)}{JAC(2,5,\infty)}$$

```
> JP:=jacprodmake(H(q)*G(q)^(11),q,80);
```

$$\frac{(JAC\,(0,5,\infty))^{12}}{(JAC\,(1,5,\infty))^{11}\, JAC\,(2,5,\infty)}$$

```
> GP:=jac2eprod(JP);
```

$$\frac{1}{(GETA\,(5,1))^{11}\, GETA\,(5,2)}$$

`jac2getaprod` — Converts a quotient of theta-function in `JAC` notation to a product of generalized eta-functions in standard notation.

EXAMPLE:
```
> jac2getaprod(JP);
```
$$\frac{1}{\eta_{5,1}\left(\tau\right)^{11}\eta_{5,2}\left(\tau\right)}$$

`GETAP2getalist` — Converts a product of generalized eta-functions into a list as described above.

EXAMPLE:
```
> GETAP2getalist(GP);
```
$$[[5,1,-11],[5,2,-1]]$$

2.3. **Processing theta-functions.** There are two main functions in the `thetaids` package for processing combinations of theta-functions.

`mixedjac2jac` — Converts a sum of quotients of theta-functions written in terms of `JAC(a,b,infinity)` to a sum with the same base `b`. The functions `jac2series` and `jacprodmake` from the `qseries` package are used.

EXAMPLE:
```
> Y1:=1+jacprodmake(G(q),q,100)*jacprodmake(H(q^2),q,100);
```
$$1+\frac{JAC\left(0,5,\infty\right)JAC\left(0,10,\infty\right)}{JAC\left(1,5,\infty\right)JAC\left(4,10,\infty\right)}$$
```
> Y2:=mixedjac2jac(Y1);
```
$$1+\frac{\left(JAC\left(0,10,\infty\right)\right)^{3}}{JAC\left(1,10,\infty\right)\left(JAC\left(4,10,\infty\right)\right)^{2}}$$

`processjacid` — Processes a theta-function identity written as a rational function of `JAC`-functions using `mixedjac2jac` and renormalizing by dividing by the term with the lowest power of q.

As an example, we consider the well-known identity

$$\theta_3(q)^4=\theta_2(q)^4+\theta_4(q)^4. \tag{2.6}$$

EXAMPLE:

```
> with(qseries):
> with(thetaids):
> F1:=theta2(q,100)^4:
> F2:=theta3(q,100)^4:
> F3:=theta4(q,100)^4:
> findhom([F1,F2,F3],q,1,0);
```

$$\{X_1 - X_2 + X_3\}$$

```
> JACID0:=qs2jaccombo(F1-F2+F3,q,100);
```

$$16\,\frac{q\,(JAC\,(0,4,\infty))^6}{(JAC\,(2,4,\infty))^2} - \frac{(JAC\,(0,4,\infty))^6\,(JAC\,(2,4,\infty))^6}{(JAC\,(1,4,\infty))^8} + (JAC\,(1,2,\infty))^4$$

```
> JACID1:=processjacid(JACID0);
```

$$-16\,\frac{q\,(JAC\,(1,4,\infty))^8}{(JAC\,(2,4,\infty))^8} + 1 - \frac{(JAC\,(1,4,\infty))^{16}}{(JAC\,(0,4,\infty))^{12}\,(JAC\,(2,4,\infty))^4}$$

```
> expand(jac2getaprod(JACID1));
```

$$-\frac{\eta_{4,1}(\tau)^{16}}{\eta_{4,2}(\tau)^4} + 1 - 16\,\frac{\eta_{4,1}(\tau)^8}{\eta_{4,2}(\tau)^8}$$

We see that (2.6) is equivalent to the identity

$$\frac{\eta_{4,1}(\tau)^{16}}{\eta_{4,2}(\tau)^4} + 16\,\frac{\eta_{4,1}(\tau)^8}{\eta_{4,2}(\tau)^8} = 1. \tag{2.7}$$

2.4. **Checking modularity.** Robins [18] has found sufficient conditions under which a generalized eta-product is a modular function on $\Gamma_1(N)$.

Theorem 2.1. [18, Theorem 3] *The function $f(\tau)$, defined in (2.4), is a modular function on $\Gamma_1(N)$ if*

(i) $\displaystyle\sum_{\substack{\delta|N\\ g}} \delta P_2(\tfrac{g}{\delta}) r_{\delta,g} \equiv 0 \pmod 2$, *and*

(ii) $\displaystyle\sum_{\substack{\delta|N\\ g}} \frac{N}{\delta} P_2(0) r_{\delta,g} \equiv 0 \pmod 2$.

The functions on the left side of (i), (ii) above are computed using the MAPLE functions `vinf` and `v0` respectively. Suppose $f(\tau)$ is given as in (2.4) and this generalized eta-product is encoded as the geta-list L. Recall that each item in the list L has the form $[\delta, g, r_{\delta,g}]$.

The syntax is `vinf(L,N)` and `v0(L,N)`. As an example we consider the two generalized eta-products in (2.7).

EXAMPLE:
```
> L1:=[[4,1,16],[4,2,-4]];
```
$$[[4,1,16],[4,2,-4]]$$
```
> vinf(L1,4),v0(L1,4);
```
$$0,2$$
```
> L2:=[[4,1,8],[4,2,-8]];
```
$$[[4,1,8],[4,2,-8]]$$
```
> vinf(L2,4),v0(L2,4);
```
$$2,0$$

The numbers 0, 2 are even and we see that both generalized eta-products in (2.7) are modular functions on $\Gamma_1(4)$ by Theorem 2.1.

`Gamma1ModFunc(L,N)` — Checks whether a given generalized eta-product is a modular function on $\Gamma_1(N)$. Here the generalized eta-product is encoded as the geta-list L. The function first checks whether each δ is a divisor of N and checks whether both `vinf(L,N)` and `v0(L,N)` are even. It returns 1 if it is a modular function on $\Gamma_1(N)$ otherwise it returns 0. If the global variable `xprint` is set to *true* then more detailed information is printed. Thus here and throughout `xprint` can be used for debugging purposes.

EXAMPLE:
```
> Gamma1ModFunc(L1,4);
```
$$1$$
```
> xprint := true:
> Gamma1ModFunc(L1,4);
* starting Gamma1ModFunc with L=[[4, 1, 16], [4, 2, -4]] and N=4
All n are divisors of 4
val0=2
which is even.
valinf=0
which is even.
It IS a modfunc on Gamma1(4)
```
$$1$$

2.5. **Cusps.** Cho, Koo and Park [6] have found a set of inequivalent cusps for $\Gamma_1(N) \cap \Gamma_0(mN)$. The group $\Gamma_1(N)$ corresponds to the case $m = 1$.

Theorem 2.2. [6, Corollary 4, p.930] *Let a, c, a', $c \in \mathbb{Z}$ with $(a, c) = (a', c') = 1$.*

(i) *The cusps $\frac{a}{c}$ and $\frac{a'}{c'}$ are equivalent mod $\Gamma_1(N)$ if and only if*

$$\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \pm \begin{pmatrix} a + nc \\ c \end{pmatrix} \pmod{N}$$

for some integer n.

(ii) *The following is a complete set of inequivalent cusps mod $\Gamma_1(N)$.*

$$\mathcal{S} = \left\{ \frac{y_{c,j}}{x_{c,i}} : 0 < c \mid N,\, 0 < s_{c,i},\, a_{c,j} \le N,\, (s_{c,i}, N) = (a_{c,j}, N) = 1, \right.$$
$$s_{c,i} = s_{c,i'} \iff s_{c,1} \equiv \pm s_{c',i'} \pmod{\tfrac{N}{c}},$$
$$a_{c,j} = a_{c,j'} \iff \begin{cases} a_{c,j} \equiv \pm a_{c,j'} \pmod{c}, & \text{if } c = \frac{N}{2} \text{ or } N, \\ a_{c,j} \equiv a_{c,j'} \pmod{c}, & \text{otherwise}, \end{cases}$$
$$\left. x_{c,i}, y_{c,j} \in \mathbb{Z} \text{ chosen s.th. } x_{c,i} \equiv c s_{c,i},\ y_{c,i} \equiv a_{c,j} \pmod{N},\ (x_{c,i}, y_{c,j}) = 1 \right\},$$

(iii) *and the fan width of the cusp $\frac{a}{c}$ is given by*

$$\kappa(\tfrac{a}{c}, \Gamma_1(N)) = \begin{cases} 1, & \text{if } N = 4 \text{ and } (c, 4) = 2, \\ \frac{N}{(c,N)}, & \text{otherwise.} \end{cases}$$

In this theorem, it is understood as usual that the fraction $\frac{\pm 1}{0}$ corresponds to $i\infty$.

`cuspequiv1`(a_1, c_1, a_2, c_2, N) — determines whether the cusps a_1/c_1 and a_2/c_2 are $\Gamma_1(N)$-equivalent using Theorem 2.2(i).

EXAMPLE:

```
> cuspequiv1(1,3,1,9,40);
```

$$false$$

```
> cuspequiv1(1,9,2,9,40);
```

$$true$$

We see that modulo $\Gamma_1(40)$ the cusps $\frac{1}{3}$ and $\frac{1}{9}$ are inequivalent and the cusps $\frac{1}{9}$ and $\frac{2}{9}$ are equivalent.

`Acmake(c,N)` — returns the set $\{a_{c,j}\}$ where c is a positive divisor of N.

`Scmake(c,N)` — returns the set $\{s_{c,i}\}$ where c is a positive divisor of N.

`newxy(x,y,N)` — returns $[x_1, y_1]$ for given $(x, y, N) = 1$ such that $x_1 \equiv x \pmod{N}$ and $y_1 \equiv y \pmod{N}$.

`cuspmake1(N)` — returns a set of inequivalent cusps for $\Gamma_1(N)$ using Theorem 2.2. Each cusp a/c in the list is represented by $[a, c]$, so that ∞ is represented by $[1, 0]$. This MAPLE procedure uses the functions `Acmake`, `Scmake` and `newxy`.

`cuspwid1(a,c,N)` — returns the width of the cusp a/c for the group $\Gamma_1(N)$ using Theorem 2.2(iii).

EXAMPLE:

```
> C10:=cuspmake1(10);
```

$$\{[0,1],[1,0],[1,2],[1,3],[1,4],[1,5],[2,5],[3,10]\}$$

```
> for L in C10 do lprint(L,cuspwid1(L[1],L[2],10));od;
    [0, 1], 10
    [1, 0], 1
    [1, 2], 5
    [1, 3], 10
    [1, 4], 5
    [1, 5], 2
    [2, 5], 2
    [3, 10], 1
```

We have the following table of cusps for $\Gamma_1(10)$.

cusp	cusp-width
0	10
∞	1
$\frac{1}{2}$	5
$\frac{1}{3}$	10
$\frac{1}{4}$	5
$\frac{1}{5}$	2
$\frac{2}{5}$	2
$\frac{3}{10}$	1

`CUSPSANDWIDMAKE(N)` — returns a set of inequivalent cusps for $\Gamma_1(N)$, and corresponding widths. Output has the form `[CUSPLIST,WIDTHLIST]`.

EXAMPLE:
```
> CUSPSANDWIDMAKE1(10);
```

$$\left[\left[oo, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{10}\right], [1, 10, 5, 10, 5, 2, 2, 1]\right]$$

2.6. **Orders at cusps.** We will use Biagioli's [5] results for theta-functions to calculate orders at cusps of generalized eta-products. We define the theta-function

$$\theta_{\delta,g}(\tau) = q^{(\delta-2g)^2/(8\delta)} \prod_{m=1}^{\infty}(1-q^{m\delta-g})(1-q^{m\delta-(g-\delta)})(1-q^{m\delta}), \tag{2.8}$$

for $0 < g < \delta$. This corresponds to Biagioli's function $f_{\delta,g}$ [5, p.277]. The classical Dedekind eta-function can be written as

$$\eta(\tau) = \theta_{3,1}(\tau), \tag{2.9}$$

and the generalized Dedekind eta-function can be written as

$$\eta_{\delta,g}(\tau) = \frac{\theta_{\delta,g}(\tau)}{\eta(\delta\tau)} = \frac{\theta_{\delta,g}(\tau)}{\theta_{3\delta,\delta}(\tau)}. \tag{2.10}$$

Biagioli [5] has calculated the invariant order of $\theta_{\delta,g}(\tau)$ at any cusp. Using (2.10) this gives a method for calculating the invariant order at any cusp of a generalized eta-product.

Theorem 2.3. [5, Lemma 3.2, p.285] *The order at the cusp $s = \frac{b}{c}$ (assuming $(b, c) = 1$) of the theta function $\theta_{g,\delta}(\tau)$ (defined above and assuming $\delta \nmid g$) is*

$$ord\,(\theta_{g,\delta}(\tau), s) = \frac{e^2}{2\delta}\left(\frac{bg}{e} - \left[\frac{bg}{e}\right] - \frac{1}{2}\right)^2, \tag{2.11}$$

where $e = (\delta, c)$ and $[\ \ ]$ is the greatest integer function.

`Bord`(δ, g, a, c) — returns the order of $\theta_{\delta,g}(\tau)$ at the cusp a/c, assuming $(a, c) = 1$ and $\delta \nmid g$.
`getacuspord`(δ, g, a, c) — returns the order of generalized eta-function $\eta_{\delta,g}(\tau)$ at the cusp a/c, assuming $(a, c) = 1$ and $\delta \nmid g$.

EXAMPLE:
```
> getacuspord(50,1,4,29);
```

$$\frac{1}{600}$$

We see that

$$\text{ord}\left(\eta_{50,1}(\tau), \frac{4}{29}\right) = \frac{1}{600}.$$

Let G be a generalized eta-product corresponding to the getalist L. The following proc calculates invariant order $\operatorname{ord}(G,\zeta)$ for any cusp ζ.

`getaprodcuspord(L,cusp)` — returns of the generalized eta-product corresponding to the geta-list L at the given cusp. The cusp is either a rational or `oo` (infinity).

EXAMPLE:
```
> GL:=[[4,1,16],[4,2,-4]];
```
$$[[4,1,16],[4,2,-4]]$$
```
> getaprodcuspord(GL,1/2);
```
$$-1$$

We see that

$$\operatorname{ord}\left(\frac{\eta_{4,1}(\tau)^{16}}{\eta_{4,2}(\tau)^4}, \frac{1}{2}\right) = -1.$$

Following [5, p.275], [17, p.91] we consider the order of a function f with respect to a congruence subgroup Γ at the cusp $\zeta \in \mathbb{Q} \cup \{\infty\}$ and denote this by

$$\operatorname{ORD}(f,\zeta,\Gamma) = \kappa(\zeta,\Gamma) \operatorname{ord}(f,\zeta). \tag{2.12}$$

`getaprodcuspORDS`(L,S,W) — returns a list of orders $\operatorname{ORD}(G,\zeta,\Gamma_1(N))$ where G is the generalized eta-product corresponding to the getalist L, $\zeta \in S$ (list of inequivalent cusps of $\Gamma_1(N)$) and W is a list of corresponding fan-widths.

EXAMPLE:
```
> CW4:=CUSPSANDWIDMAKE1(4);
```
$$\left[\left[\infty, 0, \frac{1}{2}\right], [1,4,1]\right]$$
```
> GL:=[[4,1,16],[4,2,-4]];
```
$$[[4,1,16],[4,2,-4]]$$
```
> getaprodcuspORDS(GL,CW4[1],CW4[2]);
```
$$[0,1,-1]$$

We know that the generalized eta-product

$$f(\tau) = \frac{\eta_{4,1}(\tau)^{16}}{\eta_{4,2}(\tau)^4}$$

is a modular function on $\Gamma_1(4)$. We calculated $\operatorname{ORD}(f,\zeta,\Gamma_1(4))$ at each cusp ζ of $\Gamma_1(4)$.

ζ	ORD $(f, \zeta, \Gamma_1(4)$
∞	0
0	1
$\frac{1}{2}$	-1

Observe that the total order of f with respect to $\Gamma_1(4)$ is 0:

$$\text{ORD}\,(f, \Gamma_1(4)) = \sum_{\zeta \in \mathcal{S}} \text{ORD}\,(f, \zeta, \Gamma_1(4)) = 0 + 1 - 1 = 0,$$

in agreement with the valence formula. See Theorem 2.4 below.

2.7. **Proving theta-function identities.** Our method for proving theta-function or generalized eta-product identities depends on

Theorem 2.4 (The Valence Formula [17](p.98))**.** *Let $f \neq 0$ be a modular form of weight k with respect to a subgroup Γ of finite index in $\Gamma(1) = SL_2(\mathbb{Z})$. Then*

$$ORD\,(f, \Gamma) = \frac{1}{12} \mu\, k, \tag{2.13}$$

where μ is index $\widehat{\Gamma}$ in $\widehat{\Gamma(1)}$,

$$ORD\,(f, \Gamma) := \sum_{\zeta \in R^*} ORD\,(f, \zeta, \Gamma),$$

R^* is a fundamental region for Γ, and

$$ORD\,(f, \zeta, \Gamma) = \kappa(\zeta, \Gamma)\, ord\,(f, \zeta),$$

for a cusp ζ and $\kappa(\zeta, \Gamma)$ denotes the fan width of the cusp $\zeta \pmod{\Gamma}$.

Remark. For $\zeta \in \mathfrak{h}$, ORD (f, ζ, Γ) is defined in terms of the invariant order ord (f, ζ), which is interpreted in the usual sense. See [17, p.91] for details of this and the notation used.

Since any generalized eta-product has weight $k = 0$ and has no zeros and no poles on the upper-half plane we have

Corollary 2.5. *Let $f_1(\tau)$, $f_2(\tau)$, $\ldots$, $f_n(\tau)$ be generalized eta-products that are modular functions on $\Gamma_1(N)$. Let $\mathcal{S}_N$ be a set of inequivalent cusps for $\Gamma_1(N)$. Define the constant*

$$B = \sum_{\substack{s \in \mathcal{S}_N \\ s \neq i\infty}} min(\{ORD\,(f_j, s, \Gamma_1(N)) \,:\, 1 \leq j \leq n\} \cup \{0\}), \tag{2.14}$$

and consider

$$g(\tau) := \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) + 1, \tag{2.15}$$

where each $\alpha_j \in \mathbb{C}$. Then

$$g(\tau) \equiv 0$$

if and only if

$$ORD\,(g(\tau), i\infty, \Gamma_1(N)) > -B. \tag{2.16}$$

To prove an alleged theta-function identity, we first rewrite it in the form

$$\alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) + 1 = 0, \tag{2.17}$$

where each $\alpha_i \in \mathbb{C}$ and each $f_i(\tau)$ is a generalized eta-product of level N. We use the following algorithm:

STEP 1. Use Theorem 2.1 to check that $f_j(\tau)$ is a generalized eta-product on $\Gamma_1(N)$ for each $1 \le j \le n$.

STEP 2. Use Theorem 2.2 to find a set $\mathcal{S}_N$ of inequivalent cusps for $\Gamma_1(N)$ and the fan width of each cusp.

STEP 3. Use Theorem 2.3 to calculate the invariant order of each generalized eta-product $f_j(\tau)$ at each cusp of $\Gamma_1(N)$.

STEP 4. Calculate

$$B = \sum_{\substack{s\in\mathcal{S}_N \\ s\neq i\infty}} \min(\{\mathrm{ORD}\,(f_j, s, \Gamma_1(N)) \,:\, 1 \le j \le n\} \cup \{0\}).$$

STEP 5. Show that

$$\mathrm{ORD}\,(g(\tau), i\infty, \Gamma_1(N)) > -B$$

where

$$g(\tau) = \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau).$$

Corollary 2.5 then implies that $g(\tau) \equiv 0$ and hence the theta-function identity (2.17).

To calculate the constant B in (2.14) and STEP 4 we use

`mintotORDS(L,n)` — returns the constant B in equation (2.14) where L the array of ORDS:

$$L := [\mathrm{ORD}\,(f_1), \mathrm{ORD}\,(f_2), \ldots, \mathrm{ORD}\,(f_n)],$$

where

$$\mathrm{ORD}\,(f) = [\mathrm{ORD}\,(f, \zeta_1, \Gamma_1(N)), \mathrm{ORD}\,(f, \zeta_2, \Gamma_1(N)), \ldots, \mathrm{ORD}\,(f, \zeta_m, \Gamma_1(N))]$$

and ζ_1, ζ_2, $\ldots$, ζ_m are inequivalent cusps of $\Gamma_1(N)$. Each $\mathrm{ORD}\,(f)$ is computed using `getaprodcuspORDS`.

As an example we prove Ramanujan's well-known identity

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-q^{25n})} = R(q^5) - q - \frac{q^2}{R(q^5)}, \tag{2.18}$$

where

$$R(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-2})(1-q^{5n-3})}{(1-q^{5n-1})(1-q^{5n-4})}.$$

We rewrite this identity as

$$\frac{\eta(\tau)}{\eta(25\tau)} = \frac{\eta_{25,10}(\tau)}{\eta_{25,5}(\tau)} - 1 - \frac{\eta_{25,5}(\tau)}{\eta_{25,10}(\tau)}. \tag{2.19}$$

Let

$$g(\tau) = f_1(\tau) - f_2(\tau) + f_3(\tau) + 1, \tag{2.20}$$

where

$$f_1(\tau) = \frac{\eta(\tau)}{\eta(25\tau)} = \prod_{j=1}^{12} \eta_{25,j}(\tau), \quad f_2(\tau) = \frac{\eta_{25,10}(\tau)}{\eta_{25,5}(\tau)}, \quad f_3(\tau) = \frac{1}{f_2(\tau)} = \frac{\eta_{25,5}(\tau)}{\eta_{25,10}(\tau)}.$$

STEP 1. We check that each function is a modular function on $\Gamma_1(25)$.

```
> f1:=mul(GETA(25,j), j=1..12):
> f2:=GETA(25,10)/GETA(25,5):
> f3:=1/f2:
> GP1:=GETAP2getalist(f1):
> GP2:=GETAP2getalist(f2):
> GP3:=GETAP2getalist(f3):
> Gamma1ModFunc(GP1,25),Gamma1ModFunc(GP2,25),Gamma1ModFunc(GP3,25);
```

$$1, 1, 1$$

STEP 2. We find a set of inequivalent cusps for $\Gamma_1(25)$ and their fan widths.

```
> CW25:=CUSPSANDWIDMAKE1(25):
> cusps25:=CW25[1];
```

$$[\texttt{oo}, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{2}{5}, \frac{2}{25}, \frac{3}{5}, \frac{3}{10}, \frac{3}{25}, \frac{4}{5}, \frac{4}{25}, \frac{6}{25}, \frac{7}{10}, \frac{7}{25}, \frac{8}{25}, \frac{9}{10}, \frac{9}{25}, \frac{11}{25}, \frac{12}{25}]$$

```
> widths25:=CW25[2];
```

$$[1, 25, 25, 25, 25, 5, 25, 25, 25, 25, 5, 25, 25, 5, 1, 5, 5, 1, 5, 1, 1, 5, 1, 1, 5, 1, 1, 1]$$

STEP 3. We compute $\mathrm{ORD}\left(f_j, \zeta, \Gamma_1(25)\right)$ for each j and each cusp ζ of $\Gamma_1(25)$.

```
> ORDS1:=getaprodcuspORDS(GP1,cusps25,widths25);
```

$$[-1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, -1, 0, 0, -1, 0, -1, -1, 0, -1, -1, 0, -1, -1, -1]$$

```
> ORDS2:=getaprodcuspORDS(GP2,cusps25,widths25);
```

$$[-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 1]$$

```
> ORDS3:=getaprodcuspORDS(GP3,cusps25,widths25);
```

$$[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, -1]$$

STEP 4. We calculate the constant B in (2.14).

```
> mintotORDS([ORDS1,ORDS2,ORDS3],3);
```

$$-9$$

STEP 5. To prove the identity (2.18) we need to verify that

$$\mathrm{ORD}\left(g(\tau), i\infty, \Gamma_1(25)\right) > 9.$$

```
> JACL:=map(getalist2jacprod,[GP1,GP2,GP3]):
> JACID:=JACL[1]-JACL[2]+JACL[3]+1:
> QJ:=jac2series(JACID,100):
> series(QJ,q,100);
```

$$\mathrm{O}\left(q^{99}\right)$$

This completes the proof of the identity (2.18).

STEPS 1–5 may be automated using

`provemodfuncid(JACID,N)` — returns the constant B in equation (2.14) and prints details of the verification and proof of the identity corresponding to `JACID`, which is a linear combination of symbolic `JAC`-functions, and N is the level. If `xprint=true` then more details of the verification are printed. When this function is called there is a query asking whether to verify the identity. Enter `yes` to carry out the verification.

EXAMPLE:

```
> provemodfuncid(JACID,25);
"TERM ", 1, "of ", 4, " *****************"
"TERM ", 2, "of ", 4, " *****************"
"TERM ", 3, "of ", 4, " *****************"
"TERM ", 4, "of ", 4, " *****************"
"mintotord = ", -9
"TO PROVE the identity we need to show that v[oo](ID) > ", 9
*** There were NO errors.
*** o Each term was modular function on
Gamma1(25).
*** o We also checked that the total order of
each term was zero.
*** o We also checked that the power of q was correct in
each term.
"*** WARNING: some terms were constants.  ***"
"See array CONTERMS."
To prove the identity we will need to verify if up to
q^(9).
Do you want to prove the identity?  (yes/no)
You entered yes.
We verify the identity to O(q^(59)).
RESULT: The identity holds to O(q^(59)).
CONCLUSION: This proves the identity since we had only
to show that v[oo](ID) > 9.
```

9

`provemodfuncidBATCH(JACID,N)` — is a version of `provemodfuncid` that prints less detail and does not query.

EXAMPLE:

```
> provemodfuncidBATCH(JACID,25);
```

```
*** There were NO errors.  Each term was modular function on
Gamma1(25).  Also -mintotord=9.  To prove the identity
we need to check up to O(q^(11)).
To be on the safe side we check up to O(q^(59)).
*** The identity is PROVED!
```

`printJACIDORDStable()` — prints an ORDs table for the f_j and lower bound for g after `provemodfuncid` is run. Formatted output from our example is given below. By summing the last column we see that $B = -9$, which confirms an earlier calculation using `mintotORDS`.

ζ	ORD (f_1,ζ)	ORD (f_2,ζ)	ORD (f_3,ζ)	Lower bound for ORD (g,ζ)
$\frac{1}{2}$	1	0	0	0
$\frac{1}{3}$	1	0	0	0
$\frac{1}{4}$	1	0	0	0
$\frac{1}{5}$	0	0	0	0
$\frac{1}{6}$	1	0	0	0
$\frac{1}{7}$	1	0	0	0
$\frac{1}{8}$	1	0	0	0
$\frac{1}{9}$	1	0	0	0
$\frac{1}{10}$	0	0	0	0
$\frac{1}{11}$	1	0	0	0
$\frac{1}{12}$	1	0	0	0
$\frac{2}{5}$	0	0	0	0
$\frac{2}{25}$	-1	1	-1	-1
$\frac{3}{5}$	0	0	0	0
$\frac{3}{10}$	0	0	0	0
$\frac{3}{25}$	-1	1	-1	-1
$\frac{4}{5}$	0	0	0	0
$\frac{4}{25}$	-1	-1	1	-1
$\frac{6}{25}$	-1	-1	1	-1
$\frac{7}{10}$	0	0	0	0
$\frac{7}{25}$	-1	1	-1	-1
$\frac{8}{25}$	-1	1	-1	-1
$\frac{9}{10}$	0	0	0	0
$\frac{9}{25}$	-1	-1	1	-1
$\frac{11}{25}$	-1	-1	1	-1
$\frac{12}{25}$	-1	1	-1	-1

3. Generalized Ramanujan-Robins identities

As an application of our thetaids package we show how to find and prove generalized eta-product identities due to Ramanujan and Robins, and some natural extensions. Robins [19] proved the following striking analogue of Ramanujan's identity (1.3) (or (1.5)):

$$G(3)\,H(1) - G(1)\,H(3) = 1, \tag{3.1}$$

where

$$G(n) = \frac{1}{\eta_{13,1}(n\tau)\eta_{13,3}(n\tau)\eta_{13,4}(n\tau)}, \quad H(n) = \frac{1}{\eta_{13,2}(n\tau)\eta_{13,5}(n\tau)\eta_{13,6}(n\tau)}.$$

This together with other identities due to Robins leads one to consider the functions

$$G(n,N,\chi) = G(n) := \prod_{\substack{\chi(g)=1\\ 0<g<\frac{N}{2}}} \frac{1}{\eta_{D,g}(n\tau)}, \quad H(n,N,\chi) = H(n) := \prod_{\substack{\chi(g)=-1\\ 0<g<\frac{N}{2}}} \frac{1}{\eta_{N,g}(n\tau)}, \tag{3.2}$$

where χ is non-principal real Dirichlet character mod N satisfying $\chi(-1) = 1$. We will also consider

$$G^*(n,N,\chi) = G^*(n) := \prod_{\substack{\chi(g)=1\\ 0<g<\frac{N}{2}}} \frac{1}{\eta^*_{D,g}(n\tau)}, \quad H^*(n,N,\chi) = H^*(n) := \prod_{\substack{\chi(g)=-1\\ 0<g<\frac{N}{2}}} \frac{1}{\eta^*_{N,g}(n\tau)}, \tag{3.3}$$

where

$$\eta^*_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2(g/\delta)} \prod_{m\equiv \pm g \pmod{\delta}} (1-(-q)^m). \tag{3.4}$$

We note that

$$\eta^*_{\delta,g}(\tau) = \omega_{\delta_g}\eta_{\delta,g}(\tau+\pi i),$$

where $\omega_{\delta,g}$ is a root of unity. Using the notation (3.2) (with $N = 5$ and $\chi(\cdot) = \left(\frac{\cdot}{5}\right)$, the Legendre symbol) we may rewrite Ramanujan's identities (1.2), (1.3) as

$$\begin{aligned} G(1)^{11}H(1) - G(1)H(1)^{11} &= 1 + 11G(1)^6H(1)^6,\\ H(1)G(11) - G(1)H(11) &= 1, \end{aligned}$$

respectively.

We have written a number of specialised functions for the purpose of finding and proving identities for these more general G- and H-functions. We have collected these functions into the new ramarobinsids package. Go to

http://qseries.org/fgarvan/qmaple/ramarobinsids

and follow the directions on that page. This package requires both the qseries and thetaids packages.

3.1. **Some maple functions.** `Geta(g,d,n)` — returns the generalized eta-function $\eta_{d,g}(n\tau)$ in symbolic `JAC`-form.

`GetaB(g,d,n)` — returns `Geta(g,d,n)` without the the $q^{\frac{d}{2}P_2(g/d)}$ factor.

$\eta_{d,g}(n\tau)$ in symbolic `JAC`-form.

`GetaL(L,d,n)` — returns the generalized eta-product corresponding to the geta-list in `JAC`-form with τ replaced by $n\tau$.

`GetaBL(L,d,n)` — returns the generalized eta-product `GetaL(g,d,n)` without the q-factor.

`GetaEXP(g,d,n)` — returns lowest power of q in $\eta_{d,g}(n\tau)$.

`GetaLEXP(L,d,n)` — returns lowest power of q for the generalized eta-product corresponding to `GetaL(L,d,n)`.

`MGeta(g,d,n)` — η^* analogue of `Geta(g,d,n)`

`MGetaL(L,d,n)` — η^* analogue of `GetaL(L,d,n)`

`Eeta(n)` — returns Dedekind eta-function $\eta(n\tau)$ in `JAC`-form.

EXAMPLE:

```
> with(ramarobinsids:
> Geta(1,5,2);
```

$$\frac{q^{1/30}JAC(2,10,\infty)}{JAC(0,10,\infty)}$$

```
> GetaB(1,5,2);
```

$$\frac{JAC(2,10,\infty)}{JAC(0,10,\infty)}$$

```
> GetaEXP(1,5,2);
```

$$\frac{1}{30}$$

```
> GetaL([1,3,4],13,1);
```

$$\frac{q^{1/4}}{JAC(0,13,\infty)^3}JAC(1,13,\infty)JAC(3,13,\infty)JAC(4,13,\infty)$$

```
> GetaLB([1,3,4],13,1);
```

$$\frac{JAC(1,13,\infty)JAC(3,13,\infty)JAC(4,13,\infty)}{JAC(0,13,\infty)^3}$$

```
> GetaLEXP([1,3,4],13,1);
```

$$\frac{1}{4}$$

```
> MGeta(1,5,2);
```

$$\frac{q^{1/30}JAC\,(2,10,\infty)\,JAC\,(4,40,\infty)\,(JAC\,(0,20,\infty))^2}{JAC\,(0,10,\infty)\,JAC\,(0,40,\infty)\,(JAC\,(2,20,\infty))^2}$$

```
> MGetaL([1,3,4],13,1);
```

$$\frac{\sqrt[4]{q}JAC(1,13,\infty)\,JAC(2,52,\infty)\,JAC(0,26,\infty)\,JAC(3,13,\infty)\,JAC(6,52,\infty)\,(JAC(4,26,\infty))^2\,JAC(8,}{JAC(0,13,\infty)\,JAC(0,52,\infty)\,(JAC(1,26,\infty))^2\,(JAC(3,26,\infty))^2\,JAC(4,13,\infty)\,JAC(8,52,\infty)}$$

```
> Eeta(3);
```

$$q^{1/8}JAC(0,3,\infty)$$

CHECKRAMIDF(SYMF,ACC,T) — checks whether a certain symbolic expression of G- and H-functions is an eta-product. This assumes that $G(n)$, $H(n)$, $GM(n)$, $HM(n)$ have already been defined. GM and HM are the η^* analogues of G, H. The SYMF symbolic form is written in terms of $\_G$, $\_H$, $\_GM$, $\_HM$. ACC is an upperbound on the absolute value of exponents allowed in the formal product, T is highest power of q considered. This procedure returns a list of exponents in the formal product if it is a likely eta-product otherwise it returns NULL. A number of global variables are also assigned. The main ones are

- _JFUNC: JAC-expression of SYMF.
- LQD: lowest power of q.
- RID: the conjectured eta-product.
- ebase: base of the conjectured eta-product.
- SYMID: symbolic form of the identity

EXAMPLE:

```
> with(qseries):
> with(thetaids):
> with(ramarobinsids):
> G:=j->1/GetaL([1,3,4],13,j):  H:=j->1/GetaL([2,5,6],13,j):
> GM:=j->1/MGetaL([1,3,4],13,j):  HM:=j->1/MGetaL([2,5,6],13,j):
> GE:=j->-GetaLEXP([1,3,4],13,j):  HE:=j->-GetaLEXP([2,5,6],13,j):
> GHID:=(_G(1)*_G(2)+_H(1)*_H(2))/(_G(2)*_H(1)-_G(1)*_H(2));
```

$$GHID := \frac{\_G(1)\,\_G(2) + \_H(1)\,\_H(2)}{\_G(2)\,\_H(1) - \_G(1)\,\_H(2)}$$

```
> CHECKRAMIDF(GHID,10,50);
```

$$[-2,0,-2,0,-2,0,-2,0,-2,0,-2,0,0,0,-2,0,-2,0,$$
$$-2,0,-2,0,-2,0,-2,0,-2,0,-2,0,-2,0,-2,0,-2,0,-2,0,0]$$

```
> ebase;
```

$$26$$

```
> _JFUNC;
```

$$(-q^3 JAC(1,13,\infty)\, JAC(3,13,\infty)\, JAC(4,13,\infty)\, JAC(2,26,\infty)\, JAC(6,26,\infty)\, JAC(8,26,\infty)$$
$$-\, JAC(2,13,\infty)\, JAC(5,13,\infty)\, JAC(6,13,\infty)\, JAC(4,26,\infty)\, JAC(10,26,\infty)\, JAC(12,26,\infty))$$
$$/q\,(q JAC(2,26,\infty)\, JAC(6,26,\infty)\, JAC(8,26,\infty)\, JAC(2,13,\infty)\, JAC(5,13,\infty)\, JAC(6,13,\infty)$$
$$-JAC(1,13,\infty)\, JAC(3,13,\infty)\, JAC(4,13,\infty)\, JAC(4,26,\infty)\, JAC(10,26,\infty)\, JAC(12,26,\infty))$$

```
> LDQ;
```

$$-1$$

```
> RID;
```

$$\frac{(\eta(13\,\tau))^2\,(\eta(2\,\tau))^2}{(\eta(26\,\tau))^2\,(\eta(\tau))^2}$$

```
> SYMID;
```

$$\frac{\_G(1)\,\_G(2) + \_H(1)\,\_H(2)}{\_G(2)\,\_H(1) - \_G(1)\,\_H(2)} = \frac{(\eta(13\,\tau))^2\,(\eta(2\,\tau))^2}{(\eta(26\,\tau))^2\,(\eta(\tau))^2}$$

```
> etamake(jac2series(_JFUNC,1001),q,1001);
```

$$\frac{\eta(13\,\tau)^2\,\eta(2\,\tau)^2}{\eta(26\,\tau)^2\,\eta(\tau)^2}$$

It seems that

$$\frac{G(1)\,G(2) + H(1)\,H(2)}{G(2)\,H(1) - G(1)\,H(2)} = \frac{\eta(13\,\tau)^2\,\eta(2\,\tau)^2}{\eta(26\,\tau)^2\,\eta(\tau)^2} \tag{3.5}$$

when $N = 13$ and $\chi(\cdot) = \left(\frac{\cdot}{13}\right)$, at least up to q^{1000}.

EXAMPLE:

```
> RRID1:=_JFUNC-Eeta(13)^2*Eeta(2)^2/Eeta(26)^2/Eeta(1)^2:
> JRID1:=processjacid(RRID1):
> jmxperiod;
                              26
> provemodfuncidBATCH(JRID1,26);
*** There were NO errors.  Each term was modular function on
Gamma1(26).  Also -mintotord=18.  To prove the identity
we need to check up to O(q^(20)).
To be on the safe side we check up to O(q^(70)).
*** The identity is PROVED!
```

Thus identity (3.5) is proved.

The search for and proof of such identities my be automated.

3.2. **Ten types of identities for Ramanujan's functions $G(q)$ and $H(q)$.** We consider ten types of identities. We write a MAPLE function to search for and prove identities of each type. Here we assume $N = 5$ and $\chi(\cdot) = \left(\frac{\cdot}{5}\right)$. We continue to use the notation (3.2).

In this section

$$G(1) = G(1, 5, \left(\frac{\cdot}{5}\right)) = \frac{1}{\eta_{5,1}(\tau)} = \frac{q^{-1/60}}{(q, q^4; q^5)_\infty}, \quad H(1) = H(1, 5, \left(\frac{\cdot}{5}\right)) = \frac{1}{\eta_{5,2}(\tau)} = \frac{q^{11/60}}{(q^2, q^3; q^5)_\infty}.$$

EXAMPLE:
```
> with(qseries):
> with(thetaids):
> with(ramarobinsids):
> G:=j->1/GetaL([1],5,j):  H:=j->1/GetaL([2],5,j):
> GM:=j->1/MGetaL([1],5,j):  HM:=j->1/MGetaL([2],5,j):
> GE:=j->-GetaLEXP([1],5,j):  HE:=j->-GetaLEXP([2],5,j):
```

3.2.1. *Type 1.* We consider identities of the form

$$G(a)\, H(b) \pm G(b)\, H(a) = f(\tau),$$

where $f(\tau)$ is an eta-product and a, b are positive relatively prime integers.

`findtype1(T)` — cycles through symbolic expressions

$$\_G(a)\, \_H(b) + c\, \_G(b)\, \_H(a)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $b < a$, $c \in \{-1, 1\}$, and

$$\mathtt{GE}(a) + \mathtt{HE}(b) - (\mathtt{GE}(b) + \mathtt{HE}(a)) = \frac{1}{5}\,(b - a) \in \mathbb{Z}, \tag{3.6}$$

using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. Condition (3.6) eliminates the case of fractional powers of q, which our case means $a \equiv b \mod 5$. The procedure also returns a list of `[a,b,c]` which give identities.

EXAMPLE:
```
> proveit:=true:
> findtype1(11);
*** There were NO errors.  Each term was modular function on
Gamma1(30).  Also -mintotord=8.  To prove the identity
we need to check up to O(q^(10)).
To be on the safe side we check up to O(q^(68)).
```

```
*** The identity below is PROVED!
[6, 1, -1]
```

$$\_G(6)\_H(1) - \_G(1)\_H(6) = \frac{\eta(6\tau)\eta(\tau)}{\eta(3\tau)\eta(2\tau)}$$

```
*** There were NO errors.  Each term was modular function on
Gamma1(55).  Also -mintotord=40.  To prove the identity
we need to check up to O(q^(42)).
To be on the safe side we check up to O(q^(150)).
*** The identity below is PROVED!
[11, 1, -1]
```

$$\_G(11)\_H(1) - \_G(1)\_H(11) = 1$$

$$[[6, 1, -1], [11, 1, -1]]$$

```
> myramtype1 :=findtype1(36);
```

$$myramtype1 := [[6, 1, -1], [11, 1, -1], [7, 2, -1], [16, 1, -1], [8, 3, -1], [9, 4, -1], [36, 1, -1]]$$

This also produced the following identities with proofs (some output omitted):

$$G(6)\,H(1) - G(1)\,H(6) = \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}, \qquad \Gamma_1(30), \qquad -B = 8, \tag{3.7}$$

$$G(11)\,H(1) - G(1)\,H(11) = 1, \qquad \Gamma_1(55), \qquad -B = 40, \tag{3.8}$$

$$G(7)\,H(2) - G(2)\,H(7) = \frac{\eta(\tau)\eta(14\tau)}{\eta(2\tau)\eta(7\tau)}, \qquad \Gamma_1(70), \qquad -B = 48, \tag{3.9}$$

$$G(16)\,H(1) - G(1)\,H(16) = \frac{\eta(4\tau)^2}{\eta(2\tau)\eta(8\tau)}, \qquad \Gamma_1(80), \qquad -B = 64, \tag{3.10}$$

$$G(8)\,H(3) - G(3)\,H(8) = \frac{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(24\tau)}{\eta(2\tau)\eta(3\tau)\eta(8\tau)\eta(12\tau)}, \qquad \Gamma_1(120), \qquad -B = 128, \tag{3.11}$$

$$G(9)\,H(4) - G(4)\,H(9) = \frac{\eta(\tau)\eta(6\tau)^2\eta(36\tau)}{\eta(2\tau)\eta(3\tau)\eta(12\tau)\eta(18\tau)}, \qquad \Gamma_1(180), \qquad -B = 288, \tag{3.12}$$

$$G(36)\,H(1) - G(1)\,H(36) = \frac{\eta(4\tau)\eta(6\tau)^2\eta(9\tau)}{\eta(2\tau)\eta(3\tau)\eta(12\tau)\eta(18\tau)}, \qquad \Gamma_1(180), \qquad -B = 288. \tag{3.13}$$

We have included the relevant groups $\Gamma_1(N)$ and values of B (see (2.14) and (2.16)). These identities are known and are equations (3.9), (3.5), (3.10), (3.6), (3.12), (3.14), and (3.15) in [2] respectively.

3.2.2. *Type 2.* We consider identities of the form

$$G(a)\,G(b) \pm H(a)\,H(b) = f(\tau),$$

where $f(\tau)$ is an etap-roduct and a, b are positive relatively prime integers.

`findtype2(T)` — cycles through symbolic expressions

$$\_G(a)\,\_G(b) + c\,\_H(a)\,\_H(b)$$

where $2 \le n \le T$, $ab = n$, $(a,b) = 1$, $a \le b$, $c \in \{-1, 1\}$, and

$$\texttt{GE}(a) + \texttt{GE}(b) - (\texttt{HE}(a) + \texttt{HE}(b)) = -\frac{1}{5}\,(a+b) \in \mathbb{Z}, \tag{3.14}$$

using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. Condition (3.6) eliminates the case of fractional powers of q, which our case means $a \equiv -b \mod 5$. The procedure also returns a list of `[a,b,c]` which give identities.

```
> findtype2(24);
```

$$[[1,4,-1],[1,4,1],[2,3,1],[1,9,1],[1,14,1],[1,24,1]]$$

This also produces the following identities with proofs:

$$G(1)\,G(4) - H(1)\,H(4) = \frac{\eta(10\tau)^5}{\eta(2\tau)\eta(5\tau)^2\eta(20\tau)^2}, \qquad \Gamma_1(20), \qquad -B = 4, \tag{3.15}$$

$$G(1)\,G(4) + H(1)\,H(4) = \frac{\eta(2\tau)^4}{\eta(\tau)^2\eta(4\tau)^2}, \qquad \Gamma_1(20), \qquad -B = 4, \tag{3.16}$$

$$G(2)\,G(3) + H(2)\,H(3) = \frac{\eta(2\tau)\eta(3\tau)}{\eta(\tau)\eta(6\tau)}, \qquad \Gamma_1(30), \qquad -B = 8, \tag{3.17}$$

$$G(1)\,G(9) + H(1)\,H(9) = \frac{\eta(3\tau)^2}{\eta(\tau)\eta(9\tau)}, \qquad \Gamma_1(45), \qquad -B = 24, \tag{3.18}$$

$$G(1)\,G(14) + H(1)\,H(14) = \frac{\eta(2\tau)\eta(7\tau)}{\eta(\tau)\eta(14\tau)}, \qquad \Gamma_1(70), \qquad -B = 48, \tag{3.19}$$

$$G(1)\,G(24) + H(1)\,H(24) = \frac{\eta(2\tau)\eta(3\tau)\eta(8\tau)\eta(12\tau)}{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(24\tau)}, \qquad \Gamma_1(120), \qquad -B = 128, \tag{3.20}$$

(3.21)

These identities are known and are equations (3.4), (3.3), (3.8), (3.7), (3.11), and (3.13) in [2] respectively.

3.2.3. *Type 3.* We consider identities of the form

$$\frac{G(a_1)\,G(b_1) \pm H(a_1)\,H(b_1)}{G(a_2)\,H(b_2) \pm H(a_2)\,G(b_2)} = f(\tau),$$

which are not a quotient of Type 1 and 2 identities, and where $f(\tau)$ is an eta-product, a_1, b_1 ,a_2, b_2 are positive relatively prime integers, $a_1b_1 = a_2b_2$.

`findtype3(T)` — cycles through symbolic expressions

$$\frac{\_G(a_1)\,\_G(b_1) + c_1\,\_H(a_1)\,\_H(b_1)}{\_G(a_2)\,\_H(b_2) + c_2\,\_H(a_2)\,\_G(b_2)}$$

where $2 \le n \le T$, $a_1b_1 = a_2b_2 = n$, $(a_1, b_1, c_1, d_1) = 1$, $a_1 \le b_1$, $b_2 < a_2$, $c_1, c_2 \in \{-1, 1\}$, and

$$\texttt{GE}(a_1) + \texttt{GE}(b_1) - (\texttt{HE}(a_2) + \texttt{HE}(b_2)), \quad \texttt{GE}(a_2) + \texttt{HE}(b_2) - (\texttt{HE}(a_2) + \texttt{GE}(b_2)), \quad \in \mathbb{Z}, \tag{3.22}$$

and `[`a_2, b_2, c_2`]` is not an element for the list `myramtype1` (product earlier by `findtype1`), using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[`$a_1, b_1, c_1, a_2, b_2, c_2$`]` which give identities.

```
> findtype3(126);
```

$$[[3, 7, 1, 21, 1, -1], [2, 13, 1, 26, 1, -1], [1, 34, 1, 17, 2, -1], [1, 39, 1, 13, 3, -1], [1, 54, 1, 27, 2, -1],$$
$$[7, 8, 1, 56, 1, -1], [3, 22, 1, 11, 6, -1], [2, 33, 1, 66, 1, -1], [4, 21, 1, 12, 7, -1], [1, 84, 1, 28, 3, -1],$$
$$[3, 32, 1, 96, 1, -1], [7, 18, 1, 14, 9, -1], [2, 63, 1, 126, 1, -1]]$$

This also produces the following identities with proofs:

(3.23)
$$\frac{G(3)\,G(7) + H(3)\,H(7)}{G(21)\,H(1) - H(21)\,G(1)} = 1, \qquad \Gamma_1(105), \qquad -B = 192,$$

(3.24)
$$\frac{G(2)\,G(13) + H(2)\,H(13)}{G(26)\,H(1) - H(26)\,G(1)} = 1, \qquad \Gamma_1(130), \qquad -B = 240,$$

(3.25)
$$\frac{G(1)\,G(34) + H(1)\,H(34)}{G(17)\,H(2) - H(17)\,G(2)} = \frac{\eta(2\tau)\eta(17\tau)}{\eta(\tau)\eta(34\tau)}, \qquad \Gamma_1(170), \qquad -B = 448,$$

(3.26)
$$\frac{G(1)\,G(39)+H(1)\,H(39)}{G(13)\,H(3)-H(13)\,G(3)}=\frac{\eta(3\tau)\eta(13\tau)}{\eta(\tau)\eta(39\tau)},\qquad \Gamma_1(195),\qquad -B=768,$$
(3.27)
$$\frac{G(1)\,G(54)+H(1)\,H(54)}{G(27)\,H(2)-H(27)\,G(2)}=\frac{\eta(2\tau)\eta(3\tau)\eta(18\tau)\eta(27\tau)}{\eta(\tau)\eta(6\tau)\eta(9\tau)\eta(54\tau)},\qquad \Gamma_1(270),\qquad -B=1008,$$
(3.28)
$$\star\quad \frac{G(7)\,G(8)+H(7)\,H(8)}{G(56)\,H(1)-H(56)\,G(1)}=\frac{\eta(2\tau)\eta(28\tau)}{\eta(4\tau)\eta(14\tau)},\qquad \Gamma_1(280),\qquad -B=1152,$$
(3.29)
$$\frac{G(3)\,G(22)+H(3)\,H(22)}{G(11)\,H(6)-H(11)\,G(6)}=\frac{\eta(2\tau)\eta(33\tau)}{\eta(\tau)\eta(66\tau)},\qquad \Gamma_1(330),\qquad -B=1600,$$
(3.30)
$$\frac{G(2)\,G(33)+H(2)\,H(33)}{G(66)\,H(1)-H(66)\,G(1)}=\frac{\eta(3\tau)\eta(22\tau)}{\eta(6\tau)\eta(11\tau)},\qquad \Gamma_1(330),\qquad -B=1600,$$
(3.31)
$$\star\quad \frac{G(4)\,G(21)+H(4)\,H(21)}{G(12)\,H(7)-H(12)\,G(7)}=\frac{\eta(2\tau)\eta(3\tau)\eta(7\tau)\eta(12\tau)\eta(28\tau)\eta(42\tau)}{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(14\tau)\eta(21\tau)\eta(84\tau)},\quad \Gamma_1(420),\qquad -B=2688,$$
(3.32)
$$\star\quad \frac{G(1)\,G(84)+H(1)\,H(84)}{G(28)\,H(3)-H(28)\,G(3)}=\frac{\eta(2\tau)\eta(3\tau)\eta(7\tau)\eta(12\tau)\eta(28\tau)\eta(42\tau)}{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(14\tau)\eta(21\tau)\eta(84\tau)},\quad \Gamma_1(420),\qquad -B=2688,$$
(3.33)
$$\star\quad \frac{G(3)\,G(32)+H(3)\,H(32)}{G(96)\,H(1)-H(96)\,G(1)}=\frac{\eta(2\tau)\eta(8\tau)\eta(12\tau)\eta(48\tau)}{\eta(4\tau)\eta(6\tau)\eta(16\tau)\eta(24\tau)},\qquad \Gamma_1(480),\qquad -B=3072,$$
(3.34)
$$\star\quad \frac{G(7)\,G(18)+H(7)\,H(18)}{G(14)\,H(9)-H(14)\,G(9)}=\frac{\eta(2\tau)\eta(3\tau)\eta(42\tau)\eta(63\tau)}{\eta(\tau)\eta(6\tau)\eta(21\tau)\eta(126\tau)},\qquad \Gamma_1(630),\qquad -B=5760,$$
(3.35)
$$\frac{G(2)\,G(63)+H(2)\,H(63)}{G(126)\,H(1)-H(126)\,G(1)}=\frac{\eta(3\tau)\eta(7\tau)\eta(18\tau)\eta(42\tau)}{\eta(6\tau)\eta(9\tau)\eta(14\tau)\eta(21\tau)},\qquad \Gamma_1(630),\qquad -B=5760.$$

The equations marked $\star$ appear to be new. The other equations correspond to (3.16), (3.18), (3.35), (3.22), (3.41), (3.40) and (3.39) in [2], and (1.24) in [19] respectively. We have corrected the statement of equation [19, (1.24)].

3.2.4. *Type 4.* We consider identities of the form

$$G^*(a)\,H^*(b)\pm G^*(b)\,H^*(a)=f(\tau),$$

where $f(\tau)$ is an eta-product, a, b are positive relatively prime integers, and at least one of a, b is even.

`findtype4(T)` — cycles through symbolic expressions

$$\_GM(a)\,\_HM(b) + c\,\_GM(b)\,\_HM(a)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $b < a$, $c \in \{-1, 1\}$,

$$\texttt{GE}(a) + \texttt{HE}(b) - (\texttt{GE}(b) + \texttt{HE}(a)) \in \mathbb{Z}, \tag{3.36}$$

and least one of a, b is even, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities.

```
> findtype4(24);
```

$$[[6, 1, -1]]$$

This also produces the following identity with proof:

$$G^*(6)\,H^*(1) - G^*(1)\,H^*(6) = \frac{\eta(\tau)\eta(4\tau)^3\eta(6\tau)^3\eta(24\tau)}{\eta(2\tau)^3\eta(3\tau)\eta(8\tau)\eta(12\tau)^3}, \quad \Gamma_1(120), \quad -B = 128. \tag{3.37}$$

This corresponds to equation (3.28) in [2].

3.2.5. *Type 5.* We consider identities of the form

$$G^*(a)\,G^*(b) \pm H^*(a)\,H^*(b) = f(\tau),$$

where $f(\tau)$ is an eta-product, a, b are positive relatively prime integers, and at least one of a, b is even.

`findtype5(T)` — cycles through symbolic expressions

$$\_GM(a)\,\_GM(b) + c\,\_HM(a)\,\_HM(b)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $a \le b$, $c \in \{-1, 1\}$,

$$\texttt{GE}(a) + \texttt{GE}(b) - (\texttt{HE}(a) + \texttt{HE}(b)) \in \mathbb{Z}, \tag{3.38}$$

and least one of a, b is even, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities.

```
> findtype5(24);
```

$$[[1, 4, 1], [2, 3, 1]]$$

This also produces the following identity with proof:

$$G^*(1)\,G^*(4) + H^*(1)\,H^*(4) = \frac{\eta(4\tau)^2}{\eta(2\tau)\eta(8\tau)}, \quad \Gamma_1(80), \quad -B = 64, \tag{3.39}$$

$$G^*(2)\,G^*(3) + H^*(2)\,H^*(3) = \frac{\eta(2\tau)^3\eta(3\tau)\eta(8\tau)\eta(12\tau)^3}{\eta(\tau)\eta(4\tau)^3\eta(6\tau)^3\eta(24\tau)}, \quad \Gamma_1(120), \quad -B = 128. \tag{3.40}$$

These correspond to equations (3.26) and (3.27) in [2].

3.2.6. *Type 6.* We consider identities of the form

$$G(a)\,H^*(b) \pm G^*(a)\,H(b) = f(\tau),$$

where $f(\tau)$ is an eta-product, a, b are positive relatively prime integers.
`findtype6(T)` — cycles through symbolic expressions

$$\_G(a)\,\_HM(b) + c\,\_GM(a)\,\_H(b)$$

where $2 \le n \le T$, $ab = n$, $(a,b) = 1$, $a \ge b$, $c \in \{-1, 1\}$,

$$\mathtt{GE}(a) + \mathtt{HE}(b) - (\mathtt{HE}(a) + \mathtt{HE}(b)) \in \mathbb{Z}, \tag{3.41}$$

and least one of a, b is even, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities.

EXAMPLE:
```
> findtype6(24);
```

$$[[1, 1, -1], [1, 1, 1]]$$

This also produces the following identities with proof:

$$G(1)\,H^*(1) - G^*(1)\,H(1) = 2\,\frac{\eta(20\tau)^2}{\eta(2\tau)\eta(10\tau)}, \quad \Gamma_1(20), \quad -B = 4, \tag{3.42}$$

$$G(1)\,H^*(1) + G^*(1)\,H(1) = 2\,\frac{\eta(4\tau)^2}{\eta(2\tau)^2}, \quad \Gamma_1(20), \quad -B = 4. \tag{3.43}$$

These are equivalent to equations (3.25) and (3.24) in [2].

3.2.7. *Type 7.* We consider identities of the form

$$G^*(a)\,G(b) \pm H^*(a)\,H(b) = f(\tau),$$

where $f(\tau)$ is an eta-product, a, b are positive relatively prime integers.

`findtype7(T)` — cycles through symbolic expressions

$$\_GM(a)\,\_G(b) + c\,\_HM(a)\,\_H(b)$$

where $2 \le n \le T$, $ab = n$, $(a, b) = 1$, $a \le b$, $c \in \{-1, 1\}$, and both a, b are odd, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,c]` which give identities.

EXAMPLE:
```
> findtype7(24);
```
$$[[1, 9, -1]]$$

This also produces the following identity with proof:

$$G^*(1)\,G(9) - H^*(1)\,H(9) = \frac{\eta(\tau)\eta(12\tau)\eta(18\tau)^2}{\eta(2\tau)\eta(6\tau)\eta(9\tau)\eta(36\tau)}, \quad \Gamma_1(180), \quad -B = 288. \tag{3.44}$$

This corresponds to (3.29) in [2].

3.2.8. *Type 8.* We consider identities of the form

$$G(1)^a\,H(a) \pm H(1)^a\,G(a) = f(\tau),$$

where $f(\tau)$ is an eta-product, and $a > 1$ is an integer.

`findtype8(T)` — cycles through symbolic expressions

$$\_G(1)^a\,\_H(a) + c\,\_H(1)^a\,\_G(a)$$

where $2 \le a \le T$, and $c \in \{-1, 1\}$, using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,c]` which give identities.

EXAMPLE:
```
> findtype8(24);
```
$$[[3, -1]]$$

This also produces the following identity with proof:

$$G(1)^3\,H(3) - H(1)^3\,G(3) = 3\,\frac{\eta(15\tau)^3}{\eta(\tau)\eta(3\tau)\eta(5\tau)}, \quad \Gamma_1(15), \quad -B = 4. \tag{3.45}$$

This is equivalent to equation (1.27) in Robin's thesis [19].

3.2.9. *Type 9.* We consider identities of the form

$$G(1)^a\,H(1)^b - H(1)^a\,G(1)^b + x = f(\tau),$$

where $f(\tau)$ is an eta-product, and a, b are positive integers, and $x = 0$ or $x = -1$.

`findtype9()` — determines whether

$$\_G(1)^a\,\_H(1)^b - \_H(1)^a\,\_G(1)^b + x$$

is a likely eta-product for $x = 0$ or $x = -1$ with a, b smallest such positive integers, using `CHECKRAMIDF`, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of `[a,b,x]` which give identities.

EXAMPLE:

```
> findtype9();
```

$$[[11, 1, 1]]$$

This also produces the following identity with proof:

$$G(1)^{11}\,H(1)^1 - H(1)^{11}\,G(1)^1 - 1 = 11\,\frac{\eta(5\tau)^6}{\eta(\tau)^6}, \quad \Gamma_1(5), \quad -B = 2. \tag{3.46}$$

This is equation (3.1) in [2].

3.2.10. *Type 10.* We consider identities of the form

$$\frac{G(a_1)\,H(b_1) + c_1\,H(a_1)\,G(b_1)}{G(a_2)\,H^*(b_2) + c_2\,H(a_2)\,G^*(b_2)} = f(\tau),$$

in which the numerator is not a Type 1 identity, and where $f(\tau)$ is an eta-product, a_1, b_1 ,a_2, b_2 are positive relatively prime integers, $a_1 b_1 = a_2 b_2$.

`findtype10(T)` — cycles through symbolic expressions

$$\frac{\_G(a_1)\,\_H(b_1) + c_1\,\_H(a_1)\,\_G(b_1)}{\_G(a_2)\,\_HM(b_2) + c_2\,\_H(a_2)\,\_GM(b_2)}$$

where $2 \le n \le T$, $a_1 b_1 = a_2 b_2 = n$, $(a_1, b_1, a_2, b_2) = 1$, $a_1 > b_1$, $b_2 < a_2$, $c_1, c_2 \in \{-1, 1\}$, and

$$\mathtt{GE}(a_1) + \mathtt{HE}(b_1) - (\mathtt{HE}(a_1) + \mathtt{GE}(b_1)), \quad \mathtt{GE}(a_2) + \mathtt{HE}(b_2) - (\mathtt{HE}(a_2) + \mathtt{GE}(b_2)), \quad \in \mathbb{Z}, \tag{3.47}$$

and $[a_1, b_1, c_1]$ is not an element for the list `myramtype1` (product earlier by `findtype1`), using `CHECKRAMIDF` to check whether the expression corresponds to a likely eta-product, and if so uses `provemodfuncidBATCH` to prove it. The procedure also returns a list of $[a_1, b_1, c_1, a_2, b_2, c_2]$ which give identities.

```
> qthreshold:=3000:
> findtype10(120);
```

$$[[19, 4, -1, 76, 1, 1], [28, 3, -1, 12, 7, 1], [12, 7, -1, 28, 3, 1]]$$

This also produces the following identities with proof:

(3.48)
$$\frac{G(19)\,H(4) - H(19)\,G(4)}{G(76)\,H^*(1) + H(76)\,G^*(1)} = \frac{\eta(2\tau)\eta(76\tau)}{\eta(4\tau)\eta(38\tau)}, \qquad \Gamma_1(380), \quad -B = 2160,$$

(3.49)
$$\star \quad \frac{G(28)\,H(3) - H(28)\,G(3)}{G(12)\,H^*(7) + H(12)\,G^*(7)} = \frac{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(14\tau)^2\eta(21\tau)}{\eta(2\tau)^2\eta(3\tau)\eta(7\tau)\eta(28\tau)\eta(42\tau)}, \quad \Gamma_1(420), \quad -B = 2400,$$

(3.50)
$$\star \quad \frac{G(12)\,H(7) - H(12)\,G(7)}{G(28)\,H^*(3) + H(28)\,G^*(3)} = \frac{\eta(\tau)\eta(6\tau)^2\eta(14\tau)\eta(21\tau)\eta(84\tau)}{\eta(2\tau)\eta(3\tau)\eta(7\tau)\eta(12\tau)\eta(42\tau)^2}, \quad \Gamma_1(420), \quad -B = 2400.$$

Equation (3.48) is (3.38) in [2]. The other type 10 identities appear to be new.

4. More Generalized Ramanujan-Robins identities

We consider generalized Ramanujan-Robins identities associated with non-principal real Dirichlet characters χ mod N for $N \leq 60$, that satisfy $\chi(-1) = 1$. We found David Ireland's *Dirichlet Character Table Generator* [7] useful. See the website

`http://www.di-mgt.com.au/dirichlet-character-generator.html`

4.1. **Mod 8.** There is only one non-principal character mod 8 that satisfies $\chi(-1) = 1$, namely $\chi(\cdot) = \left(\frac{8}{\cdot}\right)$. Here $\left(\frac{8}{\cdot}\right)$ is the Kronecker symbol. In this section

$$G(1) = G(1, 8, \left(\frac{8}{\cdot}\right)) = \frac{1}{\eta_{8,1}(\tau)} = \frac{q^{-11/48}}{(q, q^7; q^8)_\infty}, \quad H(1) = H(1, 8, \left(\frac{8}{\cdot}\right)) = \frac{1}{\eta_{8,3}(\tau)} = \frac{q^{13/48}}{(q^3, q^5; q^8)_\infty}.$$

These functions were considered by Robins [19, pp.16-17]. They are also related to the Göllnitz-Gordon functions [9], [10]:

$$S(q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n}\, q^{n^2} = \frac{1}{(q, q^4, q^7; q^8)_\infty},$$
$$T(q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n}\, q^{n^2+2n} = \frac{1}{(q^3, q^4, q^5; q^8)_\infty}.$$

The ratio of these two functions is the famous Ramanujan-Göllnitz-Gordon continued fraction [3, Eq.(9.3)]

$$\frac{S(q)}{T(q)} = \prod_{n=0}^{\infty} \frac{(1-q^{8n+3})(1-q^{8n+5})}{(1-q^{8n+1})(1-q^{8n+7})}$$
$$= 1 + \cfrac{q+q^2}{1+\cfrac{q^4}{1+\cfrac{q^3+q^6}{1+\cfrac{q^8}{1+\ddots}}}}.$$

Some of the identities given in this section are due Robins [19], and many are due to Huang [11]. Any identities that appear to be new are marked $\star$.

4.1.1. *Type 1 Identities.*

(4.1)
$$G(3)\,H(1) - G(1)\,H(3) = \frac{\eta(\tau)\eta(12\tau)^2}{\eta(3\tau)\eta(8\tau)\eta(24\tau)}, \qquad \Gamma_1(24), \qquad -B = 6,$$

(4.2)
$$G(3)\,H(1) + G(1)\,H(3) = \frac{\eta(2\tau)\eta(4\tau)^2\eta(6\tau)^2}{\eta(\tau)\eta(3\tau)\eta(8\tau)^2\eta(12\tau)}, \qquad \Gamma_1(24), \qquad -B = 6,$$

(4.3)
$$G(5)\,H(1) - G(1)\,H(5) = \frac{\eta(2\tau)\eta(10\tau)\eta(20\tau)}{\eta(5\tau)\eta(8\tau)\eta(40\tau)}, \qquad \Gamma_1(40), \qquad -B = 20,$$

(4.4)
$$G(7)\,H(1) - G(1)\,H(7) = \frac{\eta(4\tau)\eta(28\tau)}{\eta(8\tau)\eta(56\tau)}, \qquad \Gamma_1(56), \qquad -B = 36,$$

(4.5)
$$G(9)\,H(1) - G(1)\,H(9) = \frac{\eta(4\tau)\eta(6\tau)^2\eta(36\tau)}{\eta(3\tau)\eta(8\tau)\eta(12\tau)\eta(72\tau)}, \qquad \Gamma_1(72), \qquad -B = 60,$$

(4.6)
$$G(5)\,H(3) - G(3)\,H(5) = \frac{\eta(\tau)\eta(4\tau)\eta(6\tau)\eta(10\tau)\eta(15\tau)\eta(60\tau)}{\eta(2\tau)\eta(3\tau)\eta(5\tau)\eta(24\tau)\eta(30\tau)\eta(40\tau)}, \quad \Gamma_1(120), \quad -B = 144.$$

4.1.2. *Type 2 Identities.*

(4.7)
$$G(1)\,G(1) - H(1)\,H(1) = \frac{\eta(4\tau)^6}{\eta(\tau)\eta(2\tau)\eta(8\tau)^4}, \qquad \Gamma_1(8), \qquad -B = 1,$$

(4.8)
$$G(1)\,G(1)+H(1)\,H(1)=\frac{\eta(2\tau)^6}{\eta(\tau)^3\eta(4\tau)\eta(8\tau)^2},\qquad \Gamma_1(8),\qquad -B=1,$$
(4.9)
$$G(1)\,G(3)-H(1)\,H(3)=\frac{\eta(2\tau)^2\eta(6\tau)\eta(12\tau)^2}{\eta(\tau)\eta(3\tau)\eta(4\tau)\eta(24\tau)^2},\qquad \Gamma_1(24),\qquad -B=6,$$
(4.10)
$$G(1)\,G(3)+H(1)\,H(3)=\frac{\eta(3\tau)\eta(4\tau)^2}{\eta(\tau)\eta(8\tau)\eta(24\tau)},\qquad \Gamma_1(24),\qquad -B=6,$$
(4.11)
$$G(1)\,G(5)+H(1)\,H(5)=\frac{\eta(2\tau)\eta(4\tau)\eta(10\tau)}{\eta(\tau)\eta(8\tau)\eta(40\tau)},\qquad \Gamma_1(40),\qquad -B=20,$$
(4.12)
$$G(1)\,G(9)+H(1)\,H(9)=\frac{\eta(2\tau)\eta(3\tau)\eta(12\tau)\eta(18\tau)}{\eta(\tau)\eta(8\tau)\eta(9\tau)\eta(72\tau)},\qquad \Gamma_1(72),\qquad -B=60,$$
(4.13)
$$G(1)\,G(15)+H(1)\,H(15)=\frac{\eta(2\tau)\eta(3\tau)\eta(5\tau)\eta(12\tau)\eta(20\tau)\eta(30\tau)}{\eta(\tau)\eta(6\tau)\eta(8\tau)\eta(10\tau)\eta(15\tau)\eta(120\tau)},\qquad \Gamma_1(120),\qquad -B=144.$$

4.1.3. *Type 3 Identities.*

(4.14)
$$\star\quad \frac{G(3)\,G(5)-H(3)\,H(5)}{G(15)\,H(1)+H(15)\,G(1)}=\frac{\eta(4\tau)\eta(60\tau)}{\eta(12\tau)\eta(20\tau)},\qquad \Gamma_1(120),\qquad -B=256,$$
(4.15)
$$\frac{G(3)\,G(5)+H(3)\,H(5)}{G(15)\,H(1)-H(15)\,G(1)}=\frac{\eta(8\tau)\eta(12\tau)\eta(20\tau)\eta(120\tau)}{\eta(4\tau)\eta(24\tau)\eta(40\tau)\eta(60\tau)},\qquad \Gamma_1(120),\qquad -B=224,$$
(4.16)
$$\star\quad \frac{G(1)\,G(15)-H(1)\,H(15)}{G(5)\,H(3)+H(5)\,G(3)}=\frac{\eta(4\tau)^2\eta(6\tau)\eta(10\tau)\eta(24\tau)^2\eta(40\tau)^2\eta(60\tau)^2}{\eta(2\tau)\eta(8\tau)^2\eta(12\tau)^2\eta(20\tau)^2\eta(30\tau)\eta(120\tau)^2},\qquad \Gamma_1(120),\qquad -B=192,$$
(4.17)
$$\star\quad \frac{G(3)\,G(7)+H(3)\,H(7)}{G(21)\,H(1)-H(21)\,G(1)}=\frac{\eta(8\tau)\eta(21\tau)\eta(28\tau)\eta(168\tau)}{\eta(7\tau)\eta(24\tau)\eta(56\tau)\eta(84\tau)},\qquad \Gamma_1(168),\qquad -B=528,$$
(4.18)
$$\star\quad \frac{G(1)\,G(21)+H(1)\,H(21)}{G(7)\,H(3)-H(7)\,G(3)}=\frac{\eta(3\tau)\eta(4\tau)\eta(24\tau)\eta(56\tau)}{\eta(\tau)\eta(8\tau)\eta(12\tau)\eta(168\tau)},\qquad \Gamma_1(168),\qquad -B=528,$$

(4.19)
$$\frac{G(1)\,G(39)+H(1)\,H(39)}{G(13)\,H(3)-H(13)\,G(3)}=\frac{\eta(2\tau)\eta(3\tau)\eta(13\tau)\eta(24\tau)\eta(78\tau)\eta(104\tau)}{\eta(\tau)\eta(6\tau)\eta(8\tau)\eta(26\tau)\eta(39\tau)\eta(312\tau)},\qquad \Gamma_1(312),\quad -B=1632,$$
(4.20)
$$\frac{G(1)\,G(55)+H(1)\,H(55)}{G(11)\,H(5)-H(11)\,G(5)}=\frac{\eta(2\tau)\eta(5\tau)\eta(11\tau)\eta(40\tau)\eta(88\tau)\eta(110\tau)}{\eta(\tau)\eta(8\tau)\eta(10\tau)\eta(22\tau)\eta(55\tau)\eta(440\tau)},\qquad \Gamma_1(440),\quad -B=3680.$$

4.1.4. *Type 8 Identity.*

$$(4.21)\quad \star\quad G(1)^3\,H(3)-H(1)^3\,G(3)=3\,\frac{\eta(2\tau)^3\eta(4\tau)\eta(6\tau)\eta(24\tau)^2}{\eta(\tau)^2\eta(3\tau)\eta(8\tau)^4},\quad \Gamma_1(24),\quad -B=10.$$

4.2. **Mod 10.** There is only one real non-principal character mod 10 that satisfies $\chi(-1)=1$, namely the character χ_{10} induced by the Legendre symbol mod 5. In this section

$$G(1)=G(1,10,\chi_{10})=\frac{1}{\eta_{10,1}(\tau)}=\frac{q^{-23/60}}{(q,q^9;q^{10})_\infty},\quad H(1)=H(1,10,\chi_{10})=\frac{1}{\eta_{10,3}(\tau)}=\frac{q^{13/60}}{(q^3,q^7;q^{10})_\infty}.$$

All the identities in this section appear to be new.

4.2.1. *Type 1.*

$$(4.22)\qquad G(6)\,H(1)-G(1)\,H(6)=\frac{\eta(4\tau)\eta(5\tau)\eta(12\tau)\eta(30\tau)^3}{\eta(6\tau)\eta(10\tau)^2\eta(15\tau)\eta(60\tau)^2},\quad \Gamma_1(60),\quad -B=40.$$

4.2.2. *Type 2.*

(4.23)
$$G(2)\,G(3)-H(2)\,H(3)=\frac{\eta(4\tau)\eta(10\tau)^3\eta(12\tau)\eta(15\tau)}{\eta(2\tau)\eta(5\tau)\eta(20\tau)^2\eta(30\tau)^2},\qquad \Gamma_1(60),\quad -B=40,$$
(4.24)
$$G(1)\,G(9)-H(1)\,H(9)=\frac{\eta(2\tau)\eta(3\tau)\eta(5\tau)\eta(18\tau)\eta(30\tau)^2\eta(45\tau)}{\eta(\tau)\eta(9\tau)\eta(10\tau)^2\eta(15\tau)\eta(90\tau)^2},\quad \Gamma_1(90),\quad -B=96.$$

4.2.3. *Type 5.*

$$(4.25)\quad G^*(1)\,G^*(4)-H^*(1)\,H^*(4)=\frac{\eta(\tau)\eta(4\tau)^3\eta(10\tau)\eta(16\tau)\eta(40\tau)}{\eta(2\tau)^2\eta(5\tau)\eta(8\tau)^2\eta(20\tau)\eta(80\tau)},\quad \Gamma_1(80),\quad -B=64.$$

4.2.4. *Type 6.*

$$(4.26)\qquad G(1)\,H^*(1)-G^*(1)\,H(1)=2\,\frac{\eta(20\tau)^2}{\eta(10\tau)^2},\qquad \Gamma_1(20),\quad -B=4,$$

$$(4.27)\qquad G(1)\,H^*(1)+G^*(1)\,H(1)=2\,\frac{\eta(4\tau)^2}{\eta(2\tau)\eta(10\tau)},\quad \Gamma_1(20),\quad -B=4.$$

4.2.5. *Type 8.*

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(2\tau)\eta(5\tau)\eta(20\tau)^2}{\eta(\tau)\eta(10\tau)^3}, \quad \Gamma_1(20), \quad -B = 4, \tag{4.28}$$

$$G(1)^2\,H(2) + H(1)^2\,G(2) = 2\,\frac{\eta(4\tau)^2\eta(5\tau)}{\eta(\tau)\eta(10\tau)^2}, \quad \Gamma_1(20), \quad -B = 4, \tag{4.29}$$

(4.30)

$$G(1)^3\,H(3) - H(1)^3\,G(3) = 3\,\frac{\eta(2\tau)^3\eta(5\tau)^2\eta(6\tau)\eta(15\tau)\eta(30\tau)}{\eta(\tau)^2\eta(3\tau)\eta(10\tau)^5}, \quad \Gamma_1(30), \quad -B = 16.$$

4.3. **Mod** 12**.** There is only one non-principal character mod 12 that satisfies $\chi(-1) = 1$, namely $\chi(\cdot) = \left(\frac{12}{\cdot}\right)$. In this section

$$G(1) = G(1, 12, \left(\frac{12}{\cdot}\right)) = \frac{1}{\eta_{12,1}(\tau)} = \frac{q^{-13/24}}{(q, q^{11}; q^{12})_\infty}, \quad H(1) = H(1, 12, \left(\frac{12}{\cdot}\right)) = \frac{1}{\eta_{12,5}(\tau)} = \frac{q^{11/24}}{(q^5, q^7; q^{12})_\infty}.$$

These functions were considered by Robins [19, p17], who found (4.33), (4.34), (4.39), (4.40). The remaining identities appear to be new and are marked $\star$.

4.3.1. *Type 1.*

(4.31)

$$\star \quad G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(\tau)\eta(4\tau)\eta(6\tau)}{\eta(2\tau)\eta(12\tau)^2}, \qquad \Gamma_1(24), \quad -B = 4,$$

(4.32)

$$\star \quad G(2)\,H(1) + G(1)\,H(2) = \frac{\eta(3\tau)^2\eta(4\tau)}{\eta(\tau)\eta(12\tau)^2}, \qquad \Gamma_1(24), \quad -B = 4,$$

$$G(3)\,H(1) - G(1)\,H(3) = \frac{\eta(2\tau)\eta(18\tau)}{\eta(12\tau)\eta(36\tau)}, \qquad \Gamma_1(36), \quad -B = 12, \tag{4.33}$$

$$G(3)\,H(1) + G(1)\,H(3) = \frac{\eta(4\tau)\eta(6\tau)^5\eta(9\tau)^2}{\eta(2\tau)\eta(3\tau)^2\eta(12\tau)^3\eta(18\tau)^2}, \qquad \Gamma_1(36), \quad -B = 12, \tag{4.34}$$

(4.35)

$$\star \quad G(4)\,H(1) - G(1)\,H(4) = \frac{\eta(3\tau)\eta(16\tau)}{\eta(12\tau)\eta(48\tau)}, \qquad \Gamma_1(48), \quad -B = 24,$$

(4.36)

$$\star \quad G(5)\,H(1) - G(1)\,H(5) = \frac{\eta(4\tau)\eta(6\tau)\eta(10\tau)\eta(15\tau)}{\eta(5\tau)\eta(12\tau)^2\eta(60\tau)}, \qquad \Gamma_1(60), \quad -B = 40,$$

(4.37)

$$\star \quad G(3)\,H(2) - G(2)\,H(3) = \frac{\eta(\tau)\eta(6\tau)\eta(8\tau)\eta(9\tau)\eta(12\tau)\eta(72\tau)}{\eta(2\tau)\eta(3\tau)\eta(24\tau)^2\eta(36\tau)^2}, \quad \Gamma_1(72), \quad -B = 48,$$

(4.38)

$$\star \quad G(6)\,H(1) - G(1)\,H(6) = \frac{\eta(8\tau)\eta(9\tau)}{\eta(12\tau)\eta(72\tau)}, \qquad \Gamma_1(72), \quad -B = 60.$$

4.3.2. *Type 2.*

$$G(1)^2 - H(1)^2 = \frac{\eta(2\tau)^3\eta(6\tau)^3}{\eta(\tau)^2\eta(12\tau)^4}, \qquad \Gamma_1(12), \quad -B = 2, \tag{4.39}$$

$$G(1)^2 + H(1)^2 = \frac{\eta(2\tau)\eta(3\tau)^4\eta(4\tau)}{\eta(\tau)^2\eta(6\tau)\eta(12\tau)^3}, \qquad \Gamma_1(12), \quad -B = 2, \tag{4.40}$$

$$\star \quad G(1)\,G(2) - H(1)\,H(2) = \frac{\eta(3\tau)^2\eta(8\tau)^2}{\eta(\tau)\eta(12\tau)\eta(24\tau)^2}, \qquad \Gamma_1(24), \quad -B = 8, \tag{4.41}$$

$$\star \quad G(1)\,G(3) - H(1)\,H(3) = \frac{\eta(2\tau)\eta(4\tau)\eta(9\tau)\eta(18\tau)}{\eta(\tau)\eta(12\tau)\eta(36\tau)^2}, \qquad \Gamma_1(36), \quad -B = 18, \tag{4.42}$$

$$\star \quad G(1)\,G(5) - H(1)\,H(5) = \frac{\eta(2\tau)\eta(3\tau)\eta(20\tau)\eta(30\tau)}{\eta(\tau)\eta(12\tau)\eta(60\tau)^2}, \qquad \Gamma_1(60), \quad -B = 40. \tag{4.43}$$

4.3.3. *Type 3.*

(4.44)

$$\star \quad \frac{G(1)\,G(10) - H(1)\,H(10)}{G(5)\,G(2) - H(5)\,H(2)} = \frac{\eta(2\tau)\eta(5\tau)\eta(24\tau)^2\eta(60\tau)^2}{\eta(\tau)\eta(10\tau)\eta(12\tau)^2\eta(120\tau)^2}, \qquad \Gamma_1(120), \quad -B =$$

(4.45)

$$\star \quad \frac{G(5)\,G(7) - H(5)\,H(7)}{G(35)\,G(1) - H(35)\,H(1)} = \frac{\eta(12\tau)\eta(420\tau)}{\eta(60\tau)\eta(84\tau)}, \qquad \Gamma_1(420), \quad -B =$$

(4.46)

$$\star \quad \frac{G(1)\,G(35) - H(1)\,H(35)}{G(7)\,G(5) - H(7)\,H(5)} = \frac{\eta(3\tau)\eta(4\tau)\eta(5\tau)\eta(7\tau)\eta(60\tau)^2\eta(84\tau)^2\eta(105\tau)\eta(140\tau)}{\eta(\tau)\eta(12\tau)^2\eta(15\tau)\eta(20\tau)\eta(21\tau)\eta(28\tau)\eta(35\tau)\eta(420\tau)^2}, \qquad \Gamma_1(420), \quad -B =$$

4.3.4. *Type 4.*

$$\star \quad G^*(2)\,H^*(1) - G^*(1)\,H^*(2) = \frac{\eta(\tau)\eta(6\tau)\eta(8\tau)^3\eta(48\tau)}{\eta(2\tau)\eta(4\tau)\eta(16\tau)\eta(24\tau)^3}, \qquad \Gamma_1(48), \quad -B = 24. \tag{4.47}$$

4.3.5. *Type 5.*

$$\star \quad G^*(1)\,G^*(2) - H^*(1)\,H^*(2) = \frac{\eta(\tau)\eta(6\tau)\eta(16\tau)}{\eta(2\tau)\eta(12\tau)\eta(48\tau)}, \qquad \Gamma_1(48), \quad -B = 24. \tag{4.48}$$

4.3.6. *Type 6.*

$$\star \quad G(1)\,H^*(1) - G^*(1)\,H(1) = 2\,\frac{\eta(4\tau)^2\eta(6\tau)^2\eta(24\tau)^3}{\eta(2\tau)\eta(8\tau)\eta(12\tau)^5}, \qquad \Gamma_1(24), \quad -B = 4, \tag{4.49}$$

$$\star \quad G(1)\,H^*(1) + G^*(1)\,H(1) = 2\,\frac{\eta(4\tau)\eta(6\tau)^2\eta(8\tau)\eta(24\tau)}{\eta(2\tau)\eta(12\tau)^4}, \quad \Gamma_1(24), \quad -B = 4. \tag{4.50}$$

4.3.7. *Type 7.*

$$\star \quad G^*(1)\,G(1) - H^*(1)\,H(1) = \frac{\eta(8\tau)\eta(12\tau)^4}{\eta(4\tau)\eta(6\tau)\eta(24\tau)^3}, \quad \Gamma_1(24), \quad -B = 4, \tag{4.51}$$

$$\star \quad G^*(1)\,G(1) + H^*(1)\,H(1) = \frac{\eta(4\tau)^4\eta(6\tau)}{\eta(2\tau)^2\eta(8\tau)\eta(12\tau)\eta(24\tau)}, \quad \Gamma_1(24), \quad -B = 4. \tag{4.52}$$

4.3.8. *Type 8.*

$$\star \quad G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)^2\eta(6\tau)\eta(24\tau)^3}{\eta(\tau)\eta(8\tau)\eta(12\tau)^5}, \quad \Gamma_1(24), \quad -B = 4, \tag{4.53}$$

$$\star \quad G(1)^2\,H(2) + H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)\eta(6\tau)\eta(8\tau)\eta(24\tau)}{\eta(\tau)\eta(12\tau)^4}, \quad \Gamma_1(24), \quad -B = 4, \tag{4.54}$$

$$\star \quad G(1)^3\,H(3) - H(1)^3\,G(3) = 3\,\frac{\eta(2\tau)^2\eta(3\tau)\eta(4\tau)\eta(6\tau)\eta(9\tau)\eta(36\tau)^2}{\eta(\tau)^2\eta(12\tau)^5\eta(18\tau)}, \quad \Gamma_1(36), \quad -B = 18. \tag{4.55}$$

4.4. **Mod 13.** There is only one non-principal character mod 13 that satisfies $\chi(-1) = 1$, namely $\chi(\cdot) = \left(\frac{\cdot}{13}\right)$. In this section

$$G(1) = G(1, 13, \left(\frac{\cdot}{13}\right)) = \frac{1}{\eta_{13,1}(\tau)\,\eta_{13,3}(\tau)\,\eta_{13,4}(\tau)} = \frac{q^{-1/4}}{(q, q^3, q^4, q^9, q^{10}, q^{12}; q^{13})_\infty},$$
$$H(1) = H(1, 13, \left(\frac{\cdot}{13}\right)) = \frac{1}{\eta_{13,2}(\tau)\,\eta_{13,5}(\tau)\,\eta_{13,6}(\tau)} = \frac{q^{3/4}}{(q^2, q^5, q^6, q^7, q^8, q^{11}; q^{13})_\infty}.$$

These functions were considered by Robins [19, p.18], who found the one identity (4.56). The remaining four identities appear to be new and are marked $\star$.

4.4.1. *Type 1.*

$$G(3)\,H(1) - G(1)\,H(3) = 1, \quad \Gamma_1(39), \quad -B = 24. \tag{4.56}$$

4.4.2. *Type 3.*

$$\star \quad \frac{G(1)\,G(2)+H(1)\,H(2)}{G(2)\,H(1)-H(2)\,G(1)} = \frac{\eta(2\tau)^2\eta(13\tau)^2}{\eta(\tau)^2\eta(26\tau)^2}, \qquad \Gamma_1(26), \qquad -B=18, \tag{4.57}$$

$$\star \quad \frac{G(2)\,G(5)+H(2)\,H(5)}{G(10)\,H(1)-H(10)\,G(1)} = 1, \qquad \Gamma_1(130), \quad -B=432, \tag{4.58}$$

$$\star \quad \frac{G(1)\,G(14)+H(1)\,H(14)}{G(7)\,H(2)-H(7)\,G(2)} = \frac{\eta(2\tau)\eta(7\tau)\eta(26\tau)\eta(91\tau)}{\eta(\tau)\eta(13\tau)\eta(14\tau)\eta(182\tau)}, \quad \Gamma_1(182), \quad -B=864. \tag{4.59}$$

4.4.3. *Type 9.*

$$\star \quad G(1)^3\,H(1)-H(1)^3\,G(1)-1 = 3\,\frac{\eta(13\tau)^2}{\eta(\tau)^2}, \quad \Gamma_1(13), \quad -B=6. \tag{4.60}$$

4.5. **Mod 15.** There is only one real non-principal character mod 15 that satisfies $\chi(-1)=1$, namely the one induced by the Legendre symbol mod 5:

$$\chi_{15}(n) = \begin{cases} 1, & n \equiv \pm 1, 4 \pmod{15}, \\ -1, & n \equiv \pm 2, 7 \pmod{15}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus in this section

$$G(1) = G(1,15,\chi_{15}) = \frac{1}{\eta_{15,1}(\tau)\,\eta_{15,4}(\tau)} = \frac{q^{-17/30}}{(q,q^4,q^{11},q^{14};q^{15})_\infty},$$

$$H(1) = H(1,15,\chi_{15}) = \frac{1}{\eta_{15,2}(\tau)\,\eta_{15,7}(\tau)} = \frac{q^{7/30}}{(q^2,q^7,q^8,q^{13};q^{15})_\infty}.$$

All the identities in this section appear to be new.

4.5.1. *Type 2.*

$$G(1)\,G(4)-H(1)\,H(4) = \frac{\eta(2\tau)\eta(3\tau)\eta(10\tau)\eta(12\tau)\eta(30\tau)^2}{\eta(\tau)\eta(4\tau)\eta(15\tau)^2\eta(60\tau)^2}, \quad \Gamma_1(60), \quad -B=48. \tag{4.61}$$

4.5.2. *Type 3.*

$$\frac{G(2)\,G(3)-H(2)\,H(3)}{G(6)\,H(1)-H(6)\,G(1)} = \frac{\eta(6\tau)\eta(10\tau)\eta(15\tau)^3\eta(90\tau)}{\eta(3\tau)\eta(5\tau)\eta(30\tau)^3\eta(45\tau)}, \quad \Gamma_1(90), \quad -B=120. \tag{4.62}$$

4.5.3. *Type 6.*

$$G^*(1)\,G^*(1)-H^*(1)\,H^*(1) = 2\,\frac{\eta(4\tau)\eta(6\tau)^3\eta(10\tau)\eta(60\tau)^2}{\eta(2\tau)^2\eta(12\tau)\eta(30\tau)^4}, \quad \Gamma_1(60), \quad -B=48. \tag{4.63}$$

4.5.4. *Type 8.*

$$G(1)^2\,H(2)+H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)^2\eta(6\tau)\eta(10\tau)^2}{\eta(\tau)\eta(2\tau)\eta(15\tau)^3}, \quad \Gamma_1(30), \quad -B=12. \tag{4.64}$$

4.6. **Mod 17.** There is only one non-principal character mod 17 that satisfies $\chi(-1)=1$, namely $\chi(\cdot)=\left(\frac{\cdot}{17}\right)$. In this section

$$G(1)=G(1,17,\left(\frac{\cdot}{17}\right))=\frac{1}{\eta_{17,1}(\tau)\,\eta_{17,2}(\tau)\,\eta_{17,4}(\tau)\,\eta_{17,8}(\tau)}$$
$$=\frac{q^{-2/3}}{(q,q^2,q^4,q^8,q^9,q^{13},q^{15},q^{16};q^{17})_\infty},$$

$$H(1)=H(1,17,\left(\frac{\cdot}{17}\right))=\frac{1}{\eta_{17,3}(\tau)\,\eta_{17,5}(\tau)\,\eta_{17,6}(\tau)\,\eta_{17,7}(\tau)}$$
$$=\frac{q^{4/3}}{(q^3,q^5,q^6,q^7,q^{10},q^{11},q^{12},q^{14};q^{17})_\infty}.$$

These functions were not considered by Robins [19]. Nonetheless we find one identity.

4.6.1. *Type 1.*

$$G(2)\,H(1)-G(1)\,H(2)=1,\quad \Gamma_1(34),\quad -B=16. \tag{4.65}$$

4.7. **Mod 21.** There is only one non-principal character mod 21 that satisfies $\chi(-1)=1$, namely $\chi(\cdot)=\left(\frac{21}{\cdot}\right)$. In this section

$$G(1)=G(1,21,\left(\frac{21}{\cdot}\right))=\frac{1}{\eta_{21,1}(\tau)\,\eta_{21,4}(\tau)\,\eta_{21,5}(\tau)}=\frac{q^{-5/4}}{(q,q^4,q^5,q^{16},q^{17},q^{20};q^{21})_\infty},$$

$$H(1)=H(1,21,\left(\frac{21}{\cdot}\right))=\frac{1}{\eta_{21,2}(\tau)\,\eta_{21,8}(\tau)\,\eta_{21,10}(\tau)}=\frac{q^{3/4}}{(q^2,q^8,q^{10},q^{11},q^{13},q^{19};q^{21})_\infty}.$$

4.7.1. *Type 1.*

$$G(2)\,H(1)-G(1)\,H(2)=\frac{\eta(3\tau)\eta(6\tau)\eta(7\tau)^2}{\eta(2\tau)\eta(21\tau)^3},\quad \Gamma_1(42),\quad -B=24, \tag{4.66}$$

$$G(4)\,H(1)-G(1)\,H(4)=\frac{\eta(6\tau)^2\eta(7\tau)\eta(28\tau)}{\eta(2\tau)\eta(21\tau)\eta(42\tau)\eta(84\tau)},\quad \Gamma_1(84),\quad -B=96. \tag{4.67}$$

4.7.2. *Type 2.*

$$G(1)\,G(2)-H(1)\,H(2)=\frac{\eta(3\tau)\eta(6\tau)\eta(14\tau)^2}{\eta(\tau)\eta(42\tau)^3},\quad \Gamma_1(42),\quad -B=24. \tag{4.68}$$

4.7.3. *Type 7.*

$$G^*(1)\,G(1)-H^*(1)\,H(1)=\frac{\eta(6\tau)^2\eta(14\tau)^3\eta(84\tau)}{\eta(2\tau)\eta(28\tau)\eta(42\tau)^4},\quad \Gamma_1(84),\quad -B=96. \tag{4.69}$$

4.8. **Mod 24.** There are three real non-principal characters mod 24 that satisfy $\chi(-1) = 1$.

(i) The character $\chi_{24,1}(\cdot)$ induced by $\left(\frac{8}{\cdot}\right)$.

(ii) The character $\chi_{24,2}(\cdot) = \left(\frac{12}{\cdot}\right)$ covered previously in Section 4.3.

(iii) The character $\chi_{24,3}(\cdot) = \left(\frac{24}{\cdot}\right)$.

4.8.1. $\chi_{24,1}$. We have

$$G(1) = G(1, 24, \chi_{24,1}) = \frac{1}{\eta_{24,1}(\tau)\,\eta_{24,7}(\tau)} = \frac{q^{-25/24}}{(q, q^7, q^{17}, q^{23}; q^{24})_\infty},$$

$$H(1) = H(1, 24, \chi_{24,1}) = \frac{1}{\eta_{24,5}(\tau)\,\eta_{24,11}(\tau)} = \frac{q^{23/24}}{(q^5, q^{11}, q^{13}, q^{19}; q^{24})_\infty}.$$

Type 1.

$$G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(3\tau)\eta(12\tau)^2}{\eta(6\tau)\eta(24\tau)^2}, \quad \Gamma_1(48), \quad -B = 24, \tag{4.70}$$

$$G(2)\,H(1) + G(1)\,H(2) = \frac{\eta(4\tau)^3\eta(6\tau)^4}{\eta(2\tau)^2\eta(3\tau)\eta(8\tau)\eta(12\tau)^2\eta(24\tau)}, \quad \Gamma_1(48), \quad -B = 24, \tag{4.71}$$

$$G(3)\,H(1) - G(1)\,H(3) = \frac{\eta(4\tau)\eta(6\tau)^2\eta(9\tau)\eta(36\tau)}{\eta(3\tau)\eta(8\tau)\eta(12\tau)\eta(18\tau)\eta(72\tau)}, \quad \Gamma_1(72), \quad -B = 60. \tag{4.72}$$

Type 2.

$$G(1)\,G(1) - H(1)\,H(1) = \frac{\eta(2\tau)^2\eta(3\tau)^2\eta(4\tau)\eta(12\tau)^2}{\eta(\tau)^2\eta(6\tau)\eta(8\tau)\eta(24\tau)^3}, \quad \Gamma_1(24), \quad -B = 12. \tag{4.73}$$

Type 6.

$$G(1)\,H^*(1) - G^*(1)\,H(1) = 2\,\frac{\eta(4\tau)^2\eta(12\tau)^2\eta(48\tau)^2}{\eta(2\tau)\eta(8\tau)\eta(24\tau)^4}, \quad \Gamma_1(48), \quad -B = 24, \tag{4.74}$$

$$G(1)\,H^*(1) + G^*(1)\,H(1) = 2\,\frac{\eta(4\tau)\eta(6\tau)^2\eta(16\tau)\eta(48\tau)}{\eta(2\tau)\eta(8\tau)\eta(24\tau)^3}, \quad \Gamma_1(48), \quad -B = 24. \tag{4.75}$$

Type 7.

$$G^*(1)\,G(1) - H^*(1)\,H(1) = \frac{\eta(4\tau)^2\eta(24\tau)^2}{\eta(2\tau)\eta(8\tau)\eta(48\tau)^2}, \quad \Gamma_1(48), \quad -B = 24, \tag{4.76}$$

$$G^*(1)\,G(1) + H^*(1)\,H(1) = \frac{\eta(6\tau)^2\eta(8\tau)^2}{\eta(2\tau)\eta(12\tau)\eta(16\tau)\eta(48\tau)}, \quad \Gamma_1(48), \quad -B = 24. \tag{4.77}$$

Type 8.

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)^2\eta(12\tau)^2\eta(48\tau)^2}{\eta(\tau)\eta(6\tau)\eta(8\tau)\eta(24\tau)^4}, \quad \Gamma_1(48), \quad -B = 24, \tag{4.78}$$

$$G(1)^2\,H(2) + H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)\eta(6\tau)\eta(16\tau)\eta(48\tau)}{\eta(\tau)\eta(8\tau)\eta(24\tau)^3}, \quad \Gamma_1(48), \quad -B = 24. \tag{4.79}$$

Type 10.

(4.80)

$$\frac{G(3)\,H(2)+H(3)\,G(2)}{G(6)\,H^*(1)-H(6)\,G^*(1)} = \frac{\eta(4\tau)\eta(6\tau)^3\eta(9\tau)\eta(24\tau)^2\eta(36\tau)\eta(144\tau)^2}{\eta(2\tau)\eta(3\tau)\eta(12\tau)^2\eta(18\tau)^2\eta(48\tau)^2\eta(72\tau)^2}, \quad \Gamma_1(144), \quad -B=360.$$

4.8.2. $\chi_{24,3}$. We have

$$G(1) = G(1,24,\chi_{24,3}) = \frac{1}{\eta_{24,1}(\tau)\,\eta_{24,5}(\tau)} = \frac{q^{-37/24}}{(q,q^5,q^{19},q^{23};q^{24})_\infty},$$
$$H(1) = H(1,24,\chi_{24,3}) = \frac{1}{\eta_{24,7}(\tau)\,\eta_{24,11}(\tau)} = \frac{q^{35/24}}{(q^7,q^{11},q^{13},q^{17};q^{24})_\infty}.$$

Type 1.

$$G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(3\tau)\eta(4\tau)^2}{\eta(2\tau)\eta(24\tau)^2}, \quad \Gamma_1(48), \quad -B=24, \tag{4.81}$$

$$G(2)\,H(1) + G(1)\,H(2) = \frac{\eta(6\tau)^3\eta(8\tau)\eta(12\tau)}{\eta(2\tau)\eta(3\tau)\eta(24\tau)^3}, \quad \Gamma_1(48), \quad -B=24, \tag{4.82}$$

$$G(3)\,H(1) - G(1)\,H(3) = \frac{\eta(6\tau)^2\eta(8\tau)\eta(9\tau)\eta(36\tau)}{\eta(3\tau)\eta(18\tau)\eta(24\tau)^2\eta(72\tau)}, \quad \Gamma_1(72), \quad -B=72. \tag{4.83}$$

Type 2.

$$G(1)\,G(1) - H(1)\,H(1) = \frac{\eta(2\tau)^2\eta(3\tau)^2\eta(8\tau)\eta(12\tau)^3}{\eta(\tau)^2\eta(6\tau)\eta(24\tau)^5}, \quad \Gamma_1(24), \quad -B=12. \tag{4.84}$$

Type 6.

$$G(1)\,H^*(1) - G^*(1)\,H(1) = 2\,\frac{\eta(6\tau)^2\eta(8\tau)^2\eta(12\tau)\eta(48\tau)^3}{\eta(2\tau)\eta(16\tau)\eta(24\tau)^6}, \quad \Gamma_1(48), \quad -B=24, \tag{4.85}$$

$$G(1)\,H^*(1) + G^*(1)\,H(1) = 2\,\frac{\eta(4\tau)\eta(8\tau)\eta(12\tau)^3\eta(48\tau)^2}{\eta(2\tau)\eta(24\tau)^6}, \quad \Gamma_1(48), \quad -B=24. \tag{4.86}$$

Type 7.

$$G^*(1)\,G(1) - H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(6\tau)^2\eta(16\tau)\eta(24\tau)^3}{\eta(2\tau)\eta(8\tau)\eta(12\tau)^2\eta(48\tau)^3}, \quad \Gamma_1(48), \quad -B=24, \tag{4.87}$$

$$G^*(1)\,G(1) + H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(8\tau)\eta(12\tau)}{\eta(2\tau)\eta(48\tau)^2}, \quad \Gamma_1(48), \quad -B=24. \tag{4.88}$$

Type 8.

(4.89)

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(6\tau)\eta(8\tau)^2\eta(12\tau)\eta(48\tau)^3}{\eta(\tau)\eta(16\tau)\eta(24\tau)^6}, \quad \Gamma_1(48), \quad -B=24,$$

(4.90)
$$G(1)^2\,H(2)+H(1)^2\,G(2)=2\,\frac{\eta(3\tau)\eta(4\tau)\eta(8\tau)\eta(12\tau)^3\eta(48\tau)^2}{\eta(\tau)\eta(6\tau)\eta(24\tau)^6},\quad \Gamma_1(48),\quad -B=24.$$

Type 10.

$$\frac{G(2)\,H(1)-H(2)\,G(1)}{G(2)\,H^*(1)-H(2)\,G^*(1)}=\frac{\eta(4\tau)^2\eta(24\tau)}{\eta(8\tau)\eta(12\tau)^2},\quad \Gamma_1(48),\quad -B=40, \tag{4.91}$$

$$\frac{G(2)\,H(1)-H(2)\,G(1)}{G(2)\,H^*(1)+H(2)\,G^*(1)}=\frac{\eta(3\tau)^2\eta(12\tau)}{\eta(6\tau)^3},\quad \Gamma_1(48),\quad -B=40, \tag{4.92}$$

$$\frac{G(2)\,H(1)+H(2)\,G(1)}{G(2)\,H^*(1)-H(2)\,G^*(1)}=\frac{\eta(6\tau)^3}{\eta(3\tau)^2\eta(12\tau)},\quad \Gamma_1(48),\quad -B=40, \tag{4.93}$$

$$\frac{G(2)\,H(1)+H(2)\,G(1)}{G(2)\,H^*(1)+H(2)\,G^*(1)}=\frac{\eta(8\tau)\eta(12\tau)^2}{\eta(4\tau)^2\eta(24\tau)},\quad \Gamma_1(48),\quad -B=40. \tag{4.94}$$

4.9. **Mod** 26**.** There is only one non-principal character mod 26 that satisfies $\chi(-1)=1$, namely the character χ_{26} induced by $\left(\frac{\cdot}{13}\right)$. In this section

$$G(1)=G(1,26,\chi_{26})=\frac{1}{\eta_{26,1}(\tau)\,\eta_{26,3}(\tau)\,\eta_{26,9}(\tau)}=\frac{q^{-7/4}}{(q,q^3,q^9,q^{17},q^{23},q^{25};q^{26})_\infty},$$
$$H(1)=H(1,26,\chi_{26})=\frac{1}{\eta_{26,5}(\tau)\,\eta_{26,7}(\tau)\,\eta_{26,11}(\tau)}=\frac{q^{5/4}}{(q^5,q^7,q^{11},q^{15},q^{19},q^{21};q^{26})_\infty}.$$

We find only one identity.

4.9.1. *Type 10.*

$$\frac{G(3)\,H(2)-H(3)\,G(2)}{G(6)\,H^*(1)+H(6)\,G^*(1)}=\frac{\eta(26\tau)^3\eta(156\tau)^3}{\eta(52\tau)^3\eta(78\tau)^3},\quad \Gamma_1(156),\quad -B=576. \tag{4.95}$$

4.10. **Mod** 28**.** There is only one non-principal character mod 28 that satisfies $\chi(-1)=1$, namely the character $\chi(\cdot)=\left(\frac{28}{\cdot}\right)$. In this section

$$G(1)=G(1,28,\left(\frac{28}{\cdot}\right))=\frac{1}{\eta_{28,1}(\tau)\,\eta_{28,3}(\tau)\,\eta_{28,9}(\tau)}=\frac{q^{-17/8}}{(q,q^3,q^9,q^{19},q^{25},q^{27};q^{28})_\infty},$$
$$H(1)=H(1,28,\left(\frac{28}{\cdot}\right))=\frac{1}{\eta_{28,5}(\tau)\,\eta_{28,11}(\tau)\,\eta_{28,13}(\tau)}=\frac{q^{15/8}}{(q^5,q^{11},q^{13},q^{15},q^{17},q^{23};q^{28})_\infty}.$$

4.10.1. *Type 1.*

$$G(2)\,H(1)-G(1)\,H(2)=\frac{\eta(4\tau)^2\eta(7\tau)\eta(14\tau)}{\eta(2\tau)\eta(28\tau)^3},\quad \Gamma_1(56),\quad -B=48. \tag{4.96}$$

4.10.2. *Type 6.*

$$G^*(1)\,G^*(1) - H^*(1)\,H^*(1) = 2\,\frac{\eta(4\tau)^4\eta(14\tau)^3\eta(56\tau)^3}{\eta(2\tau)^2\eta(8\tau)\eta(28\tau)^7}, \quad \Gamma_1(56), \quad -B = 48. \tag{4.97}$$

4.10.3. *Type 7.*

$$G^*(1)\,G(1) - H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(8\tau)\eta(28\tau)^2}{\eta(2\tau)\eta(56\tau)^3}, \quad \Gamma_1(56), \quad -B = 48. \tag{4.98}$$

4.10.4. *Type 8.*

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(4\tau)^4\eta(7\tau)\eta(14\tau)^2\eta(56\tau)^3}{\eta(\tau)\eta(2\tau)\eta(8\tau)\eta(28\tau)^7}, \quad \Gamma_1(56), \quad -B = 48. \tag{4.99}$$

4.10.5. *Type 10.*

$$\frac{G(2)\,H(1) - H(2)\,G(1)}{G(2)\,H^*(1) - H(2)\,G^*(1)} = \frac{\eta(7\tau)^2\eta(28\tau)}{\eta(14\tau)^3}, \quad \Gamma_1(56), \quad -B = 72. \tag{4.100}$$

4.11. **Mod** 30**.** There is only one real non-principal character mod 30 that satisfies $\chi(-1) = 1$, namely the character χ_{30} induced by the Legendre symbol mod 5. Thus in this section

$$G(1) = G(1, 30, \chi_{30}) = \frac{1}{\eta_{30,1}(\tau)\,\eta_{30,11}(\tau)} = \frac{q^{-31/30}}{(q, q^{11}, q^{19}, q^{29}; q^{30})_\infty},$$
$$H(1) = H(1, 30, \chi_{30}) = \frac{1}{\eta_{30,7}(\tau)\,\eta_{30,13}(\tau)} = \frac{q^{41/30}}{(q^7, q^{13}, q^{17}, q^{23}; q^{30})_\infty}.$$

4.11.1. *Type 6.*

$$G^*(1)\,G^*(1) - H^*(1)\,H^*(1) = 2\,\frac{\eta(4\tau)\eta(6\tau)^2\eta(60\tau)^2}{\eta(2\tau)\eta(12\tau)\eta(30\tau)^3}, \quad \Gamma_1(60), \quad -B = 48. \tag{4.101}$$

4.11.2. *Type 8.*

(4.102)

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)\eta(5\tau)\eta(6\tau)\eta(60\tau)^2}{\eta(\tau)\eta(10\tau)\eta(12\tau)\eta(15\tau)\eta(30\tau)^2}, \quad \Gamma_1(60), \quad -B = 48.$$

4.12. **Mod** 34**.** There is only one real non-principal character mod 34 that satisfies $\chi(-1) = 1$, namely the character χ_{34} induced by the Legendre symbol mod 17.

$$G(1) = G(1, 34, \chi_{34}) = \frac{1}{\eta_{34,1}(\tau)\,\eta_{34,9}(\tau)\,\eta_{34,13}(\tau)\,\eta_{34,15}(\tau)}$$
$$= \frac{q^{2/3}}{(q, q^9, q^{13}, q^{15}, q^{19}, q^{21}, q^{25}, q^{33}; q^{34})_\infty},$$

$$H(1) = H(1, 34, \chi_{34}) = \frac{1}{\eta_{34,3}(\tau)\,\eta_{34,5}(\tau)\,\eta_{34,7}(\tau)\,\eta_{34,11}(\tau)}$$

$$= \frac{q^{-4/3}}{(q^3, q^5, q^7, q^{11}, q^{23}, q^{27}, q^{29}, q^{31}; q^{34})_\infty}.$$

4.12.1. *Type 1.*

(4.103) $$G(2)\,H(1) - G(1)\,H(2) = -1\,\frac{\eta(2\tau)^2\eta(17\tau)}{\eta(\tau)\eta(34\tau)^2}, \quad \Gamma_1(68), \quad -B = 64.$$

4.12.2. *Type 7.*

(4.104) $$G^*(1)\,G(1) - H^*(1)\,H(1) = -1\,\frac{\eta(4\tau)}{\eta(68\tau)}, \quad \Gamma_1(68), \quad -B = 64.$$

4.12.3. *Type 9.*

(4.105) $$G(1)^2\,H(1)^1 - H(1)^2\,G(1)^1 = -1\,\frac{\eta(2\tau)^2\eta(17\tau)}{\eta(\tau)\eta(34\tau)^2}, \quad \Gamma_1(34), \quad -B = 16.$$

4.13. **Mod** 40**.** There are three real non-principal characters mod 40 that satisfy $\chi(-1) = 1$.

(i) The character $\chi_{40,1}(\cdot)$ induced by $\left(\frac{\cdot}{5}\right)$. This actually a character mod 10. See Section 4.2.
(ii) The character $\chi_{40,2}(\cdot)$ induced by $\left(\frac{8}{\cdot}\right)$.
(iii) The character $\chi_{40,3}(\cdot) = \left(\frac{40}{\cdot}\right)$.

4.13.1. $\chi_{40,2}$.

$$G(1) = G(1, 40, \chi_{40,2}) = \frac{1}{\eta_{40,1}(\tau)\,\eta_{40,7}(\tau)\,\eta_{40,9}(\tau)\,\eta_{40,17}(\tau)}$$
$$= \frac{q^{-19/12}}{(q, q^7, q^9, q^{17}, q^{23}, q^{31}, q^{33}, q^{39}; q^{40})_\infty},$$

$$H(1) = H(1, 40, \chi_{40,2}) = \frac{1}{\eta_{40,3}(\tau)\,\eta_{40,11}(\tau)\,\eta_{40,13}(\tau)\,\eta_{40,19}(\tau)}$$
$$= \frac{q^{17/12}}{(q^3, q^{11}, q^{13}, q^{19}, q^{21}, q^{27}, q^{29}, q^{37}; q^{40})_\infty}.$$

Type 1.

(4.106) $$G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(4\tau)^2\eta(10\tau)^2}{\eta(2\tau)\eta(8\tau)\eta(20\tau)\eta(40\tau)}, \quad \Gamma_1(80), \quad -B = 80.$$

Type 6.

(4.107)
$$G^*(1)\,G^*(1) + H^*(1)\,H^*(1) = 2\,\frac{\eta(4\tau)^2\eta(10\tau)\eta(16\tau)\eta(20\tau)\eta(80\tau)}{\eta(2\tau)\eta(8\tau)^2\eta(40\tau)^3}, \quad \Gamma_1(80), \quad -B = 80.$$

Type 7.

$$G^*(1)\,G(1) + H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(8\tau)\eta(10\tau)}{\eta(2\tau)\eta(16\tau)\eta(80\tau)}, \quad \Gamma_1(80), \quad -B = 80. \tag{4.108}$$

Type 8.

(4.109)

$$G(1)^2\,H(2) + H(1)^2\,G(2) = 2\,\frac{\eta(4\tau)^2\eta(5\tau)\eta(16\tau)\eta(20\tau)\eta(80\tau)}{\eta(\tau)\eta(8\tau)^2\eta(40\tau)^3}, \quad \Gamma_1(80), \quad -B = 80.$$

Type 10.

$$\frac{G(2)\,H(1) - H(2)\,G(1)}{G(2)\,H^*(1) + H(2)\,G^*(1)} = 1, \quad \Gamma_1(80), \quad -B = 112. \tag{4.110}$$

4.13.2. $\chi_{40,3}$.

$$\begin{aligned} G(1) = G(1,40,\chi_{40,3}) &= \frac{1}{\eta_{40,1}(\tau)\,\eta_{40,3}(\tau)\,\eta_{40,9}(\tau)\,\eta_{40,13}(\tau)} \\ &= \frac{q^{-43/12}}{(q,q^3,q^9,q^{13},q^{27},q^{31},q^{37},q^{39};q^{40})_\infty}, \end{aligned}$$

$$\begin{aligned} H(1) = H(1,40,\chi_{40,3}) &= \frac{1}{\eta_{40,7}(\tau)\,\eta_{40,11}(\tau)\,\eta_{40,17}(\tau)\,\eta_{40,19}(\tau)} \\ &= \frac{q^{41/12}}{(q^7,q^{11},q^{17},q^{19},q^{21},q^{23},q^{29},q^{33};q^{40})_\infty}. \end{aligned}$$

Type 1.

$$G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(4\tau)\eta(8\tau)\eta(10\tau)^2}{\eta(2\tau)\eta(40\tau)^3}, \quad \Gamma_1(80), \quad -B = 112. \tag{4.111}$$

Type 6.

(4.112)

$$G^*(1)\,G^*(1) + H^*(1)\,H^*(1) = 2\,\frac{\eta(8\tau)^3\eta(10\tau)\eta(20\tau)^3\eta(80\tau)^3}{\eta(2\tau)\eta(16\tau)\eta(40\tau)^8}, \quad \Gamma_1(80), \quad -B = 112.$$

Type 7.

$$G^*(1)\,G(1) + H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(10\tau)\eta(16\tau)\eta(40\tau)}{\eta(2\tau)\eta(80\tau)^3}, \quad \Gamma_1(80), \quad -B = 112. \tag{4.113}$$

Type 8.

$$G(1)^2\,H(2) + H(1)^2\,G(2) = 2\,\frac{\eta(5\tau)\eta(8\tau)^3\eta(20\tau)^3\eta(80\tau)^3}{\eta(\tau)\eta(16\tau)\eta(40\tau)^8}, \quad \Gamma_1(80), \quad -B = 112. \tag{4.114}$$

Type 10.

$$\frac{G(2)\,H(1) - H(2)\,G(1)}{G(2)\,H^*(1) + H(2)\,G^*(1)} = 1, \quad \Gamma_1(80), \quad -B = 144. \tag{4.115}$$

4.14. **Mod 42.** There is only one real non-principal character mod 42 that satisfies $\chi(-1) = 1$, namely the one induced by the mod 21 character $\chi_{42}(\cdot) = \left(\frac{\cdot}{3}\right)\left(\frac{\cdot}{7}\right)$.

In this section

$$G(1) = G(1,42,\chi_{42}) = \frac{1}{\eta_{42,1}(\tau)\,\eta_{42,5}(\tau)\,\eta_{42,17}(\tau)} = \frac{q^{-11/4}}{(q,q^5,q^{17},q^{25},q^{37},q^{41};q^{42})_\infty},$$

$$H(1) = H(1,42,\chi_{42}) = \frac{1}{\eta_{42,11}(\tau)\,\eta_{42,13}(\tau)\,\eta_{42,19}(\tau)} = \frac{q^{13/4}}{(q^{11},q^{13},q^{19},q^{23},q^{29},q^{31};q^{42})_\infty}.$$

4.14.1. *Type 1.*

$$G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(4\tau)\eta(6\tau)^2\eta(7\tau)}{\eta(2\tau)\eta(12\tau)\eta(21\tau)\eta(42\tau)}, \quad \Gamma_1(84), \quad -B = 96. \tag{4.116}$$

4.14.2. *Type 7.*

$$G^*(1)\,G(1) - H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(6\tau)\eta(28\tau)}{\eta(2\tau)\eta(84\tau)^2}, \quad \Gamma_1(84), \quad -B = 96. \tag{4.117}$$

4.14.3. *Type 10.*

$$\frac{G(2)\,H(1) - H(2)\,G(1)}{G(2)\,H^*(1) - H(2)\,G^*(1)} = \frac{\eta(7\tau)^2\eta(28\tau)\eta(42\tau)^3}{\eta(14\tau)^3\eta(21\tau)^2\eta(84\tau)}, \quad \Gamma_1(84), \quad -B = 132. \tag{4.118}$$

4.15. **Mod 56.** There are three real non-principal characters mod 56 that satisfy $\chi(-1) = 1$.

(i) The character $\left(\frac{56}{\cdot}\right)$.
(ii) The character induced by the mod 28 character $\left(\frac{28}{\cdot}\right)$. See Section 4.10.
(iii) The character induced by the mod 8 character $\left(\frac{8}{\cdot}\right)$.

Only the third character led to new identities. In this section we assume χ is the mod 56 character induced by $\left(\frac{8}{\cdot}\right)$. Thus in this section

$$\begin{aligned} G(1,56,\chi) = G(1) &= \frac{1}{\eta_{56,1}(\tau)\,\eta_{56,9}(\tau)\,\eta_{56,15}(\tau)\,\eta_{56,17}(\tau)\,\eta_{56,23}(\tau)\,\eta_{56,25}(\tau)} \\ &= \frac{q^{11/8}}{(q,q^9,q^{15},q^{17},q^{23},q^{25},q^{31},q^{33},q^{39},q^{41},q^{47},q^{55};q^{56})_\infty}, \end{aligned}$$

$$\begin{aligned} H(1,56,\chi) = H(1) &= \frac{1}{\eta_{56,3}(\tau)\,\eta_{56,5}(\tau)\,\eta_{56,11}(\tau)\,\eta_{56,13}(\tau)\,\eta_{56,19}(\tau)\,\eta_{56,27}(\tau)} \\ &= \frac{q^{-13/8}}{(q^3,q^5,q^{11},q^{13},q^{19},q^{27},q^{29},q^{37},q^{43},q^{45},q^{51},q^{53};q^{56})_\infty} \end{aligned}$$

4.15.1. *Type 1.*

$$G(2)\,H(1)+G(1)\,H(2)=\frac{\eta(2\tau)\eta(4\tau)\eta(14\tau)}{\eta(\tau)\eta(8\tau)\eta(56\tau)},\quad \Gamma_1(112),\quad -B=144. \tag{4.119}$$

4.15.2. *Type 6.*

$$G^*(1)\,G^*(1)-H^*(1)\,H^*(1)=2\,\frac{\eta(4\tau)\eta(14\tau)\eta(16\tau)\eta(112\tau)}{\eta(8\tau)^2\eta(56\tau)^2},\quad \Gamma_1(112),\quad -B=144. \tag{4.120}$$

4.15.3. *Type 7.*

$$G^*(1)\,G(1)-H^*(1)\,H(1)=-1\,\frac{\eta(8\tau)\eta(14\tau)\eta(56\tau)}{\eta(16\tau)\eta(28\tau)\eta(112\tau)},\quad \Gamma_1(112),\quad -B=144. \tag{4.121}$$

4.15.4. *Type 8.*
(4.122)

$$G(1)^2\,H(2)-H(1)^2\,G(2)=2\,\frac{\eta(2\tau)\eta(4\tau)\eta(7\tau)\eta(16\tau)\eta(112\tau)}{\eta(\tau)\eta(8\tau)^2\eta(56\tau)^2},\quad \Gamma_1(112),\quad -B=144.$$

4.16. **Mod** 60**.** There are three real non-principal characters mod 60 that satisfy $\chi(-1)=1$.

(i) The character induced by $\left(\frac{\cdot}{5}\right)$.
(ii) The character $\chi_{60,2}(\cdot)=\left(\frac{60}{\cdot}\right)$. See Section ??.
(iii) The character $\chi_{60,3}(\cdot)$ induced by the mod 12 character $\left(\frac{12}{\cdot}\right)$.

Only (ii), (iii) seem to lead to new identities.

4.16.1. $\chi_{60,2}$. In this section

$$G(1,60,\chi_{60,2})=G(1)=\frac{1}{\eta_{60,1}(\tau)\,\eta_{60,7}(\tau)\,\eta_{60,11}(\tau)\,\eta_{60,17}(\tau)}$$
$$=\frac{q^{-35/6}}{(q,q^7,q^{11},q^{17},q^{43},q^{49},q^{53},q^{59};q^{60})_\infty},$$
$$H(1,60,\chi_{60,2})=H(1)=\frac{1}{\eta_{60,13}(\tau)\,\eta_{60,19}(\tau)\,\eta_{60,23}(\tau)\,\eta_{60,29}(\tau)}$$
$$=\frac{q^{37/6}}{(q^{13},q^{19},q^{23},q^{29},q^{31},q^{37},q^{41},q^{47};q^{60})_\infty}.$$

Type 1.

$$G(2)\,H(1)-G(1)\,H(2)=\frac{\eta(4\tau)\eta(6\tau)\eta(10\tau)\eta(30\tau)}{\eta(2\tau)\eta(60\tau)^3},\quad \Gamma_1(120),\quad -B=192. \tag{4.123}$$

Type 6.
(4.124)

$$G^*(1)\,G^*(1)-H^*(1)\,H^*(1)=2\,\frac{\eta(4\tau)\eta(12\tau)^2\eta(20\tau)^2\eta(30\tau)^3\eta(120\tau)^4}{\eta(2\tau)\eta(24\tau)\eta(40\tau)\eta(60\tau)^9},\quad \Gamma_1(120),\quad -B=192.$$

Type 8.
(4.125)

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)\eta(5\tau)\eta(12\tau)^2\eta(20\tau)^2\eta(30\tau)^4\eta(120\tau)^4}{\eta(\tau)\eta(6\tau)\eta(10\tau)\eta(15\tau)\eta(24\tau)\eta(40\tau)\eta(60\tau)^9}, \quad \Gamma_1(120), \quad -B = 192.$$

4.16.2. $\chi_{60,3}$. In this section

$$\begin{aligned} G(1,60,\chi_{60,3}) = G(1) &= \frac{1}{\eta_{60,1}(\tau)\,\eta_{60,11}(\tau)\,\eta_{60,13}(\tau)\,\eta_{60,23}(\tau)} \\ &= \frac{q^{-17/6}}{(q,q^{11},q^{13},q^{23},q^{37},q^{47},q^{49},q^{59};q^{60})_\infty}, \end{aligned}$$

$$\begin{aligned} H(1,60,\chi_{60,3}) = H(1) &= \frac{1}{\eta_{60,7}(\tau)\,\eta_{60,17}(\tau)\,\eta_{60,19}(\tau)\,\eta_{60,29}(\tau)} \\ &= \frac{q^{19/6}}{(q^7,q^{17},q^{19},q^{29},q^{31},q^{41},q^{43},q^{53};q^{60})_\infty}. \end{aligned}$$

Type 1.

$$G(2)\,H(1) - G(1)\,H(2) = \frac{\eta(4\tau)\eta(6\tau)^2\eta(10\tau)}{\eta(2\tau)\eta(12\tau)^2\eta(60\tau)}, \quad \Gamma_1(120), \quad -B = 160. \tag{4.126}$$

Type 6.
(4.127)

$$G^*(1)\,G^*(1) - H^*(1)\,H^*(1) = 2\,\frac{\eta(4\tau)\eta(6\tau)^2\eta(20\tau)^2\eta(24\tau)\eta(30\tau)\eta(120\tau)^2}{\eta(2\tau)\eta(12\tau)^3\eta(40\tau)\eta(60\tau)^4}, \quad \Gamma_1(120), \quad -B = 160.$$

Type 7.
(4.128)

$$G^*(1)\,G(1) - H^*(1)\,H(1) = \frac{\eta(4\tau)\eta(6\tau)\eta(10\tau)\eta(40\tau)\eta(60\tau)^2}{\eta(2\tau)\eta(20\tau)\eta(24\tau)\eta(30\tau)\eta(120\tau)^2}, \quad \Gamma_1(120), \quad -B = 160.$$

Type 8.
(4.129)

$$G(1)^2\,H(2) - H(1)^2\,G(2) = 2\,\frac{\eta(3\tau)\eta(4\tau)\eta(5\tau)\eta(6\tau)\eta(20\tau)^2\eta(24\tau)\eta(30\tau)^2\eta(120\tau)^2}{\eta(\tau)\eta(10\tau)\eta(12\tau)^3\eta(15\tau)\eta(40\tau)\eta(60\tau)^4}, \quad \Gamma_1(120), \quad -B = 160.$$

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